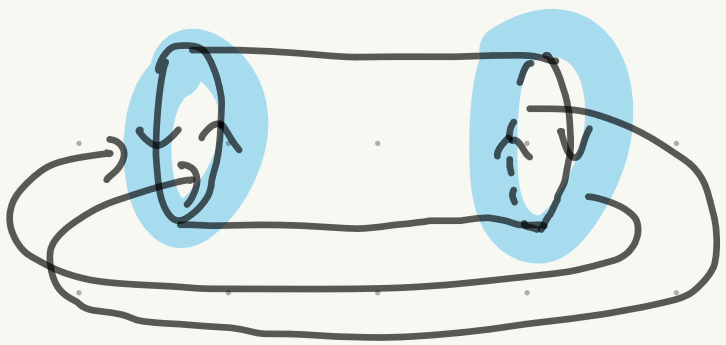
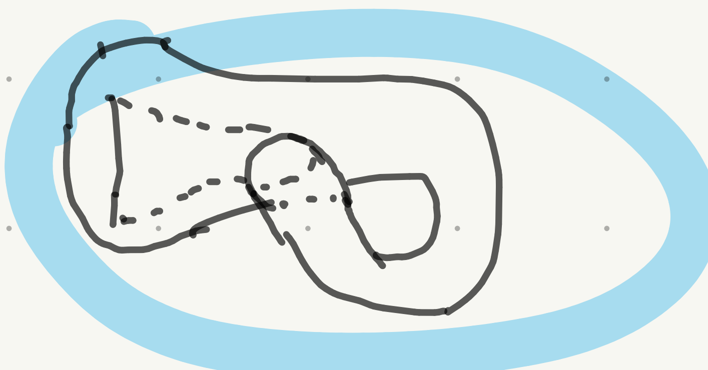
Problem set 5 (§ 2.1)

Exercise 5 simplicial homology of Klein bottle

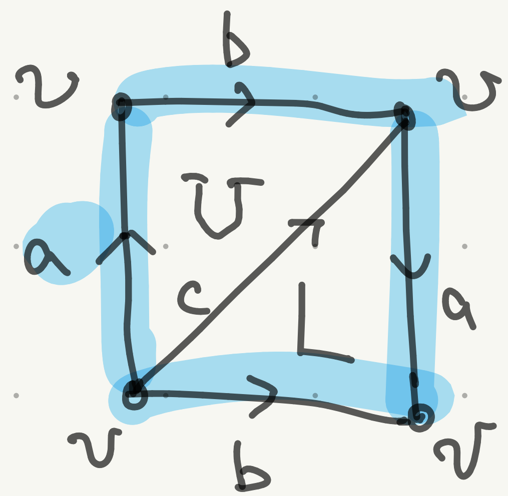
recall: the Klein bottle K is the non-orientable surface obtained by



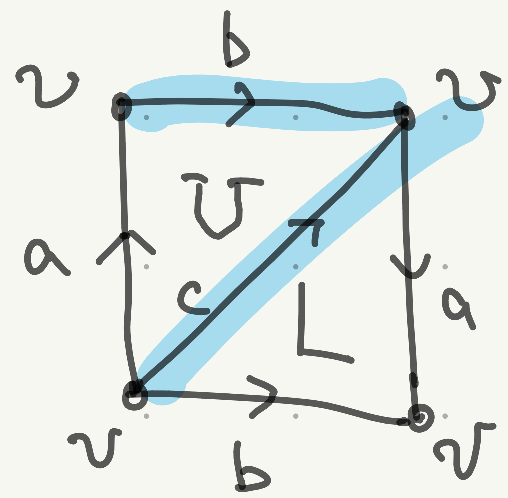
glue boundary circles in "opposite" directions



