Application of homology: Degree of $S^n \xrightarrow{f} S^n$

(§2.2)

recall $H_n(S^n) \cong \mathbb{Z}$, $f_*: H_n(S^n) \to H_n(S^n)$

Def the <u>degree</u> of $f: S^n \to S^n$ is $\deg f = d \in \mathbb{Z}$

s.t. $f_* \alpha = d\alpha$ for $\alpha \in H_n(S^n)$

Example $f: S^1 \to S^1$, $z \mapsto z^k$ has deg. k

from $\pi_1(S^1, *) \cong H_1(S^1)$ and $\underset{\text{on } \pi_1}{f_*}([\gamma]) = k[\gamma]$

Morally: $\deg f$ = "covering degree" of $S^n \xrightarrow{f} S^n$

(# of points in a typical fiber up to orientation)

Basic properties

a) $\deg(\mathrm{id}_{S^n}) = 1$ $\qquad (\mathrm{id}_{S^n})_* = \mathrm{id}_{H_\circ(S^n)}$

b) f not surjective $\Rightarrow \deg f = 0$

c) $f \simeq g \Rightarrow \deg f = \deg g$ $\qquad f_* = g_*$

d) $\deg(fg) = \deg(f)\deg(g)$ $\qquad (fg)_* = f_* g_*$

e) $U = \{(x_0, \dots, x_n) \in S^n \subset \mathbb{R}^{n+1}, \; x_n \geq 0\}$

"upper hemisphere"

$L = \{(x_0, \dots, x_n) \in S^n \subset \mathbb{R}^{n+1}, \; x_n \leq 0\}$

"lower hemisphere"

$f|_{U \cap L} = \mathrm{id}, \quad f(U) = L, \; f(L) = U$

$\Rightarrow \deg(f) = -1$

f) $\deg(-\mathrm{id}_{S^n}) = (-1)^{n+1}$

g) $\nexists x \in S^n \quad f(x) = x \quad \Rightarrow \deg(f) = (-1)^{n+1}$

Proof of (b) suppose $x_0 \notin \mathrm{img}(f)$

then f has factorization $S^n \to S^n \setminus \{x_0\} \to S^n$

f_* factors as $H_n(S^n) \to \underbrace{H_n(\underbrace{S^n \setminus \{x_0\}}_{\cong \mathbb{R}^n \text{ contr}})}_{\cong 0} \to H_n(S^n)$

so $f_* = 0$

Proof of (e)

take $\sigma_U : \Delta^n \xrightarrow{\sim} U$, $\sigma_L : \Delta^n \xrightarrow{\sim} L$

corresponding to a Δ-complex structure on S^n

with two n-simplices (so $\sigma_U|_{\partial\Delta^n} = \sigma_L|_{\partial\Delta^n}$)

$\sigma_U - \sigma_L$ is in kernel of $\Delta_n(S^n) \to \Delta_{n-1}(S^n)$

it represents a generator of $H_n^{\Delta}(S^n)$

$\rightsquigarrow$ same for $H_n^{sing}(S^n)$

- $f_*(\sigma_U - \sigma_L) = f\circ\sigma_U - f\circ\sigma_L$

- $f\circ\sigma_U : \Delta^n \to L$

  identification $L \cong D^n$ and convexity of D^n

  $f\circ\sigma_U \simeq \sigma_L$ via $t\, f\circ\sigma_U + (1-t)\,\sigma_L$

(cont.)

$\Rightarrow f_*(\sigma_U - \sigma_L) = \sigma_L - \sigma_U$ in $H_n(S^n)$

i.e. $f_* \alpha = -\alpha$ for $\alpha \in H_n(S^n)$

Proof of (f) $-id_{S^n} = f^{(0)} \circ \cdots \circ f^{(n)}$ for

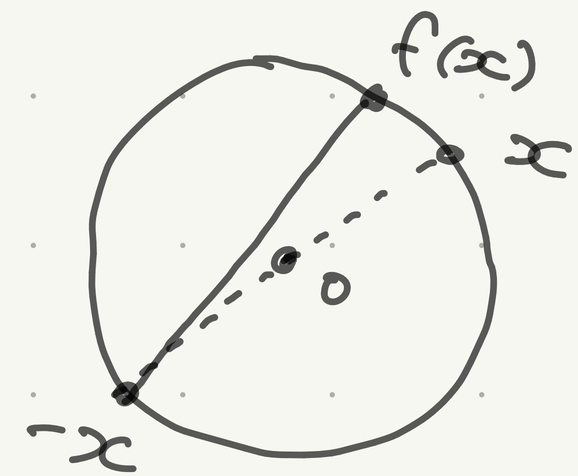
$f^{(i)} : (x_0, \dots, x_n) \mapsto (x_0, \dots, -x_i, \dots, x_n)$

