Cellular homology for cell complexes

X : cell complex (CW complex)

recall (Jan. 13) : $X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots \nearrow X$

$\uparrow$ n-skeleton

$X^n = \left(X^{n-1} \amalg \left(\coprod_{\alpha \in I_n} D^n \right) \right) /$ glue $\partial D^n = S^{n-1}$ at α-th comp.

$\uparrow$ index set for n-cells

to X^{n-1} by $\varphi_\alpha : S^{n-1} \to X^{n-1}$

$e^n_\alpha = \varphi_\alpha(\mathring{D}^n) \subset X^n$: α-th n-cell

<u>Cellular</u> <u>homology</u> $H^{CW}_\bullet(X)$: another model of homology

s.t. "chains" at degree n = lin. comb. of n-cells

$$C^{CW}_n(X) = \left\{ \sum_{\alpha \in I_n} k_\alpha \cdot e^n_\alpha : k_\alpha \in \mathbb{Z} \right\}$$

$\rightsquigarrow$ problem : how do we make sense of the boundary map $C^{CW}_n \to C^{CW}_{n-1}$?

Lem 1 (2.34 (a))

$$H_k(X^n, X^{n-1}) = \begin{cases} \bigoplus_{I_n} \mathbb{Z} \cong C_n^{CW}(X) & k = n \\ 0 & k \neq n \end{cases}$$

Proof (X^n, X^{n-1}) is good (X^{n-1} has a neighborhood $U \subset X^n$ s.t. $X^{n-1} \subset U$ is a deform. retract)

$\Rightarrow$ $H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n / X^{n-1})$

$\uparrow$ Mar. 23 (Prop 2.22)

$X^n / X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n$ from $D^n / \partial D^n \cong S^n$

$\tilde{H}_k(\bigvee_{\alpha \in I_n} S^n) \cong \bigoplus_{\alpha \in I_n} \tilde{H}_k(S^n)$ gives the claim.

$\uparrow$ Apr. 13 (Cor 2.25)

$\square$

Lem 2 (2.34 (b)) $H_k(X^n) = 0$ for $k > n$

Proof induction on n

$n = 0$: X^0 discrete $\Rightarrow H_k(X^0) = 0$ for $k > 0$

general n : look at the long exact seq

$$\cdots \to \underset{\substack{\| \\ 0 \text{ if } k+1>n \\ \text{by Lem 1}}}{H_{k+1}(X^n, X^{n-1})} \to \tilde{H}_k(X^{n-1}) \to \tilde{H}_k(X^n) \to \underset{\substack{\| \\ 0 \text{ if } k>n \\ \text{by Lem 1}}}{H_k(X^n, X^{n-1})} \to \cdots$$

$\Rightarrow H_k(X^n) \overset{?}{=} H_k(X^{n-1})$ for $k > n$ ☺

Lem 3 (2.34 (c))

$X^n \hookrightarrow X$ incl. induces $\begin{cases} H_k(X^n) \cong H_k(X) & \text{for } k < n \\ \text{surj. map } H_n(X^n) \to H_n(X) \end{cases}$

Proof for finite dim case : $X = X^N$ for some N

again look at the exact seq.

$$\cdots \to \underbrace{H_{k+1}(X^n, X^{n-1})}_{\substack{= \\ 0 \text{ if } k+1 \neq n \\ \text{by Lem 1}}} \to H_k(X^{n-1}) \to H_k(X^n) \to \underbrace{H_k(X^n, X^{n-1})}_{\substack{= \\ 0 \text{ if } k \neq n \\ \text{by Lem 1}}} \to \cdots$$

$\Rightarrow \quad H_k(X^{n-1}) \cong H_k(X^n)$ if $n < k$ or $n > k+1$

$(n-1 > k)$

$\Rightarrow \quad H_k(X^m) \cong H_k(X^{m+1}) \cong \cdots \cong H_k(X^N)$ if $m > k$

(starting $k = n$, $m = n+1$)

also $H_n(X^n) \to H_n(X^{n+1}) \cong H_n(X)$ surjective.

