Application of excision $H_n(X,A) \cong H_n(X \setminus Z, A \setminus Z)$

$Z \subset A \subset X$ "good"

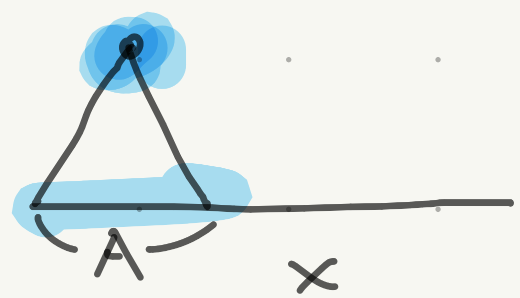
- another model of relative homology

instead of $(C_n(X)/C_n(A))_{n=0}^{\infty}$

consider the mapping cone of $A \overset{i}{\hookrightarrow} X$

i.e. $C_i = A \times I \amalg X / (a,0) \sim (a',0),\ (a,1) \sim f(a)$

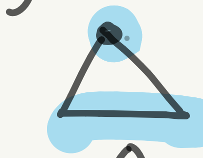
$\forall\, a, a' \in A$



Prop 1 $H_n(X,A) \cong \tilde{H}_n(C_i)$ ← reduced homology

for $\emptyset \neq A \subset X$ (if $\emptyset = A$, take $C_i = X \cup \{p\}$)

Proof Step 1 $\tilde{H}_n(C_i) \cong H_n(C_i, CA)$



A

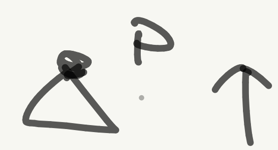
cone over A

the inclusion $CA \hookrightarrow C_i$ gives exact seq.

$$\cdots \to \underbrace{H_n(CA)}_{0} \to H_n(C_i) \xrightarrow{\sim} H_n(C_i, CA) \to \underbrace{\cdots}_{0}$$

$$\cdots \to H_0(CA) \to H_0(C_i) \to H_0(C_i, CA) \to 0$$

$H_n(CA) = 0$ $(n>0)$, $H_0(CA) \cong \mathbb{Z}$ by the contractibility of CA



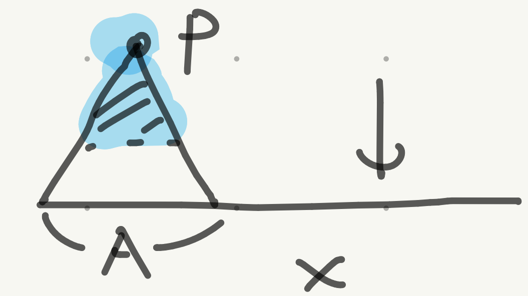
$$\Rightarrow H_n(C_i, CA) \cong H_n(C_i) \cong \tilde{H}_n(C_i) \quad (n>0)$$

$$H_0(C_i) \cong \underset{\substack{\cong \\ H_0(C_i, CA)}}{\tilde{H}_0(C_i)} \oplus \underline{\mathbb{Z}} \text{ img of } H_0(CA)$$

Step 2 $\quad H_n(Ci, CA) \cong H_n(Ci \setminus \{p\}, CA \setminus \{p\})$

for p = img of $(a, 0)$ $(a \in A)$

use excision ; p has a neigh. U



(e.g., the img of $A \times [0, \frac{1}{2})$)

for $Z = \{p\}$ $\quad \bar{Z} \subset \mathring{CA} \subset Ci$

Step 3 $\quad H_n(Ci \setminus \{p\}, CA \setminus \{p\}) \cong H_n(X, A)$

$X \hookrightarrow Ci \setminus \{p\}$ is a homotopy equiv. compat. with the subspaces A & $CA \setminus \{p\}$

($(X, A) \overset{\text{incl}}{\hookrightarrow} (Ci \setminus \{p\}, CA \setminus \{p\})$ is a homotop equivalence)

Homology of union

$(X_\alpha)_{\alpha \in I}$ collection of top. spaces

$H_n(\coprod_{\alpha \in I} X_\alpha) \cong \bigoplus_{\alpha \in I} H_n(X_\alpha)$ from def.

