Relative homology groups

Recall (singular homology)

X : topological space

$C_n(X) = \{ \sum_\sigma n_\sigma \cdot \sigma \ : \ \sigma : \Delta^n \to X \text{ cont}, \ n_\sigma \in \mathbb{Z} \}$

(finite sum) std n-simplex

$\partial_n : C_n(X) \to C_{n-1}(X)$, $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$

i-th face of σ

$H_n(X) = H_n^{sing}(X) = \ker \partial_n / \operatorname{img} \partial_{n+1}$

$A \overset{i}{\hookrightarrow} X$ incl. of subspace $\rightsquigarrow$ $H_n(A) \xrightarrow{i_*} H_n(X)$

Caution : i_* is <u>not</u> injective in general

ex. $S^{k-1} \hookrightarrow D^k$ $\quad \mathbb{Z} \cong H_{k-1}(S^{k-1}) \to H_{k-1}(D^k) = 0$

Notn./ Def. $A \subset X$ subspace

$$C_n(X, A) = C_n(X) / C_n(A)$$

(regard $\sigma : \Delta^n \to A$ as $\Delta^n \to X$)

$\partial_n : C_n(X, A) \to C_{n-1}(X, A)$ induced by ∂_n

(well-def'ed by $\partial_n(C_n(A)) \subset C_{n-1}(A)$,

$\partial_{n-1}\partial_n = 0$ from same cond. on $C_n(X)$)

n-th relative homology group of (X, A)

$$H_n(X, A) = \ker \partial_n |_{C_n(X,A)} / \operatorname{img} \partial_{n+1}$$

represented by $\alpha \in C_n(X)$

s.t. $\partial_n \alpha \in C_{n-1}(A)$

by $\alpha \in C_n(X)$ s.t.

$\alpha = \partial_{n+1} \beta + \gamma$

$\gamma \in C_n(A)$

Goals

Prop 1 there is an exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

$$\cdots \to H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X,A) \to 0$$

j_*: map induced by quot. map $C_n(X) \to C_n(X)/C_n(A)$

∂: "boundary map" (not to be confused w/ ∂_n)

Prop 2 (2.22) if $\phi \neq A \subset X$ closed,

$\exists \underset{\text{open}}{U} \supset A$ s.t. $A \subset U$ is a deform. retract

then $H_n(X,A) \cong \tilde{H}_n(X/A)$ $(= H_n(X/A)$ for $n>0)$

Getting long exact sequence (Prop. 1)

(purely algebraic)

we have a short exact sequence of chain complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d, "\partial_n"] & C_n(X) \arrow[r] \arrow[d, "\partial_n"] & C_n(X,A) \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

call $(L_\bullet, \partial)$ $\quad (M_\bullet, \partial)$ $\quad (N_\bullet, \partial)$

$\rightsquigarrow$ want $\cdots \to H_n(L_\bullet,\partial) \to H_n(M_\bullet,\partial) \to H_n(N_\bullet,\partial) \xrightarrow{\partial} H_{n-1}(L_\bullet,\partial) \to \cdots$

exact

Prop ("Snake Lemma") given a diagram of comm. grps.

$$\begin{tikzcd}
 & A_1 \arrow[r, "i"] \arrow[d, "f_1"] & A_2 \arrow[r, "j"] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "i"] & B_2 \arrow[r, "j"] & B_3 &
\end{tikzcd}$$

(squares commute

exact rows

there is a naturally defied map $\ker f_3 \xrightarrow{\partial} \operatorname{cok} f_1$ s.t.

$$\ker f_1 \xrightarrow{i_*} \ker f_2 \xrightarrow{j_*} \ker f_3 \xrightarrow{\partial} \operatorname{cok} f_1 \xrightarrow{i_*} \operatorname{cok} f_2 \xrightarrow{j_*} \operatorname{cok} f_3$$

restr. of $A_1 \to A_2$ induced by $B_1 \to B_2$

is exact

Proof Step 0 exactness at $\ker f_2$:

easy argument by diagram chasing

Step 2 exactness at $\operatorname{cok} f_2$

$\operatorname{img} i_*|_{\operatorname{cok} g_1} \subset \ker j_*|_{\operatorname{cok} g_2}$ from $ji = 0$

reverse inclusion :

in $\operatorname{cok} f_3$

take $[b] \in \operatorname{cok} g_2$ $(b \in B_2)$ s.t. $j_*([b]) = 0$

this means $j(b) = f_3(a)$ for some $a \in A_3$

$A_2 \xrightarrow{j} A_3 \to 0$ exact $\Rightarrow \exists\, a' \in A_2$ s.t. $j(a') = a$

put $b' = b - f_2(a')$:

- $[b'] = [b]$ in $\operatorname{cok} f_2$
- $j(b') = j(b) - j(f_2(a')) = j(b) - f_3(j(a')) = 0$

$\Rightarrow b' = i(b'')$ for $b'' \in B_1$ by $B_1 \xrightarrow{i} B_2 \xrightarrow{j} B_3$ ex.