Infinite dim case: look at the singular homology

each singular k-simplex $\Delta^k \xrightarrow{\sigma} X$ factors

through X^N for some (big) N

$\Rightarrow$ each chain $\alpha = \sum_\sigma m_\sigma \cdot \sigma \in C_k(X)$ is in

$C_k(X^N)$ (N could depend on α)

$\Rightarrow$ $[\alpha] \in H_k(X)$ is in the image of $H_k(X^N)$

we know $H_k(X^N) \cong H_k(X^{k+1})$

$\alpha = \partial\beta$ $\Rightarrow$ again $\exists N$ $\beta \in C_{k+1}(X^N)$

$\Rightarrow$ $[\alpha] = 0$ in $H_k(X^N)$ already

Chain complex for $H^{CW}_\bullet(X)$ (conceptual)

$$C^{CW}_n(X) = H_n(X^n, X^{n-1}) \cong \Big\{ \sum_{\alpha \in I_n} m_\alpha \cdot e^n_\alpha : m_\alpha \in \mathbb{Z} \Big\}$$

(relative singular homology) ↑ Lem 1

boundary map $d_n : C^{CW}_n(X) \to C^{CW}_{n-1}(X)$ is

$$\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[rr] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] &
\end{tikzcd}$$

∂_n from long exact seq. of (X^n, X^{n-1})

j_{n-1} from long exact seq. of (X^{n-1}, X^{n-2})

$d_{n-1} d_n = 0$ follows from

$$\begin{tikzcd}
& H_{n-1}(X^{n-1}, X^{n-2}) \arrow[dr, "\partial"] & \\
H_{n-1}(X^{n-1}) \arrow[ur, "j"] \arrow[rr, "0"'] & & H_{n-2}(X^{n-2})
\end{tikzcd}$$

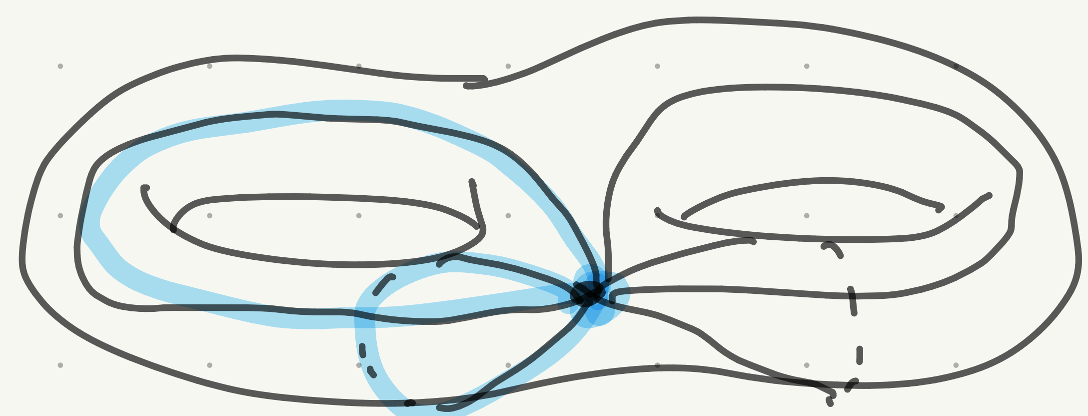
Def. $H_n^{cw}(X) = \ker\left(C_n^{cw}(X) \to C_{n-1}^{cw}(X)\right) \big/ \operatorname{img}\left(C_{n+1}^{cw}(X) \to C_n^{cw}(X)\right)$

Example $X = M_g$ orientable surface without b'dry

g holes

$\rightsquigarrow$ cell complex structure with

- one 2-cell
- $2g$ 1-cells
- one 0-cell



boundary maps are all zero $\quad H_n^{cw}(M_g) = C_n^{cw}(M_g)$

Th'm (2.35) $H_n^{CW}(X) \cong H_n(X) \left(= H_n^{sing}(X)\right)$

Proof we know $H_n(X) \cong H_n(X^{n+1})$

Step 1 $H_n(X^{n+1}) \cong H_n(X^n) / \partial_{n+1}(H_{n+1}(X^{n+1}, X^n))$

long ex. seq.

$$H_{k+1}(X^{n+1}, X^n) \xrightarrow{\partial} \tilde{H}_k(X^n) \to \tilde{H}_k(X^{n+1}) \to \underbrace{H_k(X^{n+1}, X^n)}_{=0 \text{ unless } k=n+1} \to \cdots$$

Step 2 $j_k : H_k(X^k) \to H_k(X^k, X^{k-1})$ is injective

again take the exact seq. (at $n+1=k$)

$$\underbrace{H_k(X^n)}_{=0 \text{ if } k>n} \to H_k(X^{n+1}) \xrightarrow{j} H_k(X^{n+1}, X^n) \to \cdots$$

Step 3 $\quad \operatorname{img} \partial_{n+1} \cong \operatorname{img} \underbrace{j_n \partial_{n+1}}_{\partial_{n+1}}$ from j_n injective

$\xrightarrow[j_n]{}$

Step 4 $\quad \ker \partial_n = \ker \underbrace{j_{n-1} \partial_n}_{\partial_n}$ from j_{n-1} injective

Step 5 $\quad \ker \partial_n = \operatorname{img} H_n(X^n)$ in $H_n(X^n, X^{n-1})$

from ex. seq. $\quad H_n(X^n) \to H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1})$

Summary : $H_n(X^{n+1}) \cong \underbrace{H_n(X^n)}_{\cong \ \ker \partial_n} / \underbrace{\partial_{n+1}(H_{n+1}(X^{n+1}, X^n))}_{\cong \ \operatorname{img} \partial_{n+1}}$

$\cong$

$H_n(X)$

$$\underbrace{\ker \partial_n \qquad \operatorname{img} \partial_{n+1}}_{\cong \ H_n^{CW}(X)}$$

∎

Application: if X has only even dim. cells

(or only odd dim. cells) then $H_n(X) \cong C_n^{cw}(X)$.

