Mandatory assignment

Problem 1 Equivalence of

(1) $\forall$ loop $f : S^1 \to X$ is homotopic to a const. map

(2) $\forall$ loop $f : S^1 \to X$ has an extension $D^2 \to X$

(3) $\forall$ $x_0 \in X$ $\pi_1(X, x_0) = \{e\}$

convention : in (1) we do not assume that homotopy fixes a basepoint ; more general than homotopy of paths

(3) $\Rightarrow$ (2) take $1 \in S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ as basept $* \in S^1$

suppose $f : S^1 \to X$ is given

put $x_0 = f(*)$ so f represents $[f] \in \pi_1(X, x_0)$

by assumption (3) $[f] = e$ in $\pi_1(X, x_0)$

this means $\exists\, H : I \times I \to X$

$\uparrow$ param for loop $\quad \leftarrow$ for deform

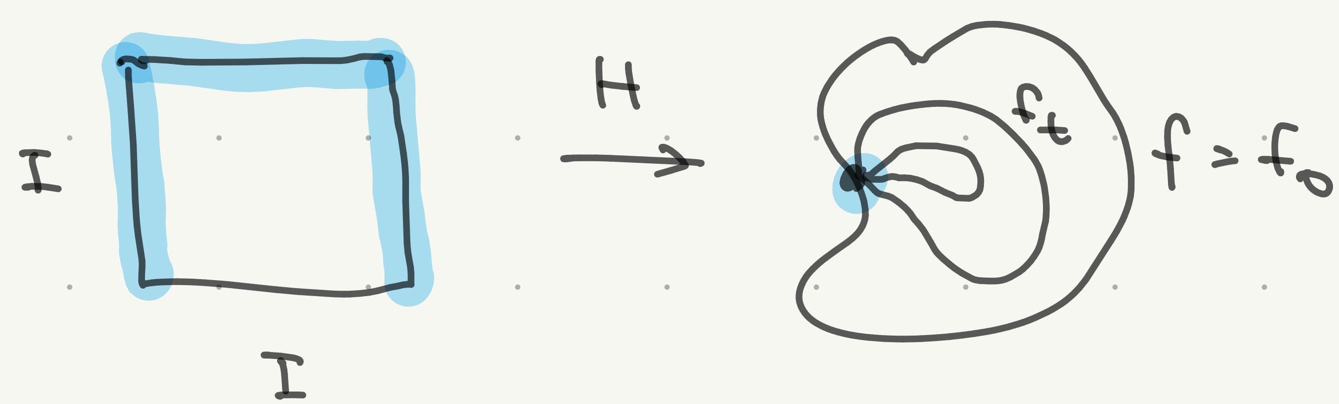
- $\forall t : H(0,t) = H(1,t)$

  $f_t(s) = H(s,t)$ is a loop based at x_0

- $\forall s : H(s,0) = f(e^{2\pi i s})$; $S^1 \cong I / 0 \sim 1$

  $f_0 = f$ $\qquad \mathbb{C} \ni e^{2\pi i s} \longleftrightarrow \overset{\in}{s}$