$\Rightarrow i_*([b'']) = [b]$

Step 3 construction of $\partial : \ker f_3 \to \operatorname{cok} f_1$

$$\begin{tikzcd}
 & a' & \ker f_3 \ni a \arrow[d] & \\
 & A_2 \arrow[u] & & \\
 A_1 \arrow[r, "i"'] \arrow[d, "f_1"] & A_2 \arrow[r, "j"] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
 0 \arrow[r] & b \in B_1 \arrow[r, "i"] \arrow[d] & B_2 \arrow[r] & B_3 \\
 & {[b] \in \operatorname{cok} f_1} & &
\end{tikzcd}$$

set $\partial(a) = [b]$ by

- choose $a' \in A_2$: $j(a') = a$
- $b \in B_1$ is the elem. s.t. $i(b) = f_2(a')$

1) $j(f_2(a')) = f_3(j(a')) = 0 \rightsquigarrow$ such b exists

2) any other choice : $a' + i(a'')$ for some $a'' \in A_1$

$\rightsquigarrow$ b is replaced by $b + f_1(a'')$

$\Rightarrow$ $[b]$ is indep. of choice of a' as above

Step 4 exactness of $\ker f_2 \xrightarrow{j^*} \ker f_3 \xrightarrow{\partial} \operatorname{cok} f_1$

- $\partial(a) = 0$ means "$b = f_1(a'')$" in above def.

$$\begin{tikzcd}
 & & \ker f_3 \ni a \arrow[d, hook] \\
a'' \in A_1 \arrow[r] \arrow[d, "f_1"] \arrow[ddd, bend right] & A_2 \ni a' \arrow[r] \arrow[d, "f_2"] & A_3 \\
B_1 \arrow[r, "i"] & B_2 & \\
b \arrow[r, mapsto] & f_2(a') &
\end{tikzcd}$$

- $j(a' - i(a'')) = a$

$\Rightarrow$ we can use $a' - i(a'')$ instead of a'

- $f_2(a' - i(a'')) = 0$

from $f_2(i(a'')) = i(f_1(a''))$

$\Rightarrow$ $a' - i(a'') \in \ker f_2$

$\Rightarrow$ a is in the image of $\ker f_2$

- if $a = j(a')$ for $a' \in \ker f_2$, we set $b = 0$

Step 5 exactness of $\ker f_3 \xrightarrow{\partial} \operatorname{cok} f_1 \xrightarrow{i_*} \operatorname{cok} f_2$

- suppose $i_*([b]) = 0$ i.e. $i(b) = f_2(a')$

  $j(a')$ is in $\ker f_3$: $f_3(j(a')) = j(f_2(a')) = j(i(b))$

  $\rightsquigarrow$ $\partial(j(a')) = [b]$ by const.

- $i_*(\partial(a)) = 0$ from $i_*([b]) = [f_2(a')] = 0$

$$
\begin{tikzcd}
 & a' \arrow[r, mapsto] & a \\
 & A_2 \arrow[r] \arrow[d] & A_1 \\
b \in B_1 \arrow[r] \arrow[d] & B_2 \ni f_2(a') \arrow[d] \arrow[ll, bend right, mapsto] & \\
\operatorname{cok} f_1 \arrow[r] & \operatorname{cok} f_2 \ni 0 &
\end{tikzcd}
$$

$[b]$

Cor. $0 \to (L_*, \partial) \xrightarrow{i} (M_*, \partial) \xrightarrow{j} (N_*, \partial) \to 0$

exact sequence of chain complexes

⇝ long exact seq.

$$\cdots \to H_n(L_\cdot, \partial) \xrightarrow{i_*} H_n(M_\cdot, \partial) \xrightarrow{j_*} H_n(N_\cdot, \partial) \xrightarrow{\partial} H_{n-1}(L_\cdot, \partial) \to \cdots$$

Proof Put $Z_n(L_\cdot) = \ker(L_n \xrightarrow{\partial} L_{n-1})$

$B_n(L_\cdot) = \mathrm{img}(L_{n+1} \to L_n)$, etc.