$C_n(\coprod X_\alpha) \cong \bigoplus C_n(X_\alpha)$

Prop 2 (Cor. 2.25) suppose $x_\alpha \in X_\alpha$ has a contractible neighborhood in X_α

($(X_\alpha, \{x_\alpha\})$ is a good pair)

then $\tilde{H}_n(\underbrace{\bigvee_{\alpha \in I} X_\alpha}_{"(\coprod X_\alpha)/x_\alpha \sim x_\beta"}) \cong \bigoplus_{\alpha \in I} \tilde{H}_n(X_\alpha)$

Proof. x_0 : img of x_α in $\bigvee_\alpha X_\alpha$

$\tilde{H}_n(\bigvee_\alpha X_\alpha) \cong H_n(\bigvee_\alpha X_\alpha, \{x_0\})$

$\{x_0\} \subset \bigvee_\alpha X_\alpha$ is a good inclusion

$U_\alpha \ni x_\alpha$ contr. in X_α $\Rightarrow$ W : img of $\bigcup_\alpha U_\alpha$ contr.

$$\Rightarrow \quad H_n\left(\bigvee_\alpha X_\alpha, \{x_0\}\right) \underset{\text{homotop.}}{\cong} H_n\left(\bigvee_\alpha X_\alpha, W\right)$$

$$\underset{\text{exc.}}{\cong} \tilde{H}_n\left(\bigvee_\alpha X_\alpha \setminus W\right) \cong \bigoplus_\alpha \tilde{H}_n(X_\alpha \setminus U_\alpha)$$

$$\bigvee_\alpha X_\alpha \setminus W \cong \coprod X_\alpha \setminus U_\alpha$$

$$\cong \bigoplus_\alpha \tilde{H}_n(X_\alpha)$$

U_α contr.

Local homology groups

Def. $x \in X$ closed point ($\{x\}$ closed in X)

<u>n-th</u> <u>local</u> <u>homology</u> at x : $H_n(U, U \setminus \{x\})$

for any open neighborhood U of x

Prop. the above def. does not depend on U

Proof enough to show

$$H_n(U, U \setminus \{x\}) \cong H_n(X, X \setminus \{x\})$$

Put $A = X \setminus \{x\}$, $B = U$ $\Rightarrow$ $A \cap B = U \setminus \{x\}$

these are open, $X = A \cup B$

(cont.) excision

$$X = \mathring{A} \cup \mathring{B} \Rightarrow H_n(X, A) \cong H_n(\overset{U}{B}, \overset{U \setminus \{x\}}{A \cap B})$$

implies the claim $\boxtimes$

Example $X = \mathbb{R}^m$ $H_n(U, U \setminus \{x\}) = \begin{cases} \mathbb{Z} & (n = m) \\ 0 & \text{otherwise} \end{cases}$

$$\cdots \to H_n(\underbrace{\mathbb{R}^m \setminus \{x\}}_{\text{homotop. to } S^{m-1}}) \to H_n(\mathbb{R}^m) \to H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \to \cdots$$

$H_n(\mathbb{R}^m \setminus \{x\}) \cong \begin{cases} \mathbb{Z} & \text{for } n = m-1 \\ 0 & \text{otherwise} \end{cases}$

$H_n(\mathbb{R}^m) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$

Comparison of simplicial and singular homology

Def. <u>relative</u> <u>simplicial</u> <u>homology</u> of Δ-cplx pair

(X, A) is the homology of $(\Delta_n(X)/\Delta_n(A))_{n=0}^{\infty}$

$\uparrow$ Δ-complex $\subset$ union of some simplices

$\Delta_n(X)$: lin. comb. of n-simplices of X

$\Delta_n(X)/\Delta_n(A)$: lin. comb. of n-simp. <u>not</u> in A

write $H_n^{\Delta}(X, A)$ $(n = 0, 1, \dots)$

We have a map of complexes $\Delta_\bullet(X) \to C_\bullet(X)$

by interpreting each n-simplex of X as

singular n-simplex $\Delta^n \longrightarrow X$

$\rightsquigarrow$ induced hom $H_n^{\Delta}(X) \to H_n^{\text{sing}}(X)$,

$H_n^{\Delta}(X, A) \to H_n^{\text{sing}}(X, A)$

Th'm 2.27 $\quad (X, A)$ Δ-complex pair

$H_n^{\Delta}(X, A) \to H_n^{sing}(X, A)$ is an isomorphism

Proof. We first check the case $A = \varnothing$

$X^k \subset X$: k-skeleton (union of n-simplices for $n \leq k$)