- $\forall s : H(s,1) = x_0$ $\qquad f_1 =$ const. loop at x_0

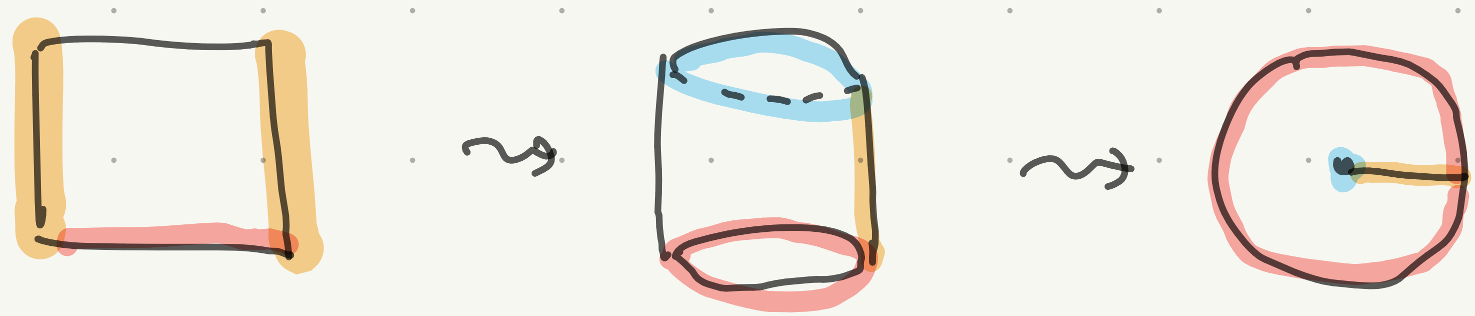


we perform gluing on domain of H to create D^2
(or collapsing subsp.)
s.t. the parameter subset for $f_0 \simeq f$ becomes
the boundary circle

approach 1 collapse the "three edges" to a pt.

$$I \times I \,/\, (0,t) \sim (1,t') \sim (s,1) \quad \forall\, t, t', s$$
$$\cong D^2$$

approach 2 cylinder by gluing $\rightsquigarrow$ collaps one bdry



in concrete formulas

$$\tilde{f}(\underbrace{re^{2\pi i s}}_{\text{polar coordinates on } D^2}) = H(s, 1-r) \qquad 0 \leq r \leq 1, \; 0 \leq s \leq 1$$

$\tilde{f}$ is well defied because

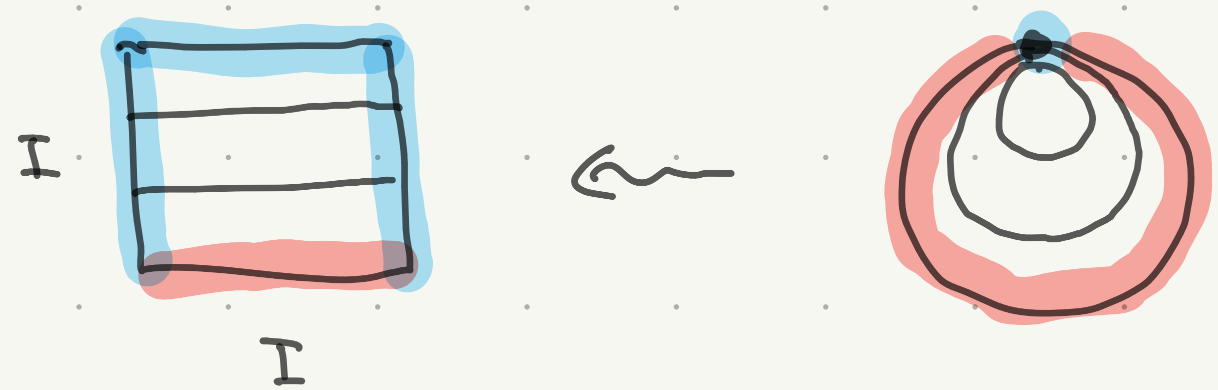
- $r = 0$ $\quad \tilde{f}(0) = H(s, 1) = x_0$
- $s = 0$ or $s = 1$ represent the same point

  $\tilde{f}(re^{2\pi 0}) = \tilde{f}(re^{2\pi i})$ from $H(0,t) = H(1,t)$

$\tilde{f}|_{\partial D^2} = f \; : \; \tilde{f}(e^{2\pi i s}) = H(s, 0) = f(e^{2\pi i s})$

$(2) \Rightarrow (3)$; $\exists\, \tilde{f} : D^2 \to X$ extending f $\Rightarrow \pi_1(X, x_0) = \{e\}$