Take

$$\begin{array}{ccccccc} & L_n / B_n(L_\cdot) & \to & M_n / B_n(M_\cdot) & \to & N_n / B_n(N_\cdot) & \to 0 \\ & \downarrow \partial_* & & \downarrow \partial_* & & \downarrow \partial_* & \\ 0 \to & Z_{n-1}(L_\cdot) & \to & Z_{n-1}(M_\cdot) & \to & Z_{n-1}(N_\cdot) & \end{array}$$

Step 1 $\quad L_n/B_n(L_\bullet) \xrightarrow{\bar{i}_*} M_n/B_n(M_\bullet) \xrightarrow{\bar{j}_*} N_n/B_n(N_\bullet) \to 0$

is exact

- exactness at $N_n/B_n(N_\bullet)$: surjectivity of $\bar{j}$
- at $M_n/B_n(M_\bullet)$: take $[a] \in M_n/B_n(M_\bullet)$

s.t. $\bar{j}_*([a]) = 0$ i.e. $\bar{j}(a) = \partial(b)$ $\quad b \in N_{n+1}$

take $a' \in M_{n+1}$ s.t. $\bar{j}(a') = b$

$$\Rightarrow \bar{j}(a - \partial a') = \bar{j}(a) - \partial(\underbrace{\bar{j}(a')}_{b}) = 0$$

so $a - \partial a' = \bar{i}(c)$ for $c \in L_n$

$$\Rightarrow \bar{i}_*([c]) = [a - \partial a'] = [a]$$

Step 2 $0 \to Z_{n-1}(L_\bullet) \xrightarrow{i_*} Z_{n-1}(M_\bullet) \xrightarrow{j_*} Z_{n-1}(N_\bullet)$ is exact

- exactness at $Z_{n-1}(L_\bullet)$: injectivity of i
- at $Z_{n-1}(M_\bullet)$: take $a \in Z_{n-1}(M_\bullet)$ s.t. $j_*(a)=0$

then $\exists\, b \in L_{n-1}$ s.t. $i(b)=a$; want $d(b)=0$

i injective $\Rightarrow$ enough to have $i(d(b))=0$

but $i(d(b)) = d(i(b)) = d(a) = 0$

$$\begin{tikzcd} L_{n-1} \arrow[r, "i"] \arrow[d, "d"] & M_{n-1} \arrow[d, "d"] \\ L_{n-2} \arrow[r, "i"'] & M_{n-2} \end{tikzcd}$$

Step 3 $\quad H_n(L_\cdot, d) \cong \ker\left(L_n / B_n(L_\cdot) \xrightarrow{d_*} Z_{n-1}(L_\cdot)\right)$

$$H_{n-1}(L_\cdot, d) \cong \operatorname{cok}\left(L_n / B_n(L_\cdot) \xrightarrow{d_*} Z_{n-1}(L_\cdot)\right)$$

from $\quad H_n \cong Z_n / B_n$

Step 4 $\quad H_n(N_\cdot, d) \xrightarrow{\partial} H_{n-1}(L_\cdot, d)$ as in claim

use the snake lemma for

$$\begin{array}{ccccccccc} & & L_n/B_n(L_\cdot) & \to & M_n/B_n(M_\cdot) & \to & N_n/B_n(N_\cdot) & \to & 0 \\ & & \downarrow d_* & & \downarrow d_* & & \downarrow d_* & & \\ 0 & \to & Z_{n-1}(L_\cdot) & \to & Z_{n-1}(M_\cdot) & \to & Z_{n-1}(N_\cdot) & & \end{array}$$

with identification from Step 3

(also for $(M_\cdot, d)$, $(N_\cdot, d)$)

Slight generalization : triples

$B \subset A \subset X \rightsquigarrow$ "triple" (X, A, B)

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0 \quad \text{exact}$$

$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ C_n(A)/C_n(B) & C_n(X)/C_n(B) & C_n(X)/C_n(A) \end{array}$$

$\Rightarrow$ long exact seq.
(Cor.)

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \cdots$$

Rem $C_n(X,\emptyset) = C_n(X)$, $H_n(X,\emptyset) = H_n(X)$

so $B = \emptyset$ subsumes the case of pairs functoriality

$f : (X, A, B) \to (X', A', B')$ map of triples;

$f : X \to X'$ cont. map, $f(A) \subset A'$, $f(B) \subset B'$

$\rightsquigarrow$ map of exact sequences

$$
\begin{tikzcd}
\cdots \arrow[r] & H_n(A,B) \arrow[r] \arrow[d, "f_*"] & H_n(X,B) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A,B) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(A',B') \arrow[r] & H_n(X',B') \arrow[r] & H_n(X',A') \arrow[r, "\partial"] & H_{n-1}(A',B') \arrow[r] & \cdots
\end{tikzcd}
$$

