Basic properties of homology

We write $H_n(X) = H_n^{sing}(X)$ for top. sp. X

(but the following properties should hold for any model of homology up to some modification)

Prop 2.6 $(X_\alpha)_{\alpha \in I}$: the path-components of X

$$H_n(X) \cong \bigoplus_{\alpha \in I} H_n(X_\alpha) \quad \text{for each } n \in \mathbb{N}$$

Proof any singular n-simplex $\Delta^n \xrightarrow{\sigma} X$ is of the form $\Delta^n \to X_\alpha \hookrightarrow X$ for unique $\alpha \in I$

$\rightsquigarrow$ boundary is also a collection of $(n-1)$-simpl.

$\sigma_\beta : \Delta^{n-1} \to X_\alpha$ for same α

(cont.) • $\underbrace{C_n(X)}_{\text{lin. comb. of } \Delta^n \xrightarrow{\sigma} X} \cong \bigoplus_{\alpha \in I} C_n(X_\alpha)$

• $\partial_n : C_n(X) \to C_{n-1}(X)$ preserves this direct sum decomposition

$\Rightarrow \ker \partial_n \cong \bigoplus_\alpha (\ker \partial_n) \cap C_n(X_\alpha)$, same for $\operatorname{img} \partial_{n+1}$

$\Rightarrow \quad H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$

Prop 2.7 $\quad X \neq \emptyset$, path-connected $\Rightarrow H_0(X) \cong \mathbb{Z}$

Proof. We have the augmentation map

$$\varepsilon : \underbrace{C_0(X)}_{\text{lin. comb. of points}} \to \mathbb{Z}, \quad \underbrace{\sum_{x \in X} n_x x}_{\text{(finite sum)}} \mapsto \sum n_x$$

We want to show $\ker \varepsilon = \operatorname{img} \partial_1$

$\ker \varepsilon \supset \operatorname{img} \partial_1$:

$\partial_1(\sigma) = \sigma(v_1) - \sigma(v_0)$ for the endpoints v_i

map $\underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ I}}{\Delta^1} \to X$

for $\sum_\sigma n_\sigma \cdot \sigma \in C_1(X)$ we have

$$\partial\left(\sum_\sigma n_\sigma \cdot \sigma\right) = \sum_\sigma n_\sigma (\sigma(v_1) - \sigma(v_0)) \in \ker \varepsilon$$

$\ker \varepsilon \subset \operatorname{img} \partial_1$: fix a basepoint $x_0 \in X$

for each $x \in X$ take $\sigma_x : \Delta^1 \to X$ s.t.

$\sigma_x(v_1) = x$, $\sigma_x(v_0) = x_0$ (possible by path-conn.)

$$\sum_x n_x \cdot x \in \ker \varepsilon \Rightarrow \partial_1\left(\sum_x n_x \cdot \sigma_x\right) = \sum_x n_x \cdot x$$

by $\sum_x n_x \cdot x_0 = 0$

Prop 2.8 $\quad H_n(\{*\}) = 0 \quad$ for $\quad n > 0$

(and $H_0(\{*\}) \cong \mathbb{Z}$ by Prop 2.7)

Proof there is a unique singular n-simplex $\Delta^n \xrightarrow{\sigma_n} \{*\}$

$$\Rightarrow \quad C_n(\{*\}) \cong \mathbb{Z}, \quad \partial_n \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} \sigma_n & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\begin{array}{ccccccccccccc}
\cdots \to & C_{2k}(\{*\}) & \longrightarrow & C_{2k-1}(\{*\}) & \longrightarrow & C_{2k-2}(\{*\}) & \to & \cdots & \to & C_1(\{*\}) & \to & C_0(\{*\}) \\
& \cong & & \cong & & \cong & & & & \cong & & \cong \\
\xrightarrow{0} & \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & \xrightarrow{\quad 0 \quad} & \mathbb{Z} & \to & \cdots & \to & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}
\end{array}$$

ker : 0 $\qquad$ ker : $\mathbb{Z}$

img : 0 $\qquad$ img : $\mathbb{Z}$

$\Downarrow \qquad\qquad \Downarrow$

$H_{2k}(\{*\}) = 0 \qquad H_{2k-1}(\{*\}) = 0$

Def **reduced homology group** $\tilde{H}_n(X)$

$\tilde{H}_n(X) = H_n(X)$ for $n > 0$

augmentation

$\tilde{H}_0(X) = \ker \varepsilon / \operatorname{img} \partial_1$ for $\varepsilon : C_0(X) \to \mathbb{Z}$

$\sum n_x \cdot x \mapsto \sum n_x$

(we can interpret $\Delta^{-1} = \emptyset$, $C_{-1}(X) \cong \mathbb{Z} \cdot (\text{unique } \Delta^{-1} \to X)$)