$\rightsquigarrow$ glue sides of



respecting orientation

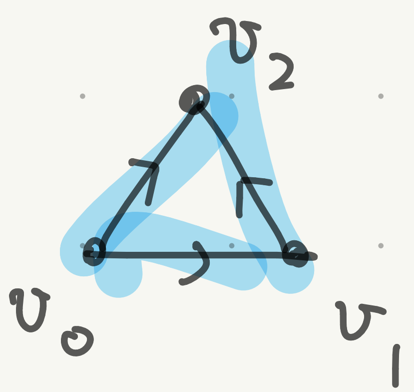


Δ-complex structure:

two 2-simplexes σ_U, σ_L

three 1-simplexes σ_a, σ_b, σ_c

one 0-simplex σ_v



Δ^2

comparing orientation on edges

$$\sigma_U\big|_{[v_1, v_2]} = \sigma_b \qquad \sigma_L\big|_{[v_1, v_2]} = \sigma_a$$

$$\sigma_U\big|_{[v_0, v_2]} = \sigma_c \qquad \sigma_L\big|_{[v_0, v_2]} = \sigma_b$$

$$\sigma_U\big|_{[v_0, v_1]} = \sigma_a \qquad \sigma_L\big|_{[v_0, v_1]} = \sigma_c$$

computing $H_n^{\Delta}(K)$:

$\Delta_n(K) = 0 \quad (\Rightarrow H_n^{\Delta}(K) = 0) \quad (n \geq 3)$

$\Delta_2(K) = \{ n_U \sigma_U + n_L \sigma_L : n_U, n_L \in \mathbb{Z} \}$

$\Delta_1(K) = \{ n_a \sigma_a + n_b \sigma_b + n_c \sigma_c : n_a, n_b, n_c \in \mathbb{Z} \}$

$\Delta_0(K) = \{ n_v \sigma_v : n_v \in \mathbb{Z} \}$

$\partial_2(\sigma_U) = \sigma_b - \sigma_c + \sigma_a \,, \quad \partial_2(\sigma_L) = \sigma_a - \sigma_b + \sigma_c$

$\partial_1(\sigma_a) = \sigma_v - \sigma_v = 0 = \partial_1(\sigma_b) = \partial_1(\sigma_c)$

$n = 2)$ $H_2^{\Delta}(K) = 0$

$\ker \partial_2 = 0$ by linear independence of $\partial_2 \sigma_U$ & $\partial_2 \sigma_L$

$n=1)$ $\quad H_1^{\Delta}(K) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$

$\ker \partial_1 = \Delta_1(K) = \mathbb{Z}\sigma_a \oplus \mathbb{Z}\sigma_b \oplus \mathbb{Z}\sigma_c$

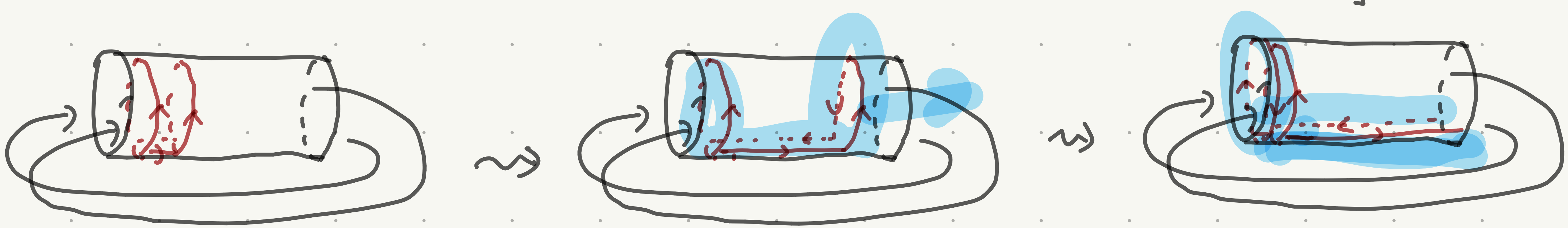
$\operatorname{img} \partial_2 =$ span of :

$\partial_2(\sigma_U) = \sigma_b - \sigma_c + \sigma_a$ & $\partial_2(\sigma_L) = \sigma_a - \sigma_b + \sigma_c$

take $\sigma_b - \sigma_c + \sigma_a,\ \sigma_a,\ \sigma_b$ as basis of $\Delta_1(K)$

$\sigma_b - \sigma_c + \sigma_a,\ 2\sigma_a$: basis of $\operatorname{img}\partial_2$

$\Rightarrow \Delta_1(K)/\operatorname{img}\partial_2 \cong \underset{\text{from } [\sigma_a]}{(\mathbb{Z}/2\mathbb{Z})} \oplus \underset{\text{from } [\sigma_b]}{\mathbb{Z}}$