Step 1 incl $X^{k-1} \subset X^k$ ~> "map of exact seqs"

$$\begin{tikzcd} \cdots \arrow[r] & H_n^{\Delta}(X^{k-1}) \arrow[r] \arrow[d] & H_n^{\Delta}(X^k) \arrow[r] \arrow[d] & H_n^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d] & H_{n-1}^{\Delta}(X^{k-1}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_n^{sing}(X^{k-1}) \arrow[r] & H_n^{sing}(X^k) \arrow[r] & H_n^{sing}(X^k, X^{k-1}) \arrow[r] & H_{n-1}^{sing}(X^{k-1}) \arrow[r] & \cdots \end{tikzcd}$$

from short exact seqs. of complexes

$$0 \to \Delta_\bullet(X^{k-1}) \to \Delta_\bullet(X^k) \to \Delta_\bullet(X^k)/\Delta_\bullet(X^{k-1}) \to 0$$

same with $C_\bullet(X^k)$, $C_\bullet(X^{k-1})$

Step 2 $\quad H^{\Delta}_{k}(X^k, X^{k-1}) \cong \bigoplus_{k\text{-simplices}} \mathbb{Z}$, $\quad H^{\Delta}_{n}(X^k, X^{k-1}) = 0 \quad (n \neq k)$

$\Delta_n(X^{k-1}) = \Delta_n(X^k)$ for $n \neq k$

$\Delta_k(X^{k-1}) = 0$ by dim. reason

$\Delta_k(X^k) \cong \bigoplus_{k\text{-simplices}} \mathbb{Z}$ by def.

$\Rightarrow \Delta_n(X^k)/\Delta_n(X^{k-1})$ is only nonzero at $n = k$

Step 3 $\quad H^{\text{sing}}_{k}(X^k, X^{k-1}) \cong \bigoplus_{k\text{-simplices}} \mathbb{Z}$, $\quad H^{\text{sing}}_{n}(X^k, X^{k-1}) = 0 \quad (n \neq k)$

$H^{\text{sing}}_{n}(X^k, X^{k-1}) \cong \tilde{H}^{\text{sing}}_{n}(X^k/X^{k-1})$ by excision

$\Delta^k_\alpha \subset X \quad (\alpha \in I)$ the k-simplices of X

$\Phi : \coprod_{\alpha \in I} \Delta^k_\alpha \to X^k$ incl.

(cont.) $\partial \Delta^k_\alpha$ is a union of $(k-1)$-simplices

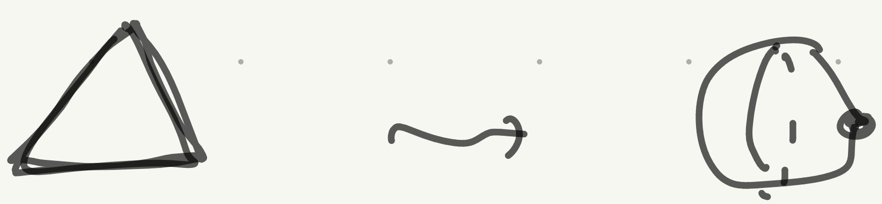
$$\Rightarrow \quad \Phi : \left(\coprod_\alpha \Delta^k_\alpha, \coprod_\alpha \partial \Delta^k_\alpha\right) \to (X^k, X^{k-1})$$

X^k is given by gluing $\coprod_\alpha \Delta^k_\alpha$ to X^{k-1}

$$\Rightarrow \quad \left(\coprod_\alpha \Delta^k_\alpha\right) \Big/ \left(\coprod_\alpha \partial \Delta^k_\alpha\right) \cong X^k / X^{k-1}$$

$$\begin{array}{c} \cong \\ \bigvee_\alpha \Delta^k_\alpha / \partial \Delta^k_\alpha \cong \bigvee_\alpha S^k \end{array}$$