Rem $H_0(X) \cong \mathbb{Z} \oplus \tilde{H}_0(X)$

Functoriality

$f: X \to Y$ continuous map

Notation $f_\#: C_n(X) \to C_n(Y)$ homomorphism

$$f_\#\left(\sum_\sigma n_\sigma \cdot \sigma\right) = \sum_\sigma n_\sigma \cdot \underbrace{(f \circ \sigma)}_{\text{sing. } n\text{-simplex in } Y}$$

Lem $f_\# \circ \partial_n = \partial_n \circ f_\#$

$$f_\#\left(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}\right) = f \circ \left(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}\right)$$

$$= (f \circ \sigma)|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$\Delta^{n-1} \to \Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$$

emb. to conv. span of $v_0, \dots, \hat{v}_i, \dots, v_n$

Prop $f_\#$ induces $f_* : H_n(X) \to H_n(Y)$

Proof $f_\#(\underbrace{\ker \partial_n}_{\text{subgrp of } C_n(X)}) \subset \underbrace{\ker \partial_n}_{\text{subgrp of } C_n(Y)}$

$f_\#(\operatorname{img} \partial_{n+1}) \subset \operatorname{img} \partial_{n+1}$

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "f_\#"] \arrow[d, "\partial_{n+1}"] & C_{n+1}(Y) \arrow[d, "\partial_{n+1}"] \\
C_n(X) \arrow[r, "f_\#"] \arrow[d, "\partial_n"] & C_n(Y) \arrow[d, "\partial_n"] \\
C_{n-1}(X) \arrow[r, "f_\#"] & C_{n-1}(Y)
\end{tikzcd}$$

$\rightsquigarrow$ $f_* : H_n(X) = \ker \partial_n / \operatorname{img} \partial_{n+1} \longrightarrow H_n(Y)$

is well-defined

Algebraic framework behind homology

Def. a <u>chain complex</u> is given by

- collection of comm. groups $(C_n)_{n\in\mathbb{Z}}$

  (or <u>graded group</u> $C_\bullet = \bigoplus_{n\in\mathbb{Z}} C_n$)

- maps $d_n : C_n \to C_{n-1}$ s.t. $d_{n-1} d_n = 0$

  (or degree (-1) map $d : C_\bullet \to C_\bullet$, $d^2 = 0$)

the <u>n-th homology</u> of such chain complex is

$$H_n(C_\bullet, d_\bullet) = \ker d_n / \operatorname{img} d_{n+1}$$

(chain) map of chain complexes $f : C_\bullet \to C'_\bullet$

$f_n : C_n \to C'_n$ s.t. $d'_n \circ f_n = f_{n-1} \circ d_n$
$n \in \mathbb{Z}$

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & C'_n \\ d_n \downarrow & & \downarrow d'_n \\ C_{n-1} & \xrightarrow{f_{n-1}} & C'_{n-1} \end{array}$$

Singular chains and singular homology as functors

$$\left[\begin{array}{l}\text{Category of} \\ \cdot\ \text{top. spaces } X \\ \cdot\ \text{cont. maps } X \xrightarrow{f} Y\end{array}\right] \longrightarrow \left[\begin{array}{l}\text{Category of} \\ \text{- chain complexes } (C_\bullet, d_\bullet) \\ \text{- map of complexes } C_\bullet \xrightarrow{f} C'_\bullet\end{array}\right]$$

$$X \longmapsto \left((C_n(X))_{n=0}^{\infty}, \partial \right)$$

$$f \longmapsto f_\#$$

$$\Big\downarrow H_\bullet$$

$$(H_n^{sing}(X))_{n=0}^{\infty} \qquad \left[\begin{array}{l}\text{Category of} \\ \text{- graded comm. grps } C_\bullet \\ \text{- degree 0 maps } C_\bullet \xrightarrow{f} C'_\bullet\end{array}\right]$$

$$f_*$$

Q. what is the analogue of homotopy for chain complexes?

Def. $(C_\bullet, \partial)$, $(C'_\bullet, \partial')$ chain complexes

$f, g : (C_\bullet, \partial) \to (C'_\bullet, \partial')$ map of complexes

a <u>chain homotopy</u> between f and g is given

by a collection of maps $h_n : C_n \to C'_{n+1}$ $n \in \mathbb{Z}$ s.t.

$$h_{n-1} \partial + \partial' h_n = f_n - g_n$$

$\exists$ such $h \rightsquigarrow$ say f and g are homotopic, $f \simeq g$

(this is an equiv. rel. on chain maps)

Lem $f, g : (C_\bullet, \partial) \to (C'_\bullet, \partial')$ map of chain cplxes

$f \simeq g \Rightarrow f_* = g_* : H_\bullet(C_\bullet, \partial) \to H_\bullet(C'_\bullet, \partial')$

Proof Take $\alpha \in \ker \partial_n \subset C_n$

$f_*([\alpha]) = g_*([\alpha]) \Leftrightarrow f(\alpha) = g(\alpha) + \partial' \beta$ for some $\beta \in C'_{n+1}$

take h implementing $f \simeq g$

$\Rightarrow f(\alpha) - g(\alpha) = (h_{n-1}\partial + \partial' h_n)(\alpha) = \partial'(h_n(\alpha))$

Th'm 2.10 X, Y top. sp. $f, g : X \to Y$ cont.

