# Excision

Morally : $A \subset X$ subspace $\Rightarrow$ relative homology
$H_n(X, A)$ represents $\tilde{H}_n(X \setminus A)$

this was true when $A$ has a neighborhood $U$
s.t. $A \subset U$ is a deformation retract (Prop. 2.22)

Key step

Th'm 2.20 (excision) $Z \subset A \subset X$ , $\overline{Z} \subset \mathring{A}$ (closure, interior)

then $X \setminus Z \hookrightarrow X$ induces $H_n(X \setminus Z, A \setminus Z) \cong H_n(X, A)$

equivalently : $A, B \subset X$ s.t. $\mathring{A} \cup \mathring{B} = X$

$B \hookrightarrow X$ induces $H_n(B, A \cap B) \cong H_n(X, A)$

We look at a cover of $X$ in the following sense

$\mathcal{U} = (U_i)_{i \in I}$ collection of subsets of $X$

s.t. $\bigcup_{i \in I} \mathring{U}_i = X$

$\rightsquigarrow C_n^{\mathcal{U}}(X) \subset C_n(X)$ subgroup generated by

singular simplexes $\sigma$ s.t. $\exists i : \sigma : \Delta^n \to U_i$

i.e. $C_n^{\mathcal{U}}(X) = \sum_{i \in I} C_n(U_i) \subset C_n(X)$

$\partial$ sends $C_n(U_i)$ to $C_{n-1}(U_i) \Rightarrow (C_n^{\mathcal{U}}(X))_{n=0}^{\infty}$

is a subcomplex of $(C_n(X))_{n=0}^{\infty}$

$H_n^{\mathcal{U}}(X)$ : $n$-th homology of this subcomplex

Prop 2.21 in the above setting

the incl. $(C_n^{\mathcal{U}}(X))_n \overset{\iota}{\hookrightarrow} (C_n(X))_n$ is a chain

homotopy equivalence ;

i.e. $\exists\, \rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ $(n = 0, 1, 2, \cdots)$

s.t. $\iota\rho \simeq \mathrm{id}_{C_\bullet(X)}$ , $\rho\iota \simeq \mathrm{id}_{C_\bullet^{\mathcal{U}}(X)}$

chain homotopy

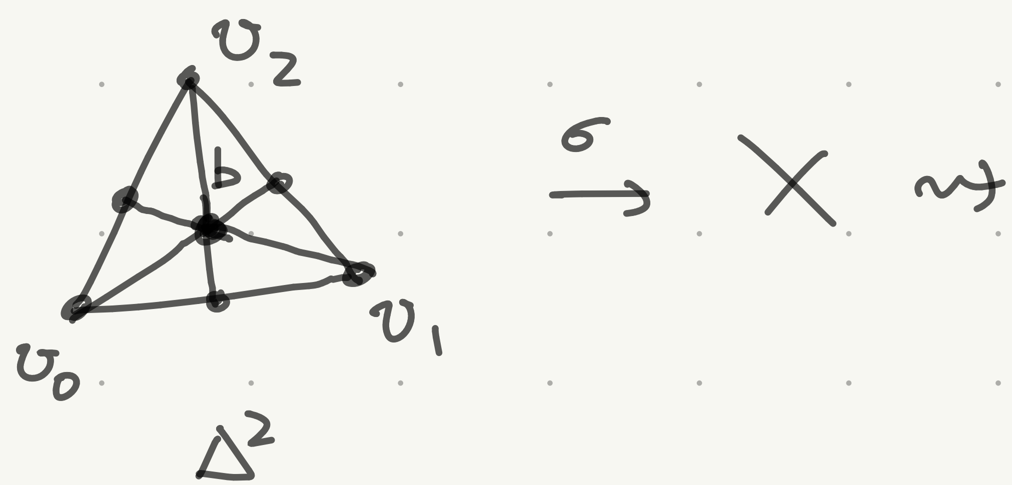
(cor. $H_n^{\mathcal{U}}(X) \cong H_n(X)$)

