Recall

$X = (X, (\sigma_\alpha : \Delta^n \to X)_{\alpha \in I_n, n \in \mathbb{N}})$: Δ-complex

$\Delta_n(X) = \{ \underbrace{\sum_{\alpha \in I_n} n_\alpha \sigma_\alpha}_{\text{finite sum}} : n_\alpha \in \mathbb{Z} \}$

$\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$, $\sigma_\alpha \mapsto \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$

"omit"

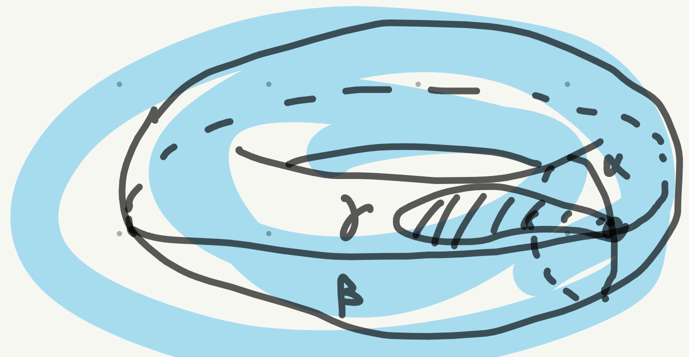
$H_n^\Delta(X) = \ker \partial_n / \operatorname{img} \partial_{n+1}$: n-th singular homology

$(H_0^\Delta(X) = \Delta_0(X) / \operatorname{img} \partial_1)$

Basic idea : $H_n^\Delta(X)$ counts the number of

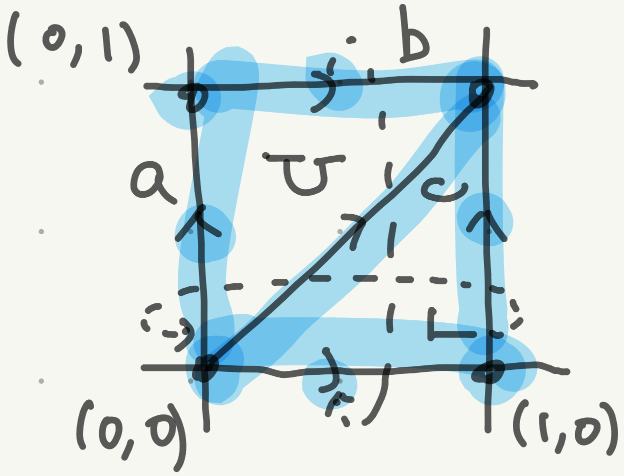
$S^n \cong \partial \Delta^{n+1} \to X$ that is <u>not</u>