$\deg(f^{(i)}) = -1$ by (e). $\deg(-id_{S^n}) \underset{(c)}{=} \prod \deg(f^{(i)})$

Proof of (g) enough to prove $f \simeq -\mathrm{id}_{S^n}$

$$f_t(x) = \frac{(1-t)\,f(x) - t\,x}{|(1-t)\,f(x) - t\,x|} \quad \text{makes sense:}$$

$(1-t)\,f(x) - t\,x \neq 0$ if $f(x) \neq x$



- $f_0(x) = f(x)$
- $f_1(x) = -x$

Thm 2.28 (hairly ball theorem)

$\exists$ nowhere vanishing vector field on S^n

$\iff$ n is odd

Proof of "$\Rightarrow$": suppose $v(x) \in T_x S^n$ is a nowhere vanishing vector field

$T_x S^n = \{ w \in \mathbb{R}^{n+1} : w \perp x \}$

$v(x)$ cont. in x, $v(x) \neq 0$

we may assume $|v(x)| = 1$ by taking $\dfrac{v(x)}{|v(x)|}$

put $f_t(x) = (\cos \pi t) x + (\sin \pi t) v(x)$

$x \perp v(x) \Rightarrow f_t(x) \in S^n$

$f_0(x) = x$, $f_1(x) = -x$ i.e. $\mathrm{id}_{S^n} \simeq -\mathrm{id}_{S^n}$

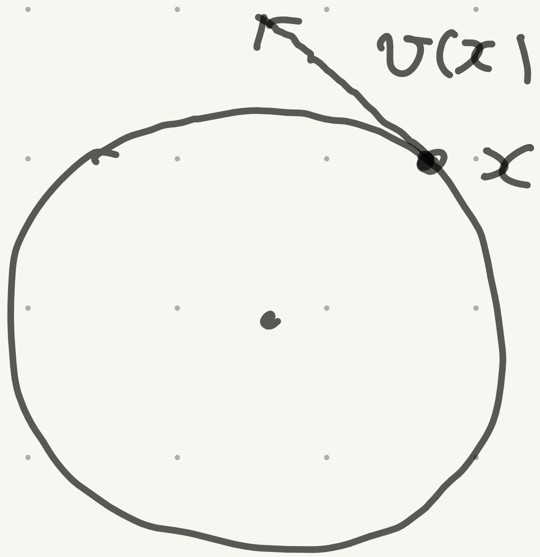
so $1 = \deg(\mathrm{id}_{S^n}) = \deg(-\mathrm{id}_{S^n}) = (-1)^{n+1}$

$\Rightarrow$ n is odd

Proof of "$\Leftarrow$": if $n = 2k+1$

$v(x_0, \dots, x_{2k+1}) = (-x_1, x_0, -x_3, x_2, \dots, -x_{2k+1}, x_{2k})$

is a nowhere vanishing vec. field



Prop 2.29 G : group, $G \curvearrowright S^n$ $n = 2k$

$gx \neq x$ for $g \neq e$ (free action)

then $G = \mathbb{Z}_2$ or $\{e\}$

Proof $\deg(g)\deg(g^{-1}) = \deg(id) = 1 \Rightarrow \deg(g) = \pm 1$

so we get $\deg : G \to \{\pm 1\} \cong \mathbb{Z}_2$ (hom.)

$g \neq e \Rightarrow \deg(g) = (-1)^{n+1} = -1$ by the property (g) $(\forall x\ f(x) \neq x \Rightarrow \deg(f) = (-1)^{n+1})$

i.e. deg is injective group hom $\blacksquare$

Rem M : compact manifold,

G : finite group, $G \curvearrowright M$ free

$\Rightarrow |G|$ divides $\chi(M) = \sum_n (-1)^n \operatorname{rk} H_n(M)$

- localization of degree

$f : S^n \to S^n$, $x \in S^n$, $y = f(x)$

U : open neighborhood of x

V : open neighborhood of y

s.t. $f(U) \subset V$, $f(U \setminus \{x\}) \subset V \setminus \{y\}$

$\rightsquigarrow$ induced map $f_* : H_n(U, U \setminus \{x\}) \to H_n(V, V \setminus \{y\})$

local homology

$H_n(U, U \setminus \{x\}) \cong \mathbb{Z}$ from local homology of $\mathbb{R}^n$

consistent generator α : img of $1 \in \mathbb{Z} \cong H_n(S^n)$

(in $H_n(U, U \setminus \{x\})$ & $H_n(V, V \setminus \{y\})$)