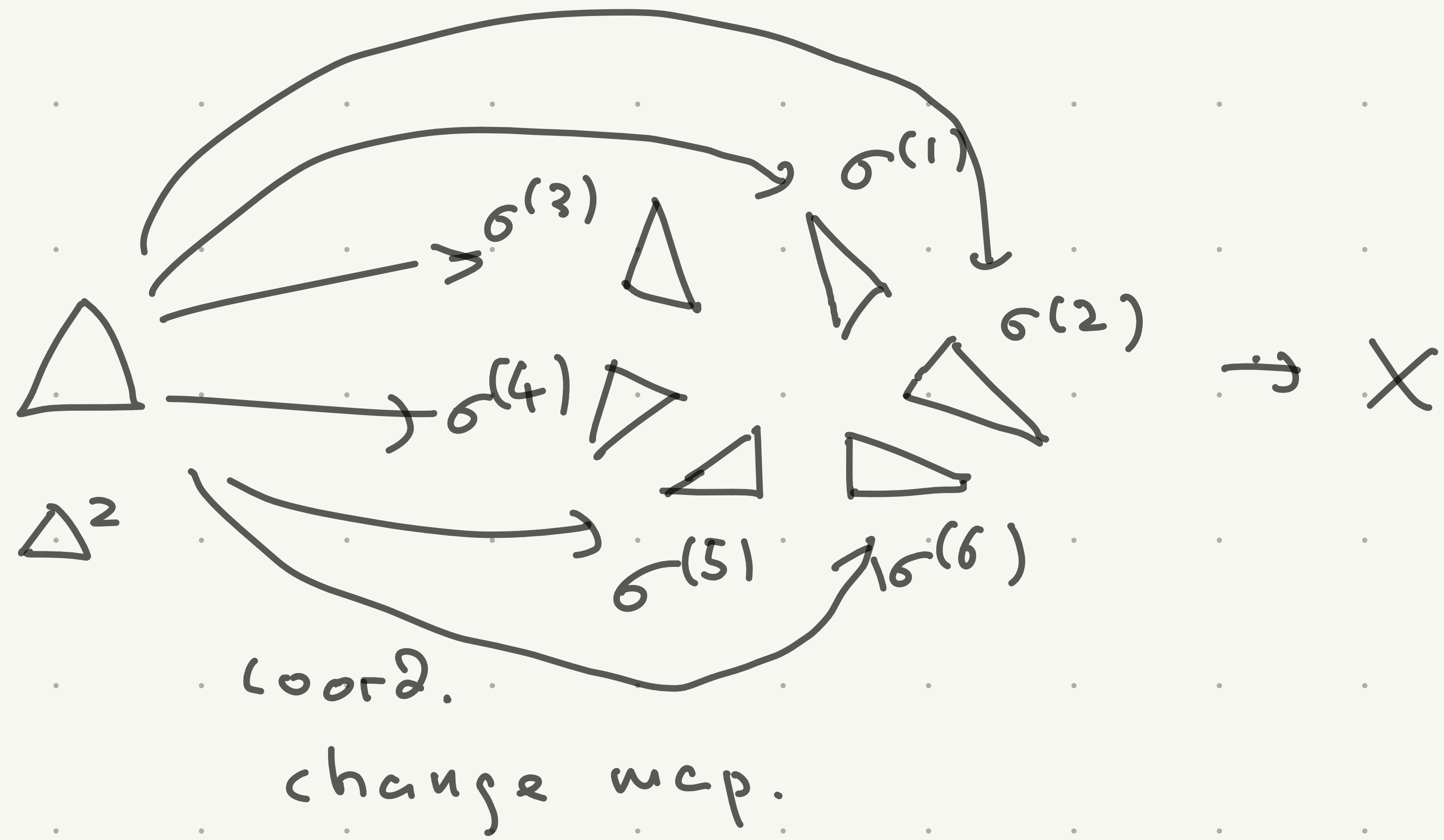
Key idea : define $\rho$ by dividing $\sigma : \Delta^n \to X$

into smaller parts $\sigma^{(1)}, \cdots, \sigma^{(k)}$ s.t.