the boundary of $B^{n+1} \cong \Delta^{n+1} \to X$

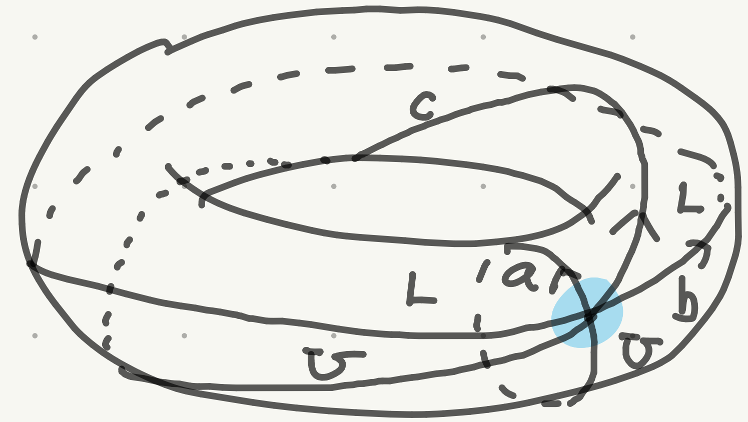


Example 1. $X = T \cong S^1 \times S^1 \cong \mathbb{R}^2 / \mathbb{Z}^2$

2-dim torus

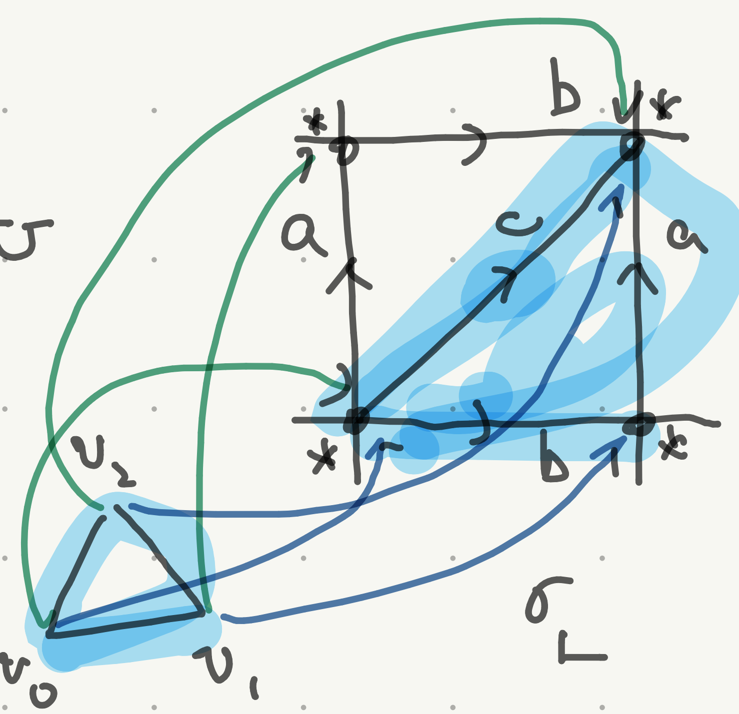


$\mathbb{R}^2$ $\rightsquigarrow$



$$[v_0, v_1, v_2] \overset{\partial_2}{\longmapsto} [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

σ_U



$$\partial_2 \sigma_U = \sigma_a - \sigma_c + \sigma_b$$

$$\partial_2 \sigma_L = \sigma_a - \sigma_c + \sigma_b$$

$$\partial_1 \sigma_a = \sigma_* - \sigma_* = 0$$

$$\partial_1 \sigma_b = 0, \quad \partial_1 \sigma_c = 0$$

$\Delta_n(T) = 0 \quad (n \geq 3)$

$\Delta_2(T) = \mathbb{Z}\sigma_U \oplus \mathbb{Z}\sigma_L$ $\Big\downarrow \partial_2 : \sigma_U, \sigma_L \mapsto \sigma_a - \sigma_c + \sigma_b$

$\Delta_1(T) = \mathbb{Z}\sigma_a \oplus \mathbb{Z}\sigma_b \oplus \mathbb{Z}\sigma_c$

$\Big\downarrow \partial_1 = 0$

$\Delta_0(T) = \mathbb{Z}\sigma_x$

so $\ker \partial_2 = \mathbb{Z}(\sigma_U - \sigma_L)$, $\operatorname{img} \partial_3 = 0$

$\rightsquigarrow H_2^\Delta(T) \cong \mathbb{Z}$

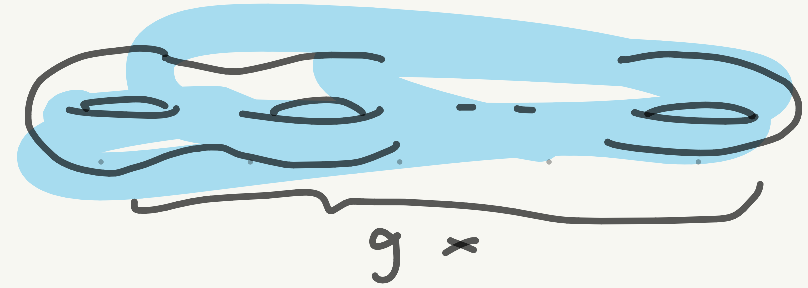
$\ker \partial_1 = \Delta_1(T)$, $\operatorname{img} \partial_2 = \mathbb{Z}(\sigma_a - \sigma_c + \sigma_b)$

$\rightsquigarrow H_1^\Delta(T) = \mathbb{Z}[\sigma_a] \oplus \mathbb{Z}[\sigma_b] \cong \mathbb{Z}^2$