$C_n^{cw}(X) \xrightarrow{\partial_n} C_{n-1}^{cw}(X)$ is always zero

$\Rightarrow H_n^{cw}(X) \cong C_n^{cw}(X)$

example $\mathbb{C}P^n$, complex Grassmannian, ..

e.g. $\mathbb{C}P^1 \cong S^2$ one 0-cell, one 2-cell

$$\begin{array}{cccc} \mathbb{C}P^n & \text{one} & \underbrace{2k\text{-cell}}_{\cong\ \mathbb{C}^k} & \text{for } k=0,\dots,n \\ \| & & & \| \\ \{[z_0:\dots:z_n] : & & & \{[z_0:\dots:z_n] \\ z_i \in \mathbb{C} & & & z_i = 0 \text{ for } i > k\} \\ [z_0:\dots:z_n] = \underline{[\lambda z_0:\dots:\lambda z_n]}\} & & & \setminus (\text{prev. step}) \end{array}$$

Concrete formula for the boundary $C_n^{CW} \xrightarrow{\partial_n} C_{n-1}^{CW}$

Prop $\partial_n(e_\alpha^n) = \sum_{\beta \in I_{n-1}} d_{\alpha\beta}\, e_\beta^{n-1}$, where $d_{\alpha\beta}$ is the degree of

$$S^{n-1} \xrightarrow[\text{gluing}]{\varphi_\alpha} X^{n-1} \xrightarrow{q_\beta} \underbrace{X^{n-1}/(X^{n-1} \setminus e_\beta^{n-1})}_{D^{n-1}/\partial D^{n-1}} \cong S^{n-1}$$

Proof ideas

- e_α^n in $H_n(X^n, X^{n-1})$: img of $H_n(D^n, \partial D^n) \ni e^n$ by the map $D^n \xrightarrow{\Phi_\alpha} X^n$
- projection $H_{n-1}(X^{n-1}, X^{n-2}) \to \mathbb{Z}\, e_\beta^{n-1} \cong \tilde{H}_{n-1}(S^{n-1})$

$X^{n-1}/X^{n-2} \cong \bigvee_{\beta' \in I_{n-1}} S^{n-1} \xrightarrow{q_\beta} S^{n-1}$ collapse S^{n-1} at $\beta' \neq \beta$

- naturality of long exact seq

for $\Phi_\alpha : (D^n, \partial D^n) \to (X^n, X^{n-1})$

$$\begin{array}{ccc} H_n(D^n, \partial D^n) & \xrightarrow{\partial} & \tilde{H}_{n-1}(\partial D^n) \\ \downarrow \Phi_{\alpha *} & & \downarrow \varphi_{\alpha *} \\ H_n(X^n, X^{n-1}) & \xrightarrow[\partial_n]{} & \tilde{H}_{n-1}(X^{n-1}) \end{array}$$

- $\partial \underline{e^n} = \underline{e^{n-1}}$ for $H_n(D^n, \partial D^n) \xrightarrow{\partial} \tilde{H}_{n-1}(\partial D^n)$

generator of $\tilde{H}_{n-1}(S^{n-1})$

generator of $H_n(D^n, \partial D^n)$

$\Rightarrow \partial_n e^n_\alpha = \varphi_{\alpha *}(e^{n-1})$

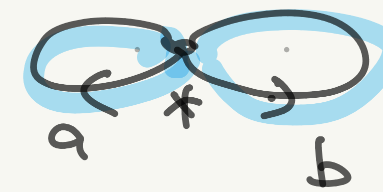
coefficient of e^{n-1}_{β} in $d\, e^n_{\alpha}$: $k \in \mathbb{Z}$ s.t.

$$
\begin{tikzcd}
e^n \in H_n(D^n, \partial D^n) \arrow[r, "\partial"] \arrow[d, "\Phi_{\alpha *}"'] \arrow[dddrr, mapsto, bend right] & \tilde{H}_{n-1}(\partial D^n) \arrow[d, "\varphi_{\alpha *}"] \arrow[dddr, bend left] & \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[rr, bend right, "d_n"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, phantom, "\cong"] \\
& & \tilde{H}_{n-1}\left(\bigvee_{\beta' \in I_{n-1}} S^{n-1}\right) \arrow[d, "q_{\beta *}"] \\
& & k e^{n-1} \in \tilde{H}_{n-1}(S^{n-1})
\end{tikzcd}
$$

picks up $\deg(q_\beta \circ \varphi_\alpha)$

Example (acyclic space $\tilde{H}_n(X) = 0$ for all n)