"unpack" the pictures from approach 1



given $[f] \in \pi_1(X, x_0)$ represented by $f : S^1 \to X$

$f(1) = x_0$, take $\tilde{f} : D^2 \to X$ extension

put $H(s,t) = \tilde{f}(t + (1-t)e^{2\pi i s})$

- $H(0,t) = \tilde{f}(1) = x_0$, $H(1,t) = \tilde{f}(1) = x_0$
- $H(s,0) = \tilde{f}(e^{2\pi i s}) = f(e^{2\pi i s})$
- $H(s,1) = \tilde{f}(1) = x_0$

(1) $\Rightarrow$ (2)

(1) $\forall$ loop $f: S^1 \to X$ is homotopic to a const. map

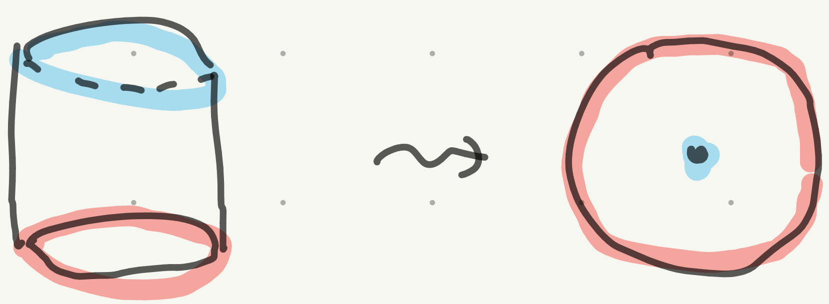
(2) $\forall$ loop $f: S^1 \to X$ has an extension $D^2 \to X$

parameter space for homotopy of maps on S^1:

cylinder $S^1 \times I$

boundary condition (const.) on one bdry comp.

$\rightsquigarrow D^2$ by collapsing that component



homotopy in (1) is given by $H: S^1 \times I \to X$

- $H(e^{2\pi i s}, 0) = f(e^{2\pi i s})$
- $H(e^{2\pi i s}, 1) = x_0$ for some $x_0 \in X$

put $\tilde{f}(re^{2\pi is}) = H(e^{2\pi is}, 1-r)$

$\tilde{f}$ is well defied because

- $r=0$ $\quad \tilde{f}(0) = H(e^{2\pi is}, 1) = x_0$

$\tilde{f}|_{\partial D^2} = f$: $\tilde{f}(e^{2\pi is}) = H(e^{2\pi is}, 0) = f(e^{2\pi is})$

Rem x_0 could be different from $f(1)$, but for the statement (2) it doesn't matter

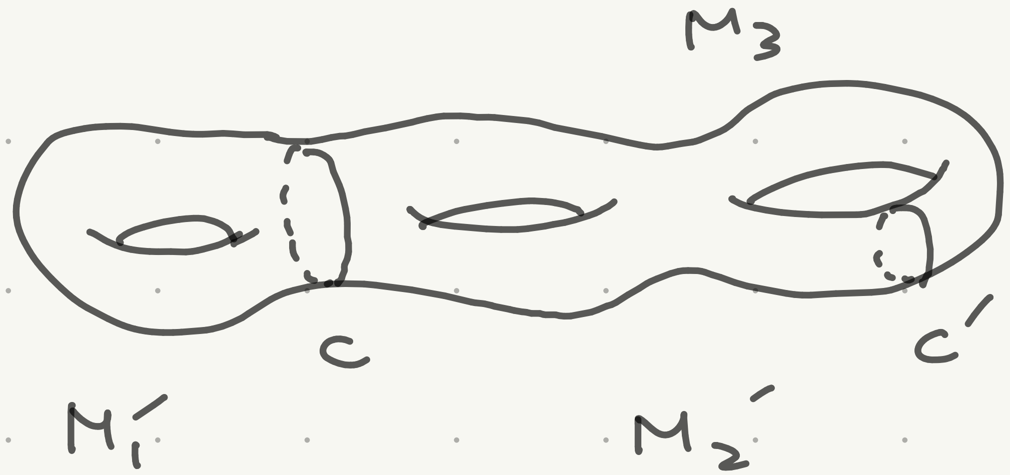
(2) $\Rightarrow$ (1) ; $f : S^1 \to X$, $\exists\, \tilde{f} : D^2 \to X$ ext. $\Rightarrow$ $f \simeq$ const.