$\forall j \ \exists i \quad \sigma^{(j)} : \Delta^n \to U_i$

to do this consistently, we perform "barycentric subdivision":




"diameter" becomes $\frac{n}{n+1}$ of the original

repeat this proceidure until we achieve
$\sigma^{(j)} : \Delta^n \to U_i$ for all $j$

Step 1 construction of $S : C_n(X) \to C_n(X)$

s.t. $S(\sigma) =$ "$\sum_j \sigma^{(j)}$"; $(\sigma^{(j)})_j$ : baryc. subdivs

for $\sigma : \Delta^n \to X$ singular $n$-simpl. in $X$

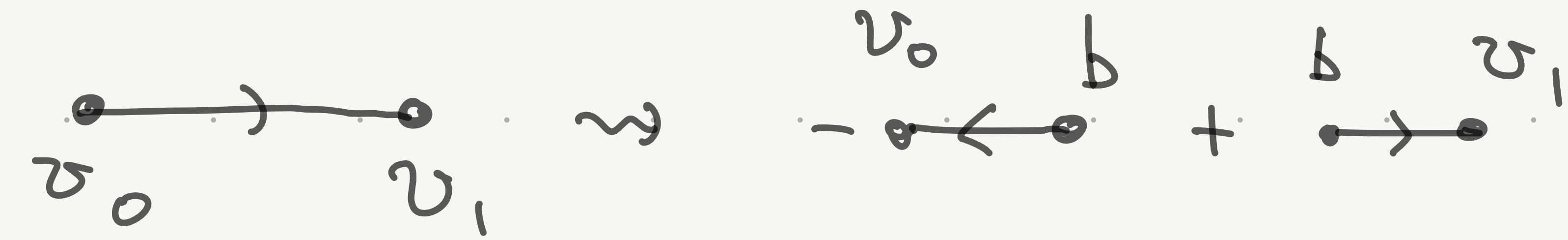
We do this by induction on $n$

$n = 0$ : $S = id_{C_0(X)}$

$n = 1$ : $b = (\frac{1}{2}, \frac{1}{2}) \in \Delta^1 = \{ (t_0, t_1) : t_i \geq 0, \sum_i t_i = 1 \}$
$\subset \mathbb{R}^2$

$( \Delta^1 = [v_0, v_1], \quad v_0 = (1,0), \; v_1 = (0,1) )$

$S(\sigma) = \sigma|_{[b, v_1]} - \sigma|_{[b, v_0]}$ (up to $\Delta^1 \to [b, v_i]$ )
$v_0 \mapsto b, \; v_1 \mapsto v_i$



general case : suppose we know $S$ on $C_{n-1}(X)$

$\sigma : \Delta^n = [v_0, \cdots, v_n] \to X$

$\partial \sigma = \sum (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$

so we can write $S(\sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]})$

as $\sum_j (-1)^{\varepsilon_{i,j}} \sigma|_{[w_0^{(i,j)}, \ldots, w_{n-1}^{(i,j)}]}$

with some points $w_k^{(i,j)} \in$ faces of $\Delta^n$

$b = (\frac{1}{n+1}, \cdots, \frac{1}{n+1}) \in \Delta^n$ barycenter

$\rightsquigarrow S(\sigma) = \sum_{i,j} (-1)^{i+\varepsilon_{i,j}} \sigma|_{[b, w_0^{(i,j)}, \ldots, w_{n-1}^{(i,j)}]}$

Step 2 $\quad S$ is an endomorphism of $(C_\bullet(X), \partial)$

i.e., $\quad S\partial = \partial S$

Put $\quad \mathrm{Con}_x[v_0', \dots, v_k'] = [x, v_0', \dots, v_k']$

for $\quad x, v_0', \dots, v_k' \in \mathbb{R}^d$

so $\quad S(\sigma) = \underbrace{\mathrm{Con}_b(S\partial\sigma)}$ $\quad$ for $\quad \sigma : \Delta^n \to X$

to be precise compose each term in

$S\partial\sigma$ with $\Delta^n \to [b, \cdots] = \mathrm{Con}_b(\text{each term})$

We have $\quad \partial \circ \mathrm{Con}_x(\alpha) = \alpha - \mathrm{Con}_x \circ \partial(\alpha)$

hence $\quad \partial S(\sigma) = S\partial(\sigma) - \mathrm{Con}_b \circ \underbrace{\partial^2}_{=0}(\sigma)$

Step 3 $S$ is chain homotopic to $\mathrm{id}_{C_\bullet(X)}$;

i.e., $\exists\, T : C_n(X) \to C_{n+1}(X)$ $(n = 0, 1, 2, \dots)$ s.t.

$$(\partial T + T\partial)(\sigma) = \sigma - S(\sigma)$$

again by induction

$n = 0$ : $\quad T\sigma = \sigma \circ (\Delta^0 \xleftarrow{} \Delta^1)$

$\uparrow \qquad$ unique

$\Delta^0 = \{pt\} \to X \qquad$ map

$\partial T\sigma = 0$, $\partial\sigma = 0$, $S(\sigma) = \sigma \Rightarrow$ above claim.