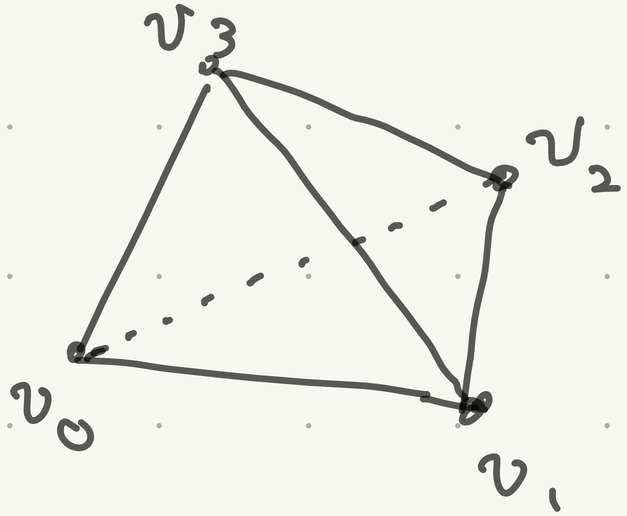


$2[\sigma_a]$

$n=0$) $H_0^{\Delta}(K) = \mathbb{Z}$

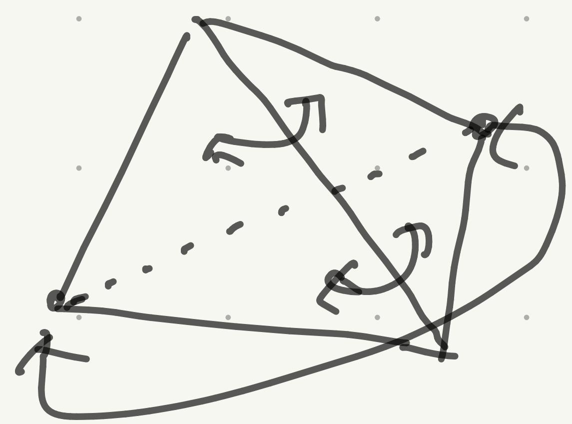
$\operatorname{img} \partial_1 = 0$, $\Delta_0(K) \cong \mathbb{Z}$

Exercise 7 Δ-complex structure of S^3

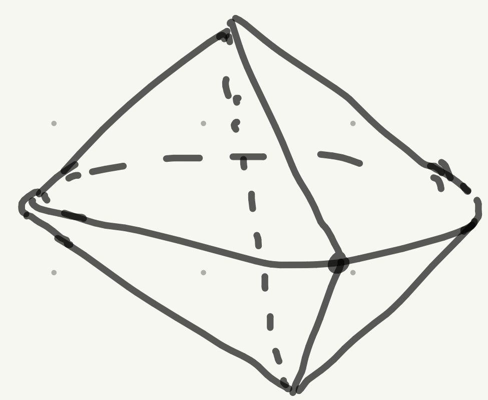


want: find pairs of faces s.t. their gluing (respecting the orientation) gives S^3

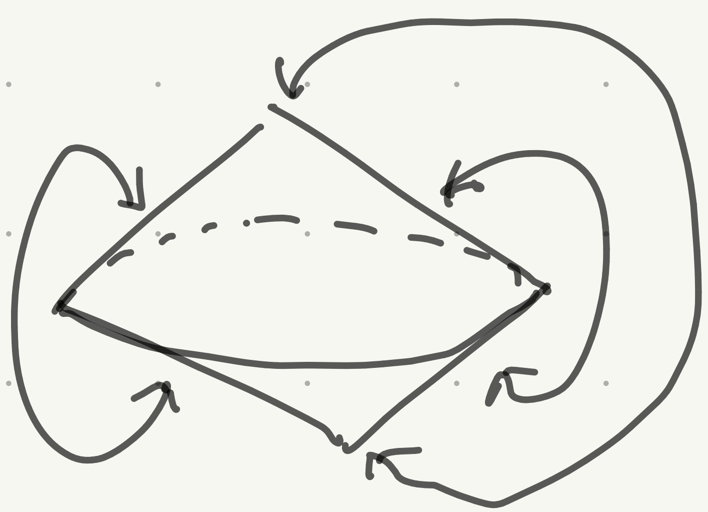
basic idea:



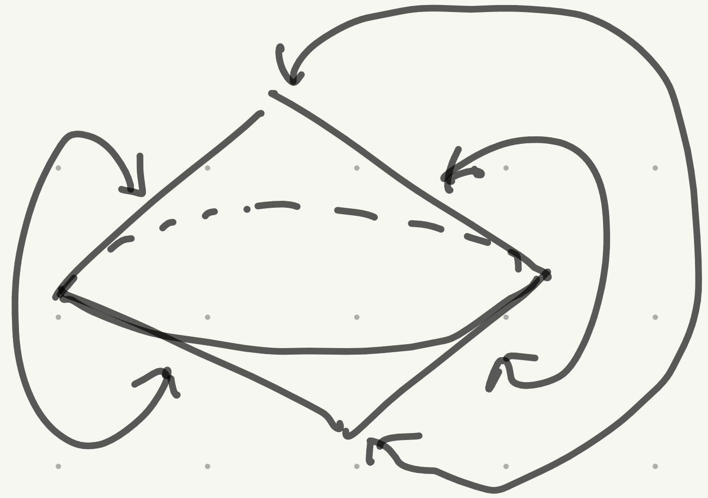
gives



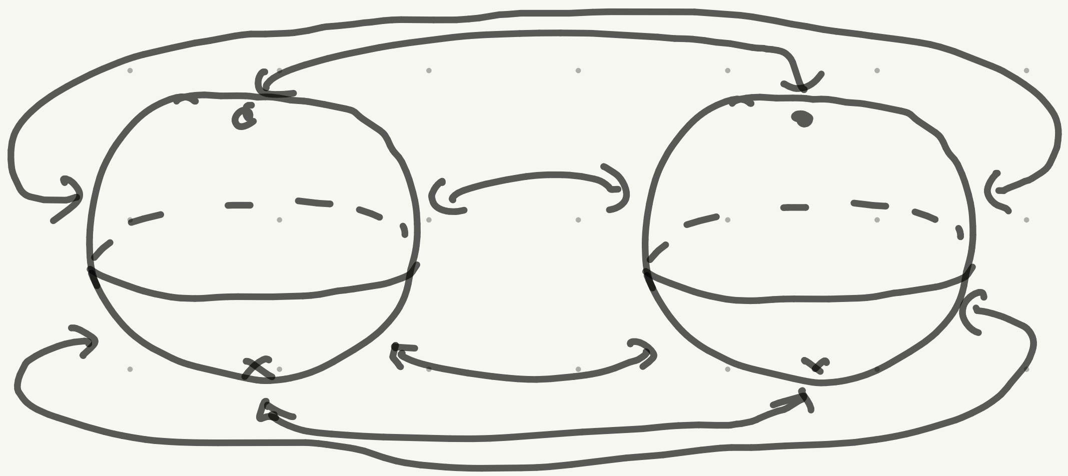
then



gives S^3

checking that  gives S^3:

same as gluing two copies of D^3



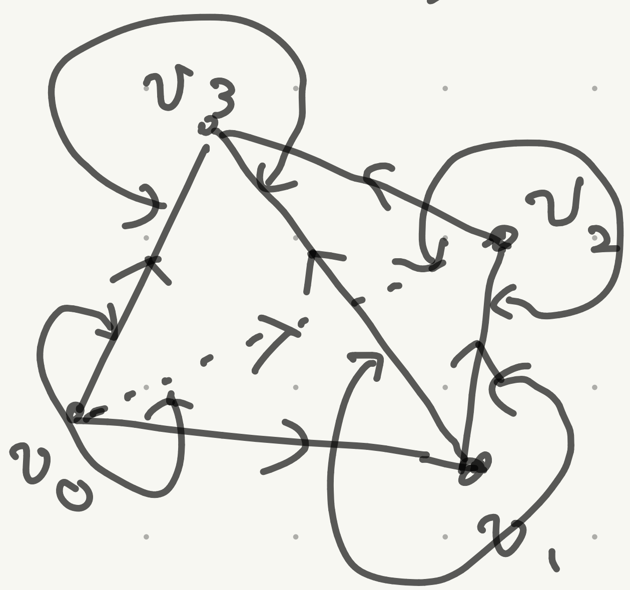
by matching hemispheres

$\cong$ one-point compactification of $\mathbb{R}^3$

$\cong S^3$

Constraint : we need to respect orientation

on edges coming from ordering of vertices



$\rightsquigarrow$ glue the face $[v_0, v_1, v_2]$

to $[v_0, v_1, v_3]$

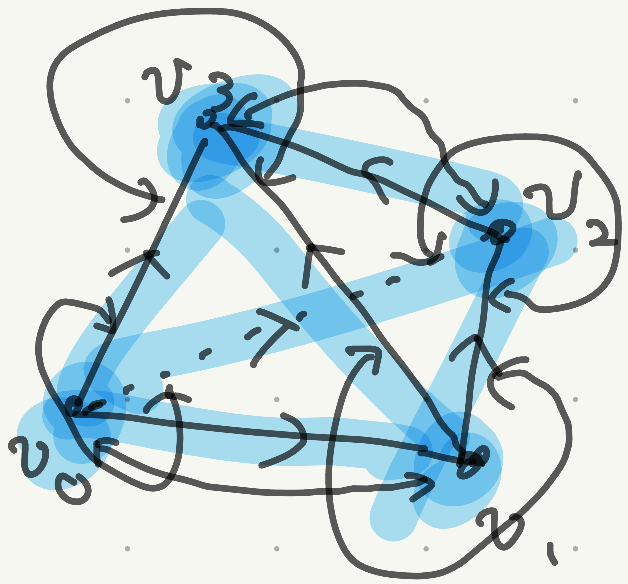
the face $[v_1, v_2, v_3]$ to $[v_0, v_2, v_3]$

i.e. one 3-simplex $\sigma^{(3)}$

two 2-simplices $\sigma_1^{(2)}$, $\sigma_2^{(2)}$

$\sigma^{(3)}|_{[v_0,v_1,v_2]} = \sigma^{(3)}|_{[v_0,v_1,v_3]} = \sigma_1^{(2)}$, etc.

In lower dimension ..



three 1-simplices

$\sigma_1^{(1)} = [v_0, v_1]$, $\sigma_2^{(1)} = [v_2, v_3]$

$\sigma_3^{(1)} = [v_1, v_2] = [v_1, v_3] = [v_0, v_3] = [v_0, v_2]$

two 0-simplices

$\sigma_1^{(0)} = [v_0] = [v_1]$, $\sigma_2^{(0)} = [v_2] = [v_3]$

chains : $\Delta_3(S^3) = \mathbb{Z}\,\sigma^{(3)}$

$\Delta_2(S^3) = \{n_1 \sigma_1^{(2)} + n_2 \sigma_2^{(2)} : n_1, n_2 \in \mathbb{Z}\}$

$\Delta_1(S^3) = \{n_1 \sigma_1^{(1)} + n_2 \sigma_2^{(1)} + n_3 \sigma_3^{(1)} : n_i \in \mathbb{Z}\}$

$\Delta_0(S^3) = \{n_1 \sigma_1^{(0)} + n_2 \sigma_2^{(0)} : n_1, n_2 \in \mathbb{Z}\}$

boundary maps

$$\partial_3 \sigma^{(3)} = \sigma^{(3)}\Big|_{[v_1,v_2,v_3]} - \sigma^{(3)}\Big|_{[v_0,v_2,v_3]} + \sigma^{(3)}\Big|_{[v_0,v_1,v_3]}$$

$$- \sigma^{(3)}\Big|_{[v_0,v_1,v_2]}$$

$$= \sigma^{(2)}_2 - \sigma^{(2)}_2 + \sigma^{(2)}_1 - \sigma^{(2)}_1 = 0$$

$$\partial_2 \sigma^{(2)}_1 = \text{"}[v_1, v_2] - [v_0, v_2] + [v_0, v_1]\text{"}$$

$$= \sigma^{(1)}_1$$

$$\text{"}[v_0, v_1, v_2] = [v_0, v_1, v_3]\text{"}$$

$$\partial_2 \sigma^{(2)}_2 = \text{"}[v_2, v_3] - [v_1, v_3] + [v_1, v_2]\text{"}$$

$$= \sigma^{(1)}_2$$

$$\partial_1 \sigma^{(1)}_1 = \text{"}[v_1] - [v_0]\text{"} = 0$$

$$\uparrow$$

"$[v_0, v_1]$"