put $H(e^{2\pi i s}, t) = \tilde{f}((1-t)\, e^{2\pi i s})$

- $H(e^{2\pi i s}, 0) = \tilde{f}(e^{2\pi i s}) = f(e^{2\pi i s})$
- $H(e^{2\pi i s}, 1) = \tilde{f}(0)$

$\Rightarrow$ $H : S^1 \times I \to X$ implements $f \simeq$ const (at $\tilde{f}(0)$)

Problem 2 $\qquad C, C' \subset M_g$ circles



- $M_g \setminus C$ has two conn. components $\cong M_h', M_k'$ punctured surfaces w/ h & k holes
- $M_g \setminus C'$ is connected

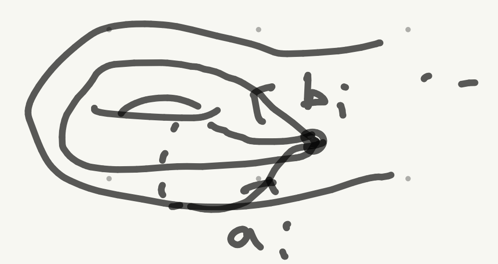
(1) M_h' does not retract onto C

<u>observation</u>

$\pi_1(M_h, *)$ has presentation

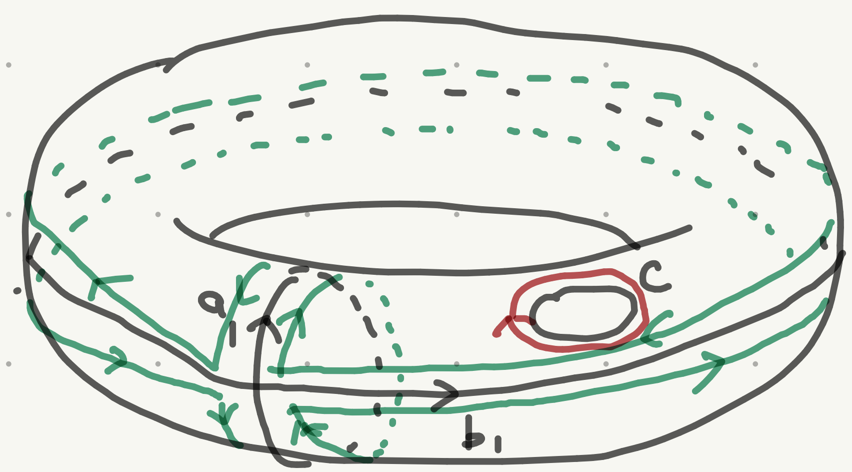
$$\langle a_1, b_1, \cdots, a_h, b_h : a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 \cdots a_h^{-1} b_h^{-1} = e \rangle$$