2-dim example with $X^1 \cong S^1 \vee S^1$



take two loops γ_1, γ_2 s.t.

$$\gamma_1 = a^5 b^{-3}, \quad \gamma_2 = b^3 (ab)^{-2} \quad \text{in } \pi_1(X^1, *)$$
$$(= b a^{-2})$$

attach 2-cells e_1^2, e_2^2 to these loops

- $\pi_1(X, *) \cong \langle a, b : a^5 b^{-3} = e = b^3 (ab)^{-2} \rangle$

- $C_2^{CW}(X) \cong \mathbb{Z}^2, \quad C_1^{CW}(X) \cong \mathbb{Z}^2, \quad C_0^{CW}(X) = \mathbb{Z}$

$$d_2 e_1^2 = 5e_a^1 - 3e_b^1 \quad ([\gamma_1] \in H_1(X^1))$$
$$d_2 e_2^2 = -2e_a^1 + e_b^1 \quad ([\gamma_2] \in H_1(X^1))$$

(cont.) $\partial_1 e^1_a = 0 = \partial_1 e^1_b$; so $(C^{cw}_n(X), \partial)$ is

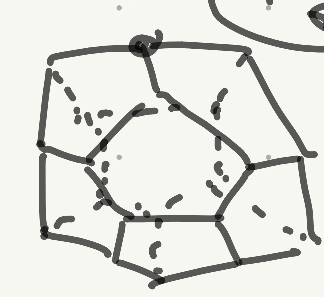
$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z}^2 & \xrightarrow{\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}} & \mathbb{Z}^2 & \xrightarrow{0} & \mathbb{Z} \\ n=3 & & n=2 & & n=1 & & n=0 \end{array}$$

determinant -1, hence invertible

$\Rightarrow$ $H^{cw}_n(X) = 0$ for $n > 0$, $H^{cw}_0(X) \cong \mathbb{Z}$

$\Rightarrow$ $\tilde{H}_n(X) = 0$ for all n

$\pi_1(X, *)$ is still nontrivial ($\Rightarrow X \not\simeq *$)

$G = \langle a, b : \underline{a^5 = e = b^3 = (ab)^2} \rangle$ is a quotient of $\pi_1(X, *)$, $G \cong$ Symm. ()

Euler characteristic $\chi(X) = \sum_{n=0}^{\infty} (-1)^n \underbrace{\mathrm{rk}\, H_n(X)}_{\text{Betti number } b_n(X)}$

(well defined if $\mathrm{rk}\, H_n(X) < \infty$ for all n and

$\mathrm{rk}\, H_n(X) = 0$ for $n \gg 0$)

Th'm 2.44 X finite cell complex

$$\chi(X) = \sum_n (-1)^n \#(n\text{-cells of } X)$$

Proof $\#(n\text{-cells of } X) = \mathrm{rk}\, C_n^{CW}(X)$

so it's enough to establish

$$\underline{\sum (-1)^n \mathrm{rk}\, H_n = \sum (-1)^n \mathrm{rk}\, C_n}$$

for chain complex $(C_\bullet, d)$, $H_n = H_n(C_\bullet, d)$

Put $Z_n = \ker(C_n \xrightarrow{\partial_n} C_{n-1})$, $B_n = \mathrm{img}(C_{n+1} \xrightarrow{\partial_{n+1}} C_n)$

so $H_n = Z_n / B_n$ $\quad (\cong \mathbb{Z}^r \oplus \underbrace{\mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_k}}_{\text{killed by } \otimes \mathbb{Q}})$

$H_n \otimes \mathbb{Q} \cong (Z_n \otimes \mathbb{Q}) / (B_n \otimes \mathbb{Q})$

$\Rightarrow \mathrm{rk}\, H_n = \mathrm{rk}\, Z_n - \mathrm{rk}\, B_n$

similarly $B_n = C_{n+1} / Z_{n+1}$ gives

$\mathrm{rk}\, B_n = \mathrm{rk}\, C_{n+1} - \mathrm{rk}\, Z_{n+1}$

$\rightsquigarrow$ alternating sum in n gives the claim $\boxtimes$

Example P : polyhedron $(\cong D^3$ as top. sp.$)$

$\#(\text{faces}) - \#(\text{edges}) + \#(\text{vertices})$ computes

$\chi(\partial P) = \chi(S^2) = 2$