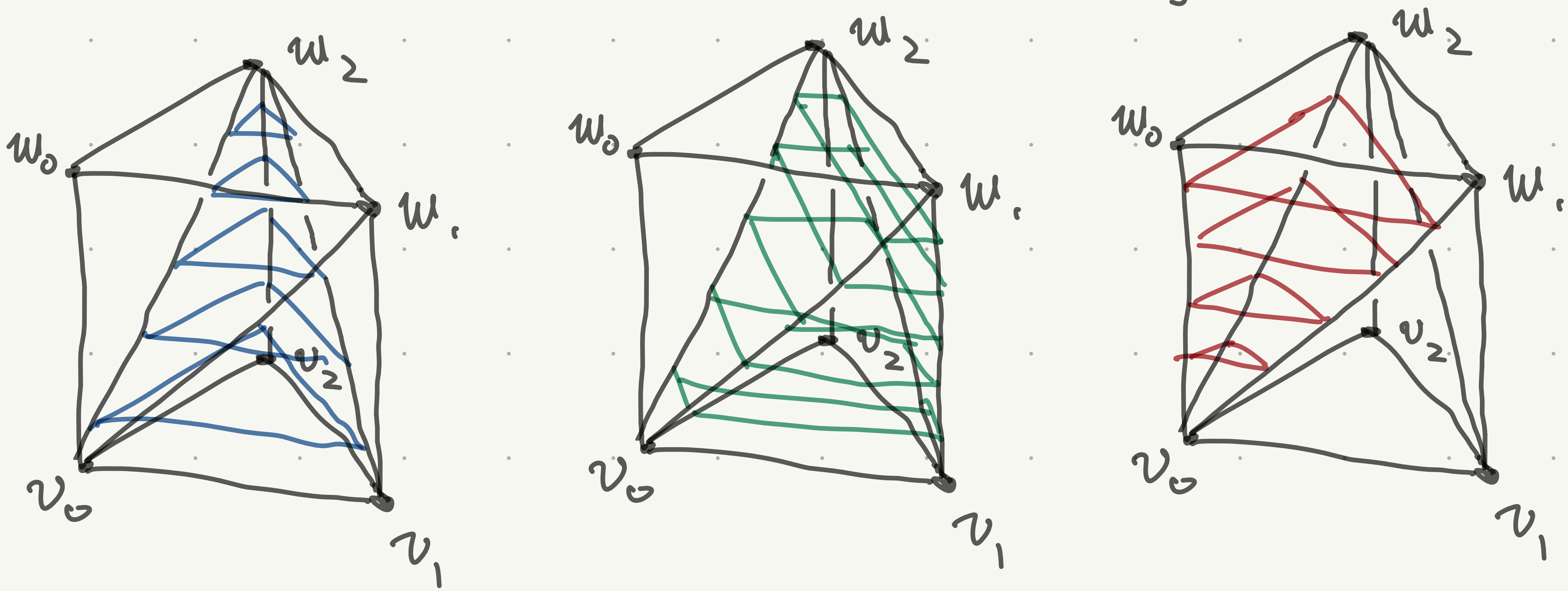
$f \simeq g \Rightarrow f_\# \simeq g_\#$ as maps $C_\bullet(X) \to C_\bullet(Y)$

$\left(\Rightarrow f_* = g_* \text{ as maps } H^{sing}_\bullet(X) \to H^{sing}_\bullet(Y) \right)$

Proof Step 1 Δ-complex structure on $\Delta^n \times I$

Write $\Delta^n \times \{0\} = [v_0, v_1, \ldots, v_n]$

$\Delta^n \times \{1\} = [w_0, w_1, \ldots, w_n]$



(n+1)-simplices in $\Delta^n \times I$

$\sigma_n^{(n+1)} : [v_0, v_1, \dots, v_n, w_n]$

$\sigma_{n-1}^{(n+1)} : [v_0, v_1, \dots, v_{n-1}, w_{n-1}, w_n]$

$\vdots$

$\sigma_0^{(n+1)} : [v_0, w_0, \dots, w_n]$

$\Delta^n \times I = \bigcup_{i=0}^{n} \sigma_i^{(n+1)}$, with only intersection at faces