- compatible with composition $(X, A, B) \xrightarrow{f} (X', A', B') \xrightarrow{g} (X'', A'', B'')$
- only depends on homotopy equiv. class of f

To prove "Prop 2":

$$H_n(X,A) \cong \tilde{H}_n(X/A) \quad \left(= H_n(X/A) \text{ for } n>0 \right)$$

for $A \subset X$ closed, s.t. $\exists\, U_{\text{open}} \supset A$: $A \subset U$ is a deform. retract

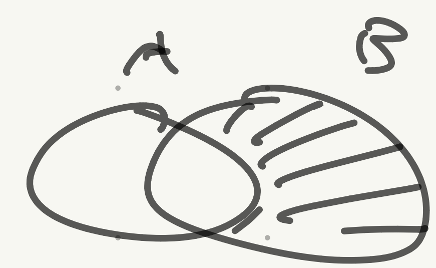
$\emptyset \neq$

We need

Thm 2.20 (Excision) $Z \subset A \subset X$ s.t. $\overline{Z} \subset \mathring{A}$

the incl. map $i : X \setminus Z \hookrightarrow X$ induces isom.

$$H_n(X\setminus Z, A \setminus Z) \cong H_n(X, A)$$



(alternative form) $A, B \subset X$, $\mathring{A} \cup \mathring{B} = X$

$\Rightarrow$ incl. $B \hookrightarrow X$ induces $H_n(B, A \cap B) \cong H_n(X, A)$

$(B = X \setminus Z)$

Excision $\Rightarrow$ Prop 2

Take $X \supset \underset{\text{open}}{U} \supset A$ s.t. $A \subset U$ is a deform. retract

i.e. $\exists\, r : U \to A$ $\quad r \circ i = id_A$, $\; i \circ r \simeq id_U$

- U/A is contractible by the induced map

$r_* : U/A \to A/A \leftarrow$ singleton space

- $(X/A) \setminus (A/A) \cong X \setminus A$, $\; (U/A) \setminus (A/A) \cong U \setminus A$

- $q : (X, U, A) \to (X/A, U/A, A/A)$

induces $H_n(X, A) \to H_n(X/A, A/A)$, etc.

Look at comm. diagram

$$\begin{tikzcd} H_n(X,A) \arrow[r] \arrow[d, "q_*"] & H_n(X,U) \arrow[d, "q_*"] & H_n(X\setminus A, U\setminus A) \arrow[l] \arrow[d, "q_*"] \\ H_n(X/A, A/A) \arrow[r] & H_n(X/A, U/A) & H_n((X/A)\setminus(A/A), (U/A)\setminus(A/A)) \arrow[l] \end{tikzcd}$$

$\cong$

$\tilde{H}_n(X/A)$

Step 1 $H_n(X,A) \to H_n(X,U)$ is iso.

long exact seq. for (X, U, A) gives

$$\cdots \to H_n(U,A) \to H_n(X,A) \to H_n(X,U) \to H_{n-1}(U,A) \to \cdots$$

homotop. equiv. $(U,A) \simeq (A,A)$ $\underbrace{\text{gives}}_{H_k(U,A)\, \cong\, H_k(A,A)}$ $H_k(U,A) = 0$

Step 2 $H_n(X/A, A/A) \to H_n(X/U, U/A)$ is iso

same args with $(\bar{U}/A, A/A) \simeq (A/A, A/A)$

Step 3 $H_n(X, \bar{U}) \leftarrow H_n(X \setminus A, \bar{U} \setminus A)$ is iso

use excision

Step 4 $H_n(X/A, \bar{U}/A) \leftarrow H_n((X/A) \setminus (A/A), (\bar{U}/A) \setminus (A/A))$

again excision for $A/A \subset \bar{U}/A \subset X/A$

Step 5 $H_n(X \setminus A, \bar{U} \setminus A) \xrightarrow{q_*} H_n((X/A) \setminus (A/A), (\bar{U}/A) \setminus (A/A))$

is iso

$(X \setminus A) \cong (X/A) \setminus (A/A)$, etc.

Prop 2 is applicable when (X, A) is a cw pair i.e.

X is (the underlying top. sp. of) a cell complex

$A \subset X$ closed, union of some cells of X (subcomplex)

$$\rightsquigarrow \cdots \to H_n(A) \to H_n(X) \to H_n(X/A) \to H_{n-1}(A)$$

$$\cdots \to H_0(A) \to H_0(X) \to \tilde{H}_0(X/A) \to 0$$

long exact