Prop. 2 gives $\tilde{H}^{sing}_n\left(\bigvee_\alpha S^k\right) \cong \bigoplus_\alpha \underbrace{\tilde{H}^{sing}_n(S^k)}_{\begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}}$



Δ^2

Step 4 $\quad H_n^{\Delta}(X^k) \to H_n^{\mathrm{sing}}(X^k)$ is iso.

by induction on k

$$\begin{tikzcd}
H^{\Delta}_{n+1}(X^k, X^{k-1}) \arrow[r] \arrow[d] & H^{\Delta}_n(X^{k-1}) \arrow[r] \arrow[d] & H^{\Delta}_n(X^k) \arrow[r] \arrow[d] & H^{\Delta}_n(X^k, X^{k-1}) \arrow[r] \arrow[d] & H^{\Delta}_{n-1}(X^{k-1}) \arrow[d] \\
H^{\mathrm{sing}}_{n+1}(X^k, X^{k-1}) \arrow[r] & H^{\mathrm{sing}}_n(X^{k-1}) \arrow[r] & H^{\mathrm{sing}}_n(X^k) \arrow[r] & H^{\mathrm{sing}}_n(X^k, X^{k-1}) \arrow[r] & H^{\mathrm{sing}}_{n-1}(X^{k-1})
\end{tikzcd}$$

rows are exact.

Steps 2 & 3, induction hyp $\Rightarrow$ vertical arrows are isom. except for the middle one

$\Longrightarrow$ the middle one is also iso.

"Five Lemma"

Step 5 $\quad H_n^\Delta(X) \to H_n^{sing}(X)$ is iso

$H_n^\Delta(X) \cong H_n^\Delta(X^k)$ for $k > n$ (by def.)

$H_n^{sing}(X^m) \cong H_n^{sing}(X^k)$ for $m > k > n$

from long ex. seq for $X^{m-1} \subset X^m$, etc.

any $\Delta^n \to X$ is contained in some X^m

$\Rightarrow H_n^{sing}(X) \cong H_n^{sing}(X^k) \quad k > n$

General case $\quad H_n^\Delta(X, A) \cong H_n^{sing}(X, A)$

apply Five Lemma to

$$\begin{array}{ccccccccc}
H_n^\Delta(A) & \to & H_n^\Delta(X) & \to & H_n^\Delta(X,A) & \to & H_{n-1}^\Delta(A) & \to & H_{n-1}^\Delta(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_n^{sing}(A) & \to & H_n^{sing}(X) & \to & H_n^{sing}(X,A) & \to & H_{n-1}^{sing}(A) & \to & H_{n-1}^{sing}(X)
\end{array}$$

Five Lemma : given a diagram of comm. grps

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

- exact rows
- commuting squares
- $\alpha, \beta, \delta, \varepsilon$ isom

then γ is also an isom.

Proof

write

$$\begin{tikzcd}
 & & C \arrow[dd, "\gamma"] \arrow[dr, "k"] & & \\
A \arrow[r, "i"] & B \arrow[ur, "j"] \arrow[dr, "j'"'] & & D \arrow[r] & E \\
 & & C' \arrow[ur, "k'"'] & &
\end{tikzcd}$$

Step 1 γ is surjective

Pick $x' \in C'$

$\exists\, x_0 : k(x_0) = k'(x')$ by exactness at D

(cont.) $\exists\, b \in B : j'(b) = x' - \gamma(x_0)$

by exactness at C'

$\Rightarrow \gamma(x_0 + j(b)) = x'$ from $\gamma j(b) = j'(b)$

Step 2 γ is injective

Pick $x \in C$ s.t. $\gamma(x) = 0$

$k(x) = k'\gamma'(x) = 0$

$\exists\, b \in B : j(b) = x$ by exactness at C

$j'(b) = \gamma j(b) = 0$

$\exists\, a \in A : i(a) = b$ by exactness at B

$\Rightarrow x = j(b) = 0$ by exactness at B $\blacksquare$