Def $\deg f|_x = d \in \mathbb{Z}$ $f_* \alpha = d\alpha$; <u>local</u> <u>degree</u> at x

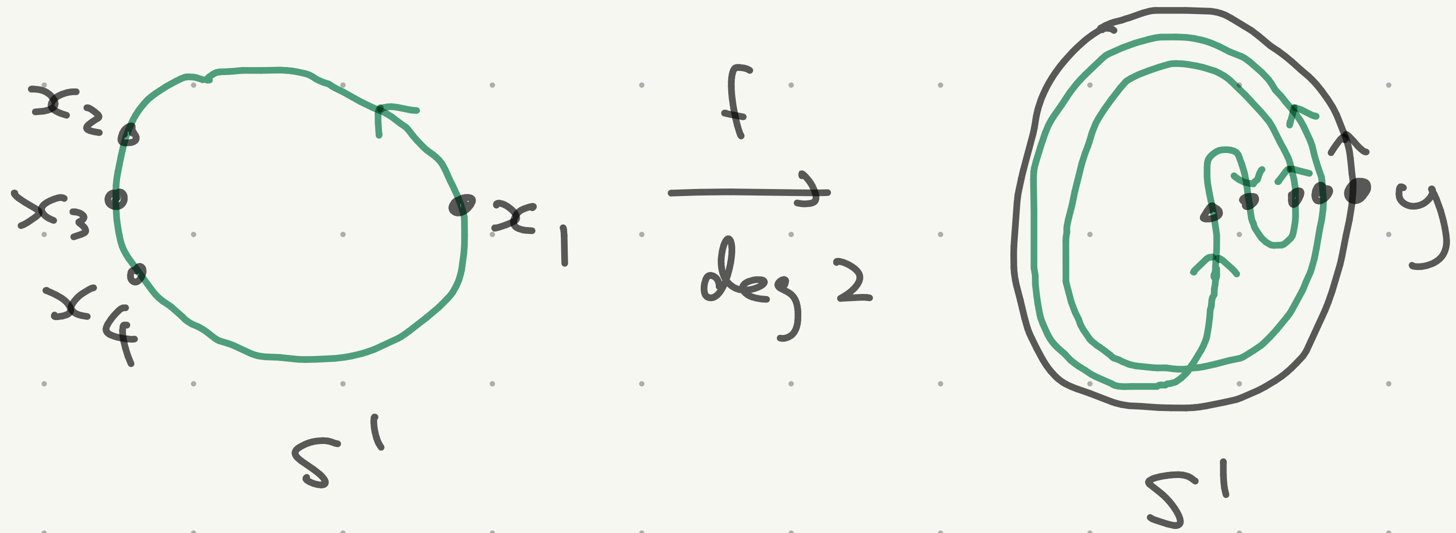
Prop 2.30 $f : S^n \to S^n$, $y \in S^n$ s.t.

$f^{-1}(y) = \{x_1, \dots, x_k\}$ is finite

Then $\deg(f) = \sum_{i=1}^{k} \underbrace{\deg f|_{x_i}}$

depends only on the local behavior of f around x_i

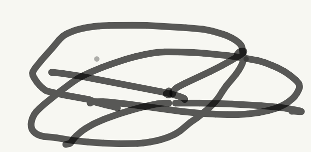
Intuition



higher dim

$f(z) = z^2 \quad z \in \mathbb{C}$

$\deg f|_0 = 2$



$\deg f|_{x_1} = \deg f|_{x_2} = \deg f|_{x_4} = 1$ pos. orientation

$\deg f|_{x_3} = -1$ neg. orientation

Proof overview: factorize $H_n(S^n) \xrightarrow{f_*} H_n(S^n)$ as

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & \begin{array}{c}\mathbb{Z} \cong H_n(S^n, S^n \setminus \{x_1\}) \\ \oplus \\ \vdots \\ \oplus \\ \mathbb{Z} \cong H_n(S^n, S^n \setminus \{x_k\})\end{array} & \longrightarrow & \mathbb{Z} \\
\text{\rotatebox{90}{$\cong$}} & & & & \text{\rotatebox{90}{$\cong$}} \\
H_n(S^n) & & & & H_n(S^n)
\end{array}$$

$$a \longmapsto a \oplus \cdots \oplus a \,, \qquad b_1 \oplus \cdots \oplus b_k \longmapsto \sum_{i=1}^{k} (\deg f|_{x_i})\, b_i$$