$(\underbrace{t_0, \dots, t_n}_{\substack{\sum t_i = 1 \\ t_i \geq 0}}, s) \in \Delta^n \times I$ $\quad 0 \leq s \leq 1$

$$\rightsquigarrow t_0 \underset{\substack{\uparrow \\ (1,0,\dots,0,0)}}{v_0} + \dots + t_i' \underset{\substack{\uparrow \\ (0,\dots,\underset{i}{1},\dots,0,0)}}{v_i} + t_i'' \underset{\substack{\uparrow \\ (0,\dots,\underset{i}{1},0,\dots,1)}}{w_i} + t_{i+1} w_{i+1} + \dots + t_n \underset{\substack{\uparrow \\ (0,\dots,0,1,1)}}{w_n}$$

$t_i'' + t_i + \dots + t_n = s$

$t_i' + t_i'' = t_i$

Step 2 construction of chain homotopy

Suppose $F : X \times I \to Y$ implements $f \simeq g$

$P : C_n(X) \to C_{n+1}(Y)$ "prism operator"

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \underbrace{F}_{Y \leftarrow X \times I} \circ \underbrace{(\sigma \times id_I)}_{X \times I \leftarrow \Delta^n \times I} \circ \underbrace{\sigma_i^{(n+1)}}_{\Delta^n \times I \leftarrow \Delta^{n+1}}$$

Step 3 $\quad g_\# - f_\# = P\partial_n + \partial_{n+1} P$ as $C_n(X) \to C_n(Y)$

$$\partial P(\sigma) = \sum_{j=0}^{n+1} (-1)^j \,(\text{omit } j\text{-th vertex}) \sum_i (-1)^i \, F \circ (\sigma \times id_I) \circ \sigma_i^{(n+1)}$$

$$= \sum_{0 \le j \le i \le n} (-1)^{i+j} F \circ (\sigma \times id_I)\big|_{[v_0, \dots, \hat{v}_j, \dots v_i, w_i, \dots, w_n]}$$

$$+ \sum_{\substack{n \ge j' \ge i \ge 0 \\ j'' - 1}} (-1)^{i+j'-1} F \circ (\sigma \times id_I)\big|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_{j'}, \dots, w_n]}$$

$j = i = 0$: $\quad F \circ (\sigma \times id_I)\big|_{[w_0, \dots, w_n]} = \underbrace{F(\sim, 1)}_{g} \circ \sigma = g_\#(\sigma)$
(in first sum)

$j' = i = n$: $\quad F \circ (\sigma \times id_I)\big|_{[v_0, \dots, v_n]} = -\underbrace{F(\sim, 0)}_{f} \circ \sigma = -f_\#(\sigma)$
(in second sum)

$i = j = k$ cancels with $j' = i = k-1$
(first sum) (second sum)

remaining terms:

$$\sum_{0\le j<i\le n} (-1)^{i+j} F\circ(\sigma\times id_I)\big|_{[v_0,\dots,\hat{v}_j,\dots v_i,w_i,\dots,w_n]}$$

$$+ \sum_{n\ge j'>i\ge 0} (-1)^{i+j'-1} F\circ(\sigma\times id_I)\big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_{j'},\dots,w_n]}$$

$$= -P\partial(\sigma) \qquad \text{(check!)}$$

for example: P(skip 0-th vertex of σ) is

$$\sum_{i=0}^{n-1} (-1)^i F\circ(\sigma|_{[v_1,\dots,v_n]}\times id_I)\circ\sigma_i^{(n)}$$

$$= \sum_i (-1)^i F\circ(\sigma\times id_I)\big|_{[v_1,\dots,v_{i+1},w_{i+1},\dots,w_n]}$$

$\boxtimes$

Cor $\quad X\simeq Y \Rightarrow H_n^{sing}(X)\cong H_n^{sing}(Y) \quad (n\in\mathbb{N})$

comparison of $\pi_1(X, x_0)$ and $H_1(X)$

$f : I \to X$, $f(0) = x_0 = f(1)$ loop based at x_0

$\rightsquigarrow$ can be interpreted as a singular 1-chain

$\partial f = x_0 - x_0 = 0$

$\rightsquigarrow$ defines a class $[f] \in H_1^{\text{sing}}(X)$

Thm (2A.1) $\pi_1(X, x_0) \to H_1(X)$, $[f] \mapsto [f]$ is well defined

- kernel: the commutator subgroup $[G, G]$ for $G = \pi_1(X, x_0)$ (normal subgroup generated by $aba^{-1}b^{-1}$)

- if X is path-conn. this map is surjective

in particular $H_1(X) = G^{ab} = G/[G, G]$ for $G = \pi_1(X, x_0)$