$$\partial_1 \sigma^{(1)}_2 = \text{"}[v_3] - [v_2]\text{"} = 0$$

$$\partial_1 \sigma^{(1)}_3 = \text{"}\underline{[v_2] - [v_1]}\text{"} = \underline{\sigma^{(0)}_2 - \sigma^{(0)}_1}$$

homology

$$H^\Delta_3(S^3) \cong \mathbb{Z} \quad \text{from} \quad \ker \partial_3 = \Delta_3(S^3) \cong \mathbb{Z}$$

$$H^\Delta_2(S^3) \cong 0 \quad \text{from} \quad \ker \partial_2 = 0$$

$$H^\Delta_1(S^3) \cong 0 \quad \text{from} \quad \ker \partial_1 = \mathbb{Z}\sigma^{(1)}_1 \oplus \mathbb{Z}\sigma^{(1)}_2$$

$$= \operatorname{img} \partial_2$$

$$H^\Delta_0(S^3) = (\mathbb{Z}\sigma^{(0)}_1 \oplus \mathbb{Z}\sigma^{(0)}_2)/\mathbb{Z}(\sigma^{(0)}_1 - \sigma^{(0)}_2) \cong \mathbb{Z}$$

Recall: long exact seq.

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X/A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

$$\rightsquigarrow \quad 0 \to M \to \underline{H_n(X)} \to M' \to 0$$

if $M' = \mathbb{Z}^d \Rightarrow H_n(X) \cong M \oplus \mathbb{Z}^d$

if $M' \cong \mathbb{Z}/k\mathbb{Z}$

$H_n(X)$ can have several possibilities

$\rightsquigarrow$ "extension problem"

Exercise 14 extension problem

$$\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z} = \{ [n] : n \in \mathbb{Z},\ [n+mk] = [n] \}$$

1) Is there a short exact seq. of the form

$$0 \to \mathbb{Z}_4 \xrightarrow{f} \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0 \quad ?$$

YES : consider f s.t. $f([1]) = \underbrace{([2],[1])}_{\text{order } 4}$

$G = (\mathbb{Z}_8 \oplus \mathbb{Z}_2)/\mathrm{img}\, f$ has order 4

$\Rightarrow$ isom. to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

$[([1],[0])]$ is not of order 2 in G

$$\big(([1],[0]) + ([1],[0]) = ([2],[0]) \notin \mathrm{img}\, f \big)$$

(cont.) G cannot be $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

2) abelian groups A appearing in ex. seq.

$$0 \to \mathbb{Z}_{p^m} \xrightarrow{f} A \to \mathbb{Z}_{p^n} \to 0 \qquad (p : \text{prime})$$

"basic" example $A = \mathbb{Z}_{p^{m+n}}$

$$f([k]) = [p^n k] \qquad (p^n) = f([1]) \text{ has ord. } p^m \text{ in } \mathbb{Z}_{p^{m+n}}$$

"trivial" example $A = \mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n}$

$$f([k]) = ([k], [0])$$

structure of finitely generated comm. grps

$$\mathbb{Z}^d \oplus \mathbb{Z}_{p_1^{e_1}} \oplus \mathbb{Z}_{p_2^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}}$$

order p^{m+n}, contains an elem. of ord. p^m

$$\mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_k}}, \qquad e_1 + \cdots + e_k = m+n$$

$$e_1 \geq e_2 \geq \cdots, \quad m \leq e_1$$

take $M \rightsquigarrow M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ on the exact seq

$$0 \to \mathbb{Z}_{p^m} \to \underset{\text{as above}}{A} \to \mathbb{Z}_{p^n} \to 0$$

$\rightsquigarrow$ exact seq. $\overset{\text{dim } 1}{\mathbb{Z}_p} \to \underbrace{\overbrace{\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}^{k \text{ factors}}}_{\text{dim } k} \to \overset{\text{dim } 1}{\mathbb{Z}_p} \to 0$ over $\mathbb{Z}_p$ field

right exactness of $\sim \otimes_{\mathbb{Z}} \mathbb{Z}_p$ & $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p$

$$M \otimes_{\mathbb{Z}} \mathbb{Z}_p = \{ \underset{\substack{\cap \\ M}}{m} \otimes \underset{\substack{\cap \\ \mathbb{Z}_p}}{x} : \underset{k \in \mathbb{Z}}{km \otimes x = m \otimes kx} \}$$

$$pm \otimes [i] = m \otimes p \cdot [i] = m \otimes [0] = 0$$

$$\Rightarrow \quad \mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p$$

$$\underline{[1] \otimes [1]} \longleftrightarrow [1]$$

(cont.) $k = 1, 2$.