"$\ker \partial_0$" $= \Delta_0(T)$, $\operatorname{img} \partial_1 = 0$

$\rightsquigarrow H_0^\Delta(T) \cong \mathbb{Z}$

Example 2. $X = M_g$



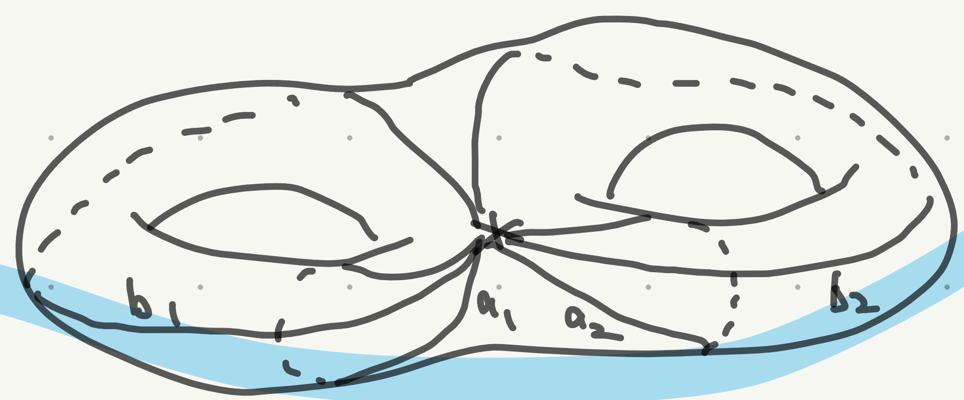
for 2-dim cell : we only care what ∂_2 is (and not img ∂_3 as it will be 0)

we take non-triangle 2-cell

one 0-cell (basept) $\rightsquigarrow \partial_1 = 0$, so we put orientation on 1-cell to capture ∂_2(2-cell)

- one 0-cell $*$
- $2g$ 1-cells $a_1, b_1, \ldots, a_g, b_g$

  meridian around first hall, longitude ...
- one 2-cell f



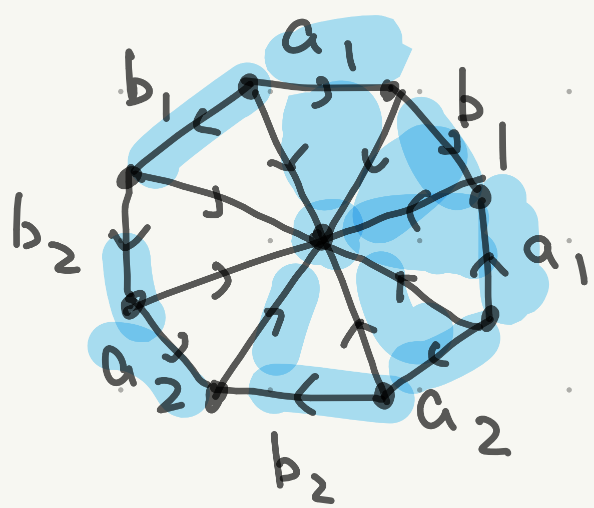
$$\partial_2 \sigma_f = \sigma_{a_1} + \sigma_{b_1} - \sigma_{a_1} - \sigma_{b_1} + \cdots = 0$$

$$\partial_1 \sigma_{a_i} = 0 = \partial \sigma_{b_i}$$

$$\rightsquigarrow \quad H_2^{``\Delta"}(M_g) \cong \mathbb{Z}, \quad H_1^{``\Delta"}(M_g) \cong \mathbb{Z}^{2g}, \quad H_0^{``\Delta"}(M_g) \cong \mathbb{Z}$$

same as chains

more honest model



M_2 with

two 0-cells $\qquad \Delta_0(M_g) = \mathbb{Z}^2$

twelve 1-cells $\qquad \Delta_1(M_g) = \mathbb{Z}^{12}$

eight 2-cells $\qquad \Delta_2(M_g) = \mathbb{Z}^8$

img ∂_2 : "7-dim", img ∂_1 : "1-dim"

ker ∂_2 : "1-dim", ker ∂_1 : "11-dim" (cont.) $H_2^{\Delta}(M_g) \cong \mathbb{Z}$, $H_1^{\Delta}(M_g) \cong \mathbb{Z}^{2g}$, $H_0^{\Delta}(M_g) \cong \mathbb{Z}$

with analogous triangulation on M_g

connection to Euler characteristic

P : polyhedron (maybe polygon ⋯)

$$C_2 = \bigoplus_{f:\text{faces}} \mathbb{Z} f = \Big\{ \sum_{f:\text{faces}} n_f f \;:\; n_f \in \mathbb{Z} \Big\}$$

$$C_1 = \bigoplus_{e:\text{edges}} \mathbb{Z} e$$

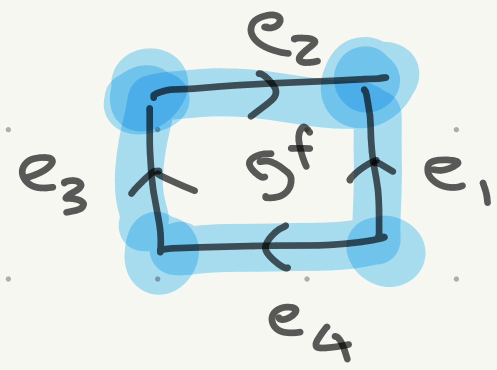
$$C_0 = \bigoplus_{v:\text{vertices}} \mathbb{Z} v$$