a_i : "meridian"
b_i : "longitudu"
} along the i-th hole



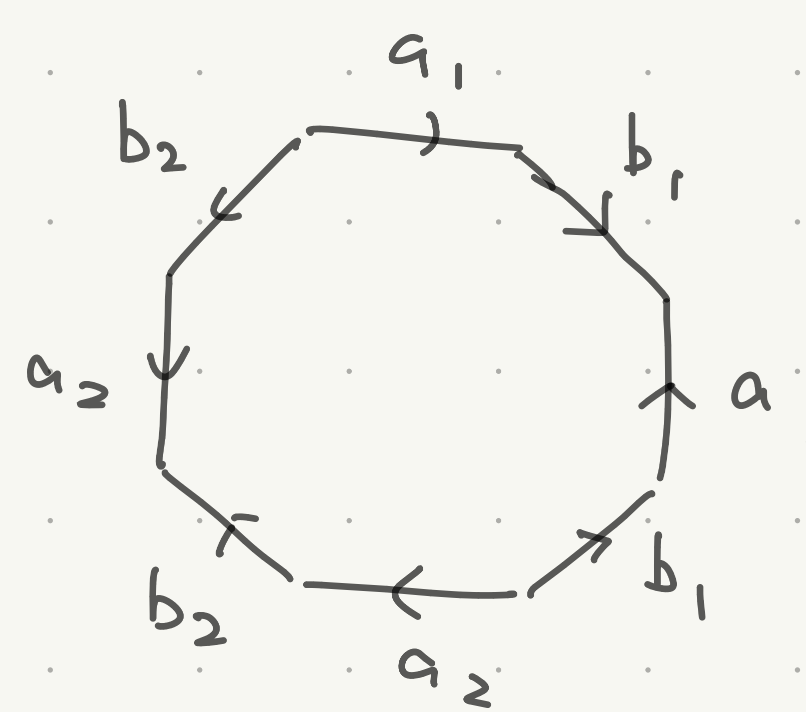
(cont.) in $\pi_1(M_h', *)$ we have

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 \cdots a_h^{-1} b_h^{-1} = C$$

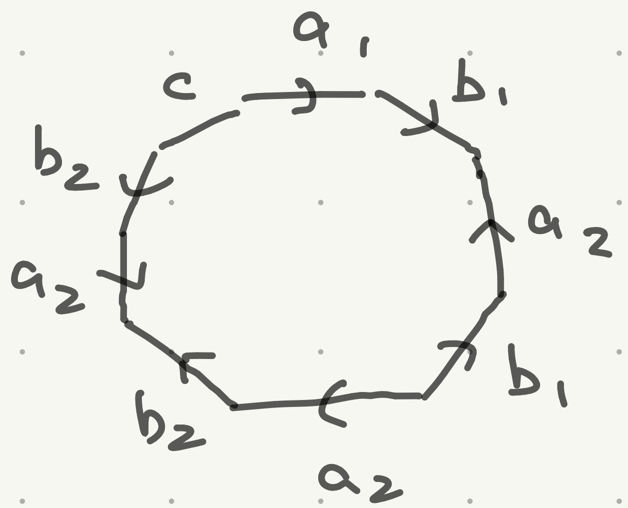


example for $h = 1$

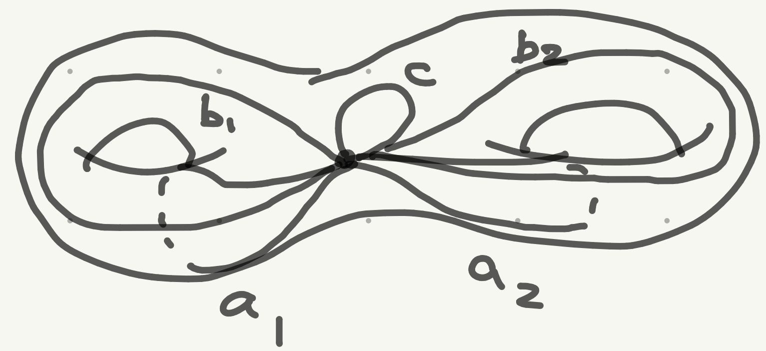
more formally recall that M_h was obtained by gluing pairs of edges of a $(4h)$-gon with orientation configuration like this $\rightarrow$



(cont.) inserting an edge (between a_1 and b_n)



and doing same gluing will give M'_n



why does this obstruct C to be a retract of M'_n ?

$\nexists\, r : C \overset{i}{\hookrightarrow} M'_n \xrightarrow{r} C \quad r \circ i = id_C$; if such r existed

the induced will be

$$\begin{tikzcd} \mathbb{Z} \cong \pi_1(C,*) \arrow[r, "i_*"] \arrow[rr, bend right, "id"'] & \pi_1(M'_n,*) \arrow[r, "r_*"] & \pi_1(C,*) \end{tikzcd}$$