$\underline{\mathbb{Z}_{p^{m+n}}}$, $\mathbb{Z}_{p^{m+n-1}} \oplus \mathbb{Z}_p$, $\cdots$, $\mathbb{Z}_{p^{\max(m,n)}} \oplus \mathbb{Z}_{p^{\min(m,n)}}$

$f : \mathbb{Z}_{p^m} \to \mathbb{Z}_{p^{m+n-k}} \oplus \mathbb{Z}_{p^k}$, $[i] \mapsto ([p^{n-k} i], [i])$

will do $(k \le m, n)$

quot. is cyclic

image of $([0], [i])$ = img of $(-[p^{n-k} i], [0])$

candidates of A in $0 \to \mathbb{Z} \xrightarrow{f} A \to \mathbb{Z}_{p^n} \to 0$

"basic" ex. $A = \mathbb{Z}$, $f(i) = p^n i$

"trivial" ex. $A = \mathbb{Z} \oplus \mathbb{Z}_{p^n}$, $f(i) = i \oplus 0$

generally as before

$$\mathbb{Z}^d \oplus \mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{e_k}}$$

Step 1 $d = 1$: take $- \otimes_{\mathbb{Z}} \mathbb{Q}$; $\mathbb{Q}$ is flat over $\mathbb{Z}$ so

$$0 \to \underset{\mathbb{Q}}{\underset{\wr\wr}{\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}}} \to \underset{\mathbb{Q}^d}{\underset{\wr\wr}{A \otimes_{\mathbb{Z}} \mathbb{Q}}} \to \underset{0}{\underset{\wr\wr}{\mathbb{Z}_{p^n} \otimes_{\mathbb{Z}} \mathbb{Q}}} \to 0$$

is exact $\Rightarrow$ $d = 1$

Step 2 no i with $p_i \neq p$

otherwise $\sim \otimes_{\mathbb{Z}} \mathbb{Z}_{p_i}$ gives an exact seq.

$$\begin{array}{ccccc}
\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_{p_i} & \to & A \otimes_{\mathbb{Z}} \mathbb{Z}_{p_i} & \to & \mathbb{Z}_{p^{e_n}} \otimes \mathbb{Z}_{p_i} \to 0 \\
\cong & & \cong & & \cong \\
\mathbb{Z}_{p_i} & & \mathbb{Z}_{p_i} \oplus \underbrace{\mathbb{Z}_{p_i^{e_i}} \otimes_{\mathbb{Z}} \mathbb{Z}_{p_i}}_{\substack{\cong \\ 0 \text{ if } p_i \neq p_1 \\ \mathbb{Z}_{p_i} \text{ if } p_i = p_1}} \oplus \cdots \oplus \underbrace{\mathbb{Z}_{p_i^{e_i}} \otimes \mathbb{Z}_{p_i}}_{\substack{\cong \\ \mathbb{Z}_{p_i}}} \oplus \cdots & & 0
\end{array}$$

has $\dim_{\mathbb{Z}_{p_i}} \geq 2$

Step 3 $k \leq 1$ take $\sim \otimes_{\mathbb{Z}} \mathbb{Z}_p$, compare

dim over $\mathbb{Z}_p$

Step 4 $\quad A = \mathbb{Z} \oplus \mathbb{Z}_{p^e} \Rightarrow e \leq n$

$(0, [i])$ is not in the image of $\mathbb{Z}$

but $e > n \Rightarrow 0 \neq (0, [p^n])$ is in the kernel of $A \to \mathbb{Z}_{p^n}$

Step 5 $\quad$ any $e \leq n$ would give

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z}_{p^e} \to \mathbb{Z}_{p^n} \to 0$$

$$f(i) = (p^{n-e} i, [i])$$

then
- $(p^n, [0]) = f(p^e) \in \operatorname{img} f$
- $(j, [0]) \in \operatorname{img} f \Rightarrow p^n \mid j$

$\Rightarrow (\mathbb{Z} \oplus \mathbb{Z}_{p^e}) / \operatorname{img} f \cong \mathbb{Z}_{p^n}$