Put orientation on faces & edges

$$\partial_2 : C_2 \to C_1, \quad \partial_2 f = \sum_{e \in \partial f} \pm e$$

according to ori.

$$\partial_1 : C_1 \to C_0, \quad \partial_1 e = v_r - v_s \qquad v_s \xrightarrow{e} v_r$$

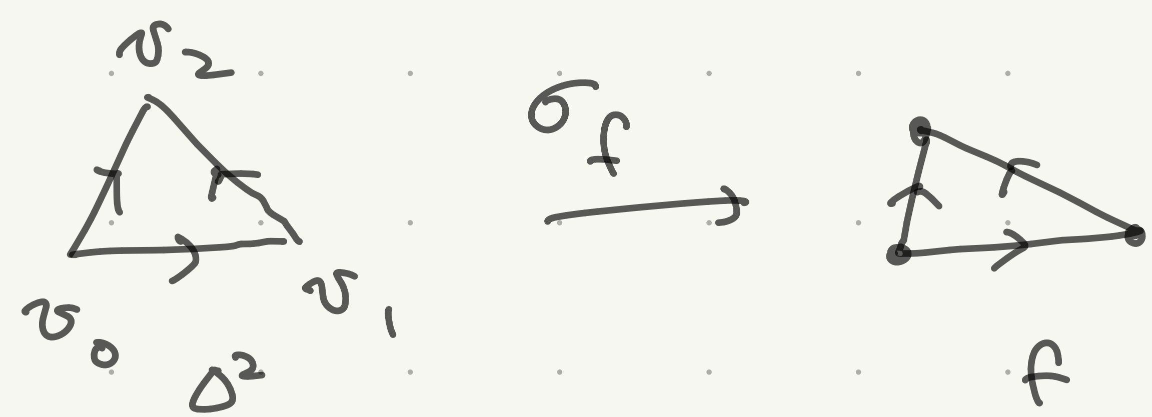


$$\partial_2 f = e_1 - e_2 - e_3 - e_4$$

$$H_2(P) = \ker \partial_2, \quad H_1(P) = \ker \partial_1 / \operatorname{img} \partial_2, \quad H_0(P) = \overset{\operatorname{cok} \partial_1}{C_0 / \operatorname{img} \partial_1}$$

when the faces are triangles we get Δ-complex

by $\sigma_f : \Delta^2 \xrightarrow{\cong} f$



(choose orientation on edges so this can be done consistently)

so $\partial_2 f$ recovers $\partial \sigma_f$ for simplicial homology

$\rightsquigarrow H_i(P) \cong H_i^{\Delta}(P)$

To pick up "dimension" of $H_i(P)$, etc.

Def. M commutative group ($\mathbb{Z}$-module)

its <u>rationalization</u> is

$$M \otimes_{\mathbb{Z}} \mathbb{Q} = \{ m \otimes q : m \in M, q \in \mathbb{Q} \text{ with rel.}$$

$$\underbrace{(m + \cdots + m)}_{k \text{ times}} \otimes q = m \otimes kq \}$$

$\rightsquigarrow$ vector space over $\mathbb{Q}$ with $\quad m \otimes \frac{n}{d} = nm \otimes \frac{1}{d}$

$$\left(m_1 \otimes \frac{n_1}{d_1}\right) + \left(m_2 \otimes \frac{n_2}{d_2}\right) = (d_2 n_1 m_1 + d_1 n_2 m_2) \otimes \frac{1}{d_1 d_2}$$

the <u>rank</u> of M: $\operatorname{rk} M = \dim_{\mathbb{Q}} (M \otimes_{\mathbb{Z}} \mathbb{Q})$

$$\operatorname{rk} (\mathbb{Z}^k \oplus \underline{\mathbb{Z}/p_1^{e_1} \oplus \cdots \oplus \mathbb{Z}/p_n^{e_n}}) = k$$