r_* is a group hom $\Rightarrow$ it sends $a_i b_i a_i^{-1} b_i^{-1}$ to

$r(a_i)\, r(b_i)\, r(a_i)^{-1} r(b_i)^{-1} = e$

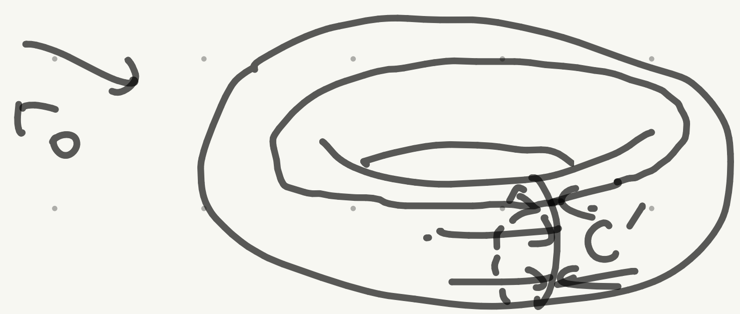
($\pi_1(C,*) \cong \mathbb{Z}$ is commutative) (cont.) on the other hand

$S^1 \xrightarrow{f} C$ corresponding to $1 \in \mathbb{Z}$ gives

$$i_*[f] = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n^{-1} b_n^{-1}$$

by the observation $\rightsquigarrow$ we have $r_* i_* [f] = e \neq [f]$

(2) M_g retracts to any meridian circle C'



"collapse" shaded part to get a torus, then use $T \cong C' \times C''$ forget C''-component to define $T \xrightarrow{r_1} C'$

$r = r_1 \circ r_0$

Problem 3

Recall: $\tilde{X} \xrightarrow{p} X$ is a covering when

$\forall\, x \in X \quad \exists \underset{\text{open}}{U} \ni x \; : \; p^{-1}(U) = \bigcup_{i \in I} V_i$ ← some index set

- V_i open set in $\tilde{X}$
- $i \neq j \Rightarrow V_i \cap V_j = \varnothing$
- $\forall i \quad p|_{V_i} : V_i \to U$ homeo

call such U an evenly covered open neighborhood of x

(1) $A \subset X \Rightarrow \tilde{A} = p^{-1}(A) \xrightarrow{p} A$ is a covering

fix $x \in A$; we need to find an evenly covered open neigh. U' of x (for the ind. top. on A)

take $U \subset_{\text{open}} X$, $(V_i)_{i \in I}$ as above

set $U' = U \cap A$, $V_i' = V_i \cap \tilde{A}$

- U' is an open set of A by def. of the induced topology
- V_i' is an open set of $\tilde{A}$ (same)
- $(p')^{-1}(U') = \bigcup_{i \in I} V_i'$ for $p' = p|_{\tilde{A}} : \tilde{A} \to A$ by unpacking defs.
- $V_i' \cap V_j' = \emptyset$ for $i \neq j$ from $V_i \cap V_j = \emptyset$

(cont.) • $p'|_{V'_i} : V'_i \to U'$ is homeo.

bijectivity, continuity } consequence of defs.

open-ness of $p'|_{V'_i} : V'_i \to U'$

an open set of V'_i is of the form

$V'_i \cap \tilde{W}$ for some open $\tilde{W} \subset \tilde{X}$

$V_i \cap \tilde{W}$ is open in $\tilde{X}$ & we may assume $\tilde{W} \subset V_i$

$\Rightarrow W = p(\tilde{W})$ is an open set of U

$\rightsquigarrow$ we get an open set $U' \cap W$ of U'

$p'(V'_i \cap \tilde{W}) = U' \cap W$, so $p'|_{V'_i}$ is open.