The intermediate group: $\bigoplus_{i=1}^{k} H_n(S^n, S^n \setminus \{x_i\})$

Step 1 this group is $H_n(S^n, S^n \setminus \underbrace{\{x_1, \dots, x_k\}}_{f^{-1}(y)})$

take small neighborhood U_i of x_i s.t.

$U_i \cap U_j = \emptyset$ for $i \neq j$

By excision $H_n(S^n, S^n \setminus \{x_i\}) \cong H_n(U_i, U_i \setminus \{x_i\})$

(cont.)

$$H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) = H_n\left(\bigcup_{i=1}^k U_i, \bigcup_{i=1}^k U_i \setminus \{x_1, \dots, x_k\}\right)$$

$$\underset{U_i \cap U_j = \emptyset}{\cong} \bigoplus_{i=1}^k H_n(U_i, U_i \setminus \{x_i\}) \cong \bigoplus_{i=1}^k H_n(S^n, S^n \setminus \{x_i\})$$

Step 2 the natural map $H_n(S^n) \to H_n(S^n, S^n \setminus \{x_1, \dots, x_k\})$ becomes $\mathbb{Z} \to \mathbb{Z}^k$, $a \mapsto (a, \dots, a)$ up to

- $H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) \cong \bigoplus_i H_n(S^n, S^n \setminus \{x_i\})$
- fix a generator $\alpha \in H_n(S^n)$ $(\alpha \leftrightarrow 1 \in \mathbb{Z})$
- use its img. $\bar{\alpha} \in H_n(S^n, S^n \setminus \{x_i\})$ as the generator $(\bar{\alpha} \leftrightarrow 1 \in \mathbb{Z})$

Step 3 $\quad f_*: H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) \to H_n(S^n, S^n \setminus \{y\})$

(well defied by $f(S^n \setminus \{x_1, \dots, x_k\}) \subset S^n \setminus \{y\}$)

becomes $\mathbb{Z}^k \to \mathbb{Z}$, $(b_1, \dots, b_k) \mapsto \sum_{i=1}^{k} (\deg f|_{x_i}) \cdot b_i$

up to $H_n(S^n, S^n \setminus \{y\}) \ni \bar{\alpha} \longleftrightarrow 1 \in \mathbb{Z}$

- $H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{f_*} H_n(S^n, S^n \setminus \{y\})$ is $b \mapsto (\deg f|_{x_i}) b$

  by def. of $\deg f|_{x_i}$

- $H_n(U_i, U_i \setminus \{x_i\}) \xrightarrow{i_*} H_n(\bigcup_{i=1}^{k} U_i, \bigcup_{i=1}^{k} U_i \setminus \{x_1, \dots, x_k\})$

  is the inclusion of direct summand

- $H_n(\bigcup_{i=1}^{k} U_i, \bigcup_{i=1}^{k} U_i \setminus \{x_1, \dots, x_k\}) \cong H_n(S^n, S^n \setminus \{x_1, \dots, x_k\})$

Summary

$$\begin{array}{ccccc}
H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \cong \mathbb{Z} & \xrightarrow{\sim} & H_n(S^n, S^n \setminus \{y\}) \\
& \text{gives } \deg f & & & \uparrow \sum_{i=1}^{k} f_* \quad \text{gives } \deg f|x_i \\
\downarrow & & & & \\
H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) & \cong & \bigoplus_{i=1}^{k} H_n(S^n, S^n \setminus \{x_i\}) & &
\end{array}$$

$$H_n(S^n, S^n \setminus \{x_1, \dots, x_k\}) \cong \mathbb{Z}^k$$

$\boxed{\Sigma}$

Example $\quad S^3 \cong SU(2) = \{ X \in M_2(\mathbb{C}) : X^* X = I_2, \ \det X = 1 \}$

$SU(2) \to SU(2)$, $X \mapsto X^k$ has degree k

small neighborhood of $I_2 \sim$ small neigh. of $0 \in \mathbb{R}^3$

manifold structure

$X \mapsto X^k$ around $I_2 \longleftrightarrow v \mapsto kv$ around $0 \in \mathbb{R}^3$