general case : suppose we know $T$ on $C_{n-1}(X)$

$b = \left(\frac{1}{n+1}, \dots, \frac{1}{n+1}\right) \in \Delta^n$ barycenter

$T\sigma = \mathrm{Con}_b(\sigma - T\partial\sigma)$ as before

$$\partial T \sigma = \partial \, \mathrm{Con}_b(\sigma - T \partial \sigma)$$

$$= \sigma - T\partial\sigma - \mathrm{Con}_b \, \partial(\sigma - T\partial\sigma)$$

$\partial \, \mathrm{Con}_x \alpha = \alpha - \mathrm{Con}_x \partial \alpha$

$$= \sigma - T\partial\sigma - \mathrm{Con}_b(S\partial\sigma - T\partial^2\sigma)$$

ind. hyp. for $S$ & $T$

$$= \sigma - T\partial\sigma - S(\sigma)$$

$S(\sigma) = \mathrm{Con}_b(S(\partial\sigma))$

Step 4 $\quad S^k \simeq \mathrm{id}_{C_\bullet(X)}$

this follows from compatibility of chain homotopy and compos. of chain maps

$$\left( S \simeq \mathrm{id}_{C_\bullet(X)} \Rightarrow S^k \simeq \mathrm{id}^k_{C_\bullet(X)} = \mathrm{id}_{C_\bullet(X)} \right)$$

concretely : $D_k = \sum_{i=0}^{k-1} T S^i$ satisfies

$$\partial D_k + D_k \partial = \mathrm{id} - S^k$$

Step 5 construction of $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$

and $D : C_n(X) \to C_{n+1}(X)$ s.t. $\rho \circ \iota = id_{C_\bullet^{\mathcal{U}}(X)}$,

$\partial D + D \partial = id_{C_\bullet(X)} - \iota \circ \rho$

for each $\sigma : \Delta^n \to X$, let $m(\sigma)$ be the smallest integer s.t. $S^{m(\sigma)}(\sigma) \in C_n^{\mathcal{U}}(X)$

such $m(\sigma)$ exists because

$S(\sigma)$ is a sum of "smaller bits" of $\sigma$

& $\forall\, x \in \Delta^n$ $\exists\, \varepsilon$ s.t. $\varepsilon$-ball around $x$ is mapped to some $U_i$ under $\sigma$

$\rightsquigarrow$ Put $D(\sigma) = D_{m(\sigma)}(\sigma) = \sum_{i=0}^{m(\sigma)-1} T S^i(\sigma)$

(cont.) and $\rho(\sigma) = \sigma - \partial D(\sigma) - D\partial(\sigma)$

by const. $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ and

$id - \iota \circ \rho = (\partial D + D \partial)(\sigma)$

if $\sigma : \Delta^n \to U_i$ for some $i$,

$m_\sigma = 0$, $m_{\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}} = 0$

$\Rightarrow D(\sigma) = 0 = D(\partial\sigma) \Rightarrow \rho \circ \iota(\sigma) = \sigma$ $\boxed{\text{E}}$

Proof of Thm 2.20

apply Prop 2.21 to $\mathcal{U} = \{A, B\}$

$\rightsquigarrow C_\bullet^{\mathcal{U}}(X) \overset{\iota}{\hookrightarrow} C_\bullet(X)$ is a chain homotopy equiv.

$C_\bullet(A)$ is a subcomplex of both, and

$\iota$, $\rho$, and $D$ preserve this

$\Rightarrow$ we get $C_\bullet^{\mathcal{U}}(X)/C_\bullet(A) \underset{\rho}{\overset{\iota}{\rightleftarrows}} \overbrace{C_\bullet(X)/C_\bullet(A)}^{\text{computes } H_\bullet(X,A)}$

still chain homotopy equivalent

$C_n^{\mathcal{U}}(X)/C_n(A)$ has basis $\{\sigma : \Delta^n \to B,$ img not contained in $A \cap B\}$

(cont.) hence $C_n^{\mathcal{U}}(X)/C_n(A) \cong \underbrace{C_n(B)/C_n(A \cap B)}$

computes $H_\bullet(B, A \cap B)$

$\rightsquigarrow H_n(B, A \cap B) \cong H_n(X, A)$ $\boxtimes$