(2) $p_i : \tilde{X}_i \to X_i$ covering $(i = 1, 2)$

$\Rightarrow$ so is $p_1 \times p_2 : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$

fix $(x, y) \in X_1 \times X_2$; take

$U^{(1)} \ni x$ evenly covered, $p_1^{-1}(U^{(1)}) = \bigcup_{i \in I} V_i^{(1)}$

$U^{(2)} \ni y$ evenly covered, $p^{-1}(U^{(2)}) = \bigcup_{j \in J} V_j^{(2)}$

claim : $U^{(1)} \times U^{(2)}$ is evenly covered :

$(p_1 \times p_2)^{-1}(U^{(1)} \times U^{(2)}) = \bigcup_{i \in I, j \in J} V_i^{(1)} \times V_j^{(2)}$

- $(V_i^{(1)} \times V_j^{(2)}) \cap (V_{i'}^{(1)} \times V_{j'}^{(2)}) = \emptyset$ if $(i, j) \neq (i', j')$
- $p_1 \times p_2 : V_i^{(1)} \times V_j^{(2)} \to U^{(1)} \times U^{(2)}$ homeo

(3) $\tilde{X}_1 \times_X \tilde{X}_2 \to X$ is a covering

if $p_i : \tilde{X}_i \to X$ is a one $(i = 1, 2)$

$$\tilde{X}_1 \times_X \tilde{X}_2 = \{ (\tilde{x}, \tilde{y}) : \tilde{x} \in \tilde{X}_1, \tilde{y} \in \tilde{X}_2, p_1(\tilde{x}) = p_2(\tilde{y}) \}$$

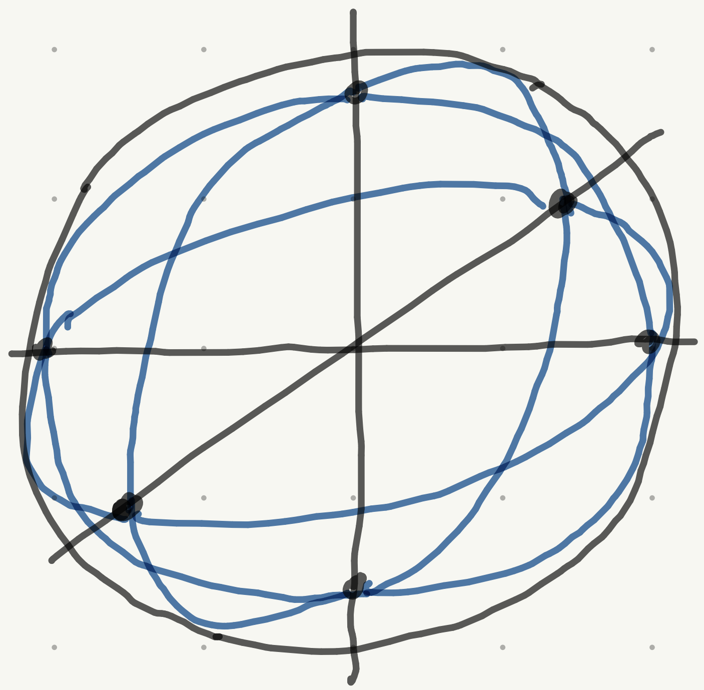
can be regarded as $(p_1 \times p_2)^{-1}(A)$ for

$A = \Delta(X) = \{ (x, x) : x \in X \} \subset X \times X$

$\tilde{X}_1 \times \tilde{X}_2 \xrightarrow{p_1 \times p_2} X \times X$ is a covering by (2)

its restriction to A is still a covering by (1)