Prop. M_0, M_1, M_2 : comm. grps

$\partial_2 : M_2 \to M_1$, $\partial_1 : M_1 \to M_0$ homs s.t. $\partial_1 \partial_2 = 0$

$H_2 = \ker \partial_2$, $H_1 = \ker \partial_1 / \operatorname{img} \partial_2$, $H_0 = \operatorname{cok} \partial_1$

Then

$$\operatorname{rk} M_0 - \operatorname{rk} M_1 + \operatorname{rk} M_2 = \operatorname{rk} H_0 - \operatorname{rk} H_1 + \operatorname{rk} H_2$$

Proof Step 1 : $\partial_i : M_i \to M_{i-1}$ induces lin. map

$$\tilde{\partial}_i : M_i \otimes \mathbb{Q} \to M_{i-1} \otimes \mathbb{Q}$$

We have $\ker \tilde{\partial}_i = (\ker \partial_i) \otimes \mathbb{Q}$, $\operatorname{cok} \tilde{\partial}_i = (\operatorname{cok} \partial_i) \otimes \mathbb{Q}$

("$\mathbb{Q}$ is flat over $\mathbb{Z}$")

$\tilde{\partial}_i (m \otimes \frac{1}{d}) = 0 \Rightarrow \exists k : k \partial_i m = 0$

$\Rightarrow m \otimes \frac{1}{d} = km \otimes \frac{1}{kd} \in (\ker \partial_i) \otimes \mathbb{Q}$

cok is easier

Step 2 $\operatorname{rk} H_1 = \dim \ker \tilde{\partial}_1 - \dim \operatorname{img} \tilde{\partial}_2$, etc.

$$\operatorname{rk} H_1 = \dim_{\mathbb{Q}} H_1 \otimes \mathbb{Q}$$

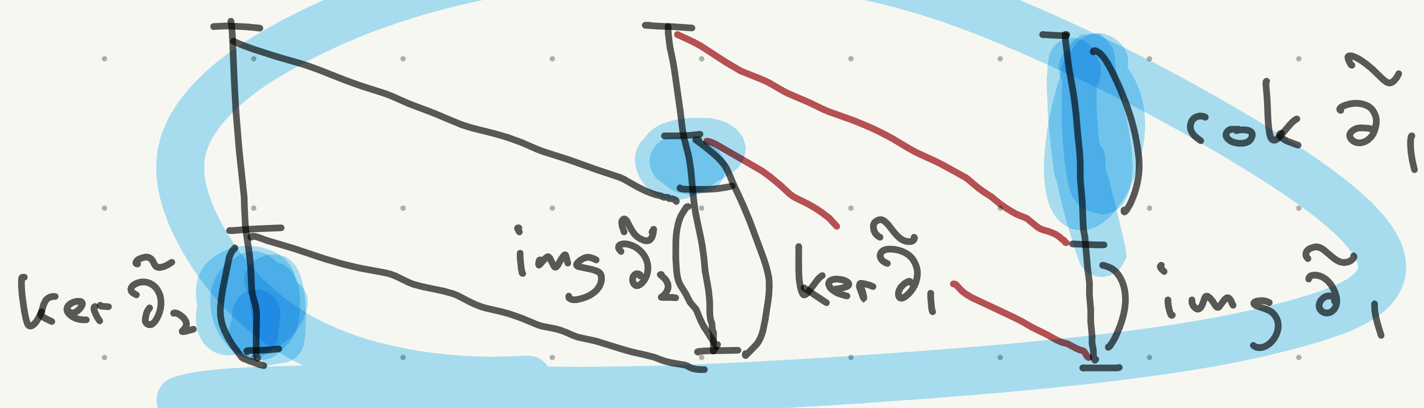
$$H_1 \otimes \mathbb{Q} \cong (\ker \tilde{\partial}_1)/(\operatorname{img} \tilde{\partial}_2)$$

$$(A/B) \otimes \mathbb{Q} \cong (A \otimes \mathbb{Q})/(B \otimes \mathbb{Q}) \quad \text{for } B < A$$

So $\dim H_1 = \dim \ker \tilde{\partial}_1 - \dim \operatorname{img} \tilde{\partial}_2$

Step 3 $\operatorname{rk} M_0 - \operatorname{rk} M_1 + \operatorname{rk} M_2 =$

$$\underbrace{\dim \ker \tilde{\partial}_2}_{\operatorname{rk} H_2} - \underbrace{(\dim \ker \tilde{\partial}_1 - \dim \operatorname{img} \tilde{\partial}_2)}_{\operatorname{rk} H_1} + \underbrace{\dim \operatorname{cok} \tilde{\partial}_1}_{\operatorname{rk} H_0}$$



Back to polyhedron P, $C_2 \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{f:\text{ faces}} \mathbb{Q} f$, etc.

$\operatorname{rk} C_2 = |\text{faces}|$, $\operatorname{rk} C_1 = |\text{edges}|$, $\operatorname{rk} C_0 = |\text{vertices}|$

So Prop implies

$\chi(P) = |\text{faces}| - |\text{edges}| + |\text{vertices}|$ is equal to

$\operatorname{rk} H_2 - \operatorname{rk} H_1 + \operatorname{rk} H_0$

invariance of $\chi(P)$ under homotopy would follow

from invariance of $H_i(P)$

Singular homology

More universal construction of homology by incorpolating "singular" simplices

i.e. take arbitrary cont. maps $\Delta^n \xrightarrow{\sigma} X$

(and do not assume that $\forall x \in X \ \exists! \sigma \ \ x \in \sigma(\mathring{\Delta}^n)$)

Def X : topological space $n \in \mathbb{N}$

a singular n-simplex in X is given by a continuous map $\sigma : \Delta^n \to X$

Δ^n: std. n-simplex

the group of singular n-chains :

$$C_n(X) = \bigoplus_{\sigma : \text{sing. } n\text{-chain}} \mathbb{Z}\sigma \cong \{ \underbrace{\sum_{\sigma} n_\sigma \cdot \sigma}_{\text{fin. sum}} : n_\sigma \in \mathbb{Z} \}$$

boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} : \Delta^{n-1} \to \Delta^n \xrightarrow{\sigma} X$$

emb. into convex span of $v_0, \dots, \hat{v}_i, \dots, v_n$

Lem $\partial_n \partial_{n+1} = 0$

same proof as before

Def. the n-th <u>singular</u> <u>homology group</u> of X

$$H_n^{\text{sing}}(X) = H_n(X) = \ker \partial_n / \operatorname{img} \partial_{n+1}$$

$H^\Delta(X)$ vs. $H^{sing}(X)$

computability

- $H^\Delta(X)$: can be (in principle) directly computed from cell complex presentation and face maps $\partial^n_i : I_n \to I_{n-1}$ ← ind. for n-simplexes, ← for $(n-1)$-simplexes
  "i-th face" index map
- $H^{sing}(X)$: impossible to compute directly (cannot enumerate $\Delta^n \xrightarrow{\sigma} X$, etc.)

naturality

- $H^\Delta(X)$: a priori depends on Δ-complex structure
  not clear what happens when we change it
  & not flexible under cont. deform., maps

(cont.)

- $H^{sing}(X)$: functorial for cont. maps

$X \xrightarrow{f} Y$ cont. map

$\rightsquigarrow$ $H_n^{sing}(X) \xrightarrow{f_*} H_n^{sing}(Y)$ induced by

$C_n(X) \to C_n(Y)$, $\sum_\sigma n_\sigma \cdot \sigma \mapsto \sum_\sigma n_\sigma \cdot f \circ \sigma$

$(gf)_* = g_* f_*$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$

$(id_X)_* = id_{H^{sing}(X)}$

Questions

Δ-complex $\qquad$ underlying top. space

- when do we have $H_n^{\Delta}(X) \cong H_n^{\text{sing}}(X)$?

  will be OK for Δ-complexes

  $\rightsquigarrow$ justifies $H_n(X) = H_n^{\Delta}(X) = H_n^{\text{sing}}(X)$

  at least for top. sp. w/ Δ-complex structure

- is $f_*: H_n(X) \to H_n(Y)$ invariant under continuous deform. of $X \xrightarrow{f} Y$?

  will be YES $\quad (f_t)_{t\in I} \quad (f_0)_* = (f_1)_*: H(X) \to H(Y)$

- "relative" version $H_n(X, A)$ for CW pairs

  $A \subset X$ subcomplex ; relating $H(X)$, $H(A)$

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots$$

induced by $A \to X$ | induced by $(X,\phi) \to (X,A)$ | boundary map

<u>exact</u> <u>sequence</u>

$$\cdots \to M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} M_{n-2} \to \cdots$$ is <u>exact</u> if

comm. grp.

img f_n = ker f_{n-1} for all n