Problem 4



$$v_i^{(\pm)} = (0, \dots, \underset{i\text{-th}}{\pm 1}, \dots, 0) \in S^n$$

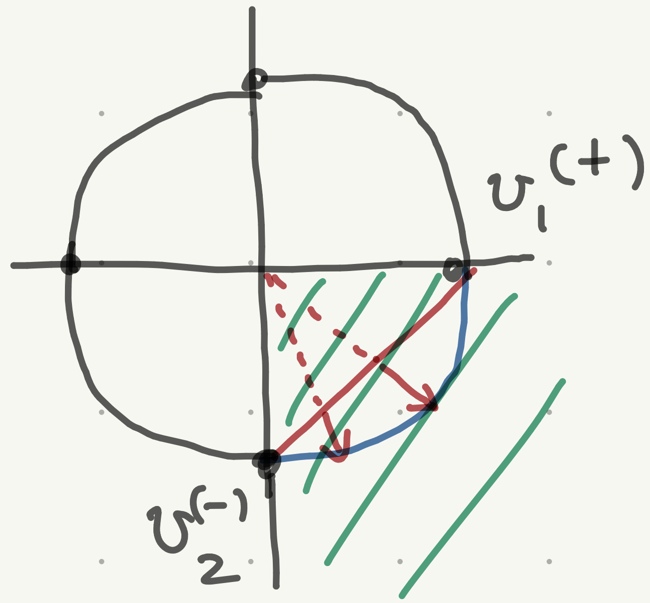
(1) Δ-complex structure on S^n with $v_i^{(\pm)}$ as vertices :

given k vertices on different axes

$$\alpha = (v_{i_0}^{(\varepsilon_0)}, \dots, v_{i_k}^{(\varepsilon_k)}) : i_0 < \dots < i_k, \ \varepsilon_j = \pm$$

$$\Delta_\alpha^k = S^n \cap \left\{ \sum_{j=0}^{k} t_j v_{i_j}^{(\varepsilon_j)} : t_j \geq 0 \right\}$$

(cont.)



$\sigma_\alpha : \Delta^k \to \Delta^k_\alpha$ "projection" from origin between

$\left\{ \sum_{j=0}^{k} t_j v_{i_j}^{(\varepsilon_j)} : t_j \geq 0, \ \sum_{j=0}^{k} t_j = 1 \right\}$

and Δ^k_α

if $\beta = (v_{i'_0}^{(\varepsilon'_0)}, \dots, v_{i'_\ell}^{(\varepsilon'_\ell)})$ is a subsequence of α

$\sigma_\alpha |_{\Delta^\ell} = \sigma_\beta$ up to $\Delta^\ell \to \Delta^k$,

$(t_0, \dots, t_\ell) \mapsto (0, \dots, t_0, \dots, t_1, \dots)$

$\uparrow$ pos. of $v_{i'_0}^{(\varepsilon'_0)}$ in α

(compatibility for faces)

(2) Δ-complex structure on $\mathbb{R}P^n = S^n / x \sim (-x)$

consider $\mathbb{Z}_2 = \{e, g\}$ acting on S^n by

$g\,x = -x$ so $\mathbb{R}P^n \cong \mathbb{Z}_2 \backslash S^n$

g moves each simplex of (1) to a different one

if $\alpha = (v_{i_0}^{(\varepsilon_0)}, \dots, v_{i_k}^{(\varepsilon_k)})$,

$g\,\Delta^k_\alpha = \Delta^k_{\alpha'}$ for $\alpha' = (v_{i_0}^{(-\varepsilon_0)}, \dots, v_{i_k}^{(-\varepsilon_k)})$

$\rightsquigarrow$ label for k-cells of $\mathbb{R}P^n$: quotient of

$\{\alpha = (v_{i_0}^{(\varepsilon_0)}, \dots, v_{i_k}^{(\varepsilon_k)})$ as above$\} / \alpha \sim \alpha'$

(cont.) formally: the set of params for k-cells:

$\coprod_{\{i_0 < \cdots < i_k\} \subset \{0, \ldots, n\}} \mathbb{Z}_2^{\,k} / \mathbb{Z}_2$ $\qquad \mathbb{Z}_2 \curvearrowright \mathbb{Z}_2^{\,k}$ diag. action

$\bar{\Delta}_\alpha^k$: ims of Δ_α^k in $\mathbb{R}P^n$

so $\bar{\Delta}_\alpha^k = \bar{\Delta}_{\alpha'}^k$, and this is the only overlap of index

$\bar{\sigma}_\alpha : \Delta^k \to \bar{\Delta}_\alpha^k$ compos. of σ_α and $S^n \to \mathbb{R}P^n$

β subseq of $\alpha \Rightarrow \beta'$ subseq of α'

so this is compat. with taking faces