Problem set 6

Section 2.1

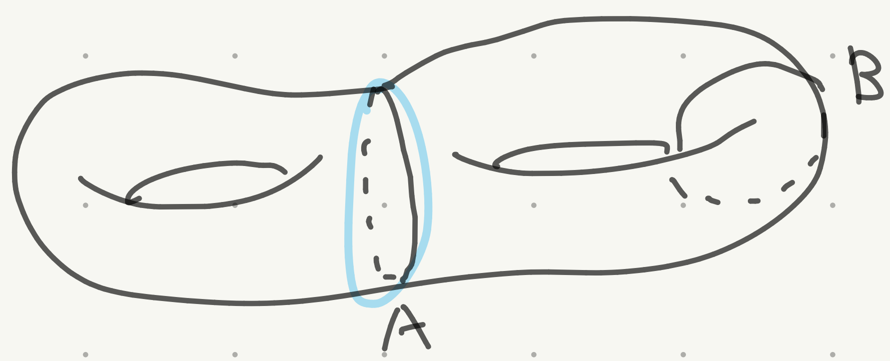
Exercise 17 compute $H_n(X, A)$

(a) for $X = S^2$, A finite subset, and

for $X = S^1 \times S^1$, A finite subset

(X path connected $\Rightarrow$ precise location of points $a \in A$ does not matter)

(b) $X = M_2$ (genus 2 closed orientable surface)



also compute $H_n(X, B)$

(a) look at the long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \cdots$$

$$\cdots \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X,A) \to 0$$

A finite $\Rightarrow$ $H_n(A) = 0$ for $n > 0$, $\tilde{H}_0(A) = \mathbb{Z}^{|A|-1}$

$\Rightarrow$ we get $H_n(X) \cong H_n(X,A)$ for $n > 1$

extension $H_1(X) \to H_1(X,A) \to \mathbb{Z}^{|A|-1}$

and $H_0(X,A) = 0$

Lem any extension of the form $M \xrightarrow{i} N \xrightarrow{p} \mathbb{Z}^d$ splits

i.e. $N \cong M \oplus \mathbb{Z}^d$

Proof $e_1, \ldots, e_d$: (standard) basis of $\mathbb{Z}^d$

choose $v_i \in N$ s.t. $p(v_i) = e_i$

(cont.) and define $s : \mathbb{Z}^d \to N$ by $s(e_i) = v_i$

then $N \to M \oplus \mathbb{Z}^d$, $v \mapsto (i^{-1}(\underbrace{v - sp(v)}_{\text{in } \ker p = \operatorname{img} i}), p(v))$

is an isomorphism

So we get $H_1(X, A) \cong H_1(X) \oplus \mathbb{Z}^{|A|-1}$

For $X = S^2$ $H_2(S^2) \cong H_0(S^2) \cong \mathbb{Z}$, $H_n(S^2) = 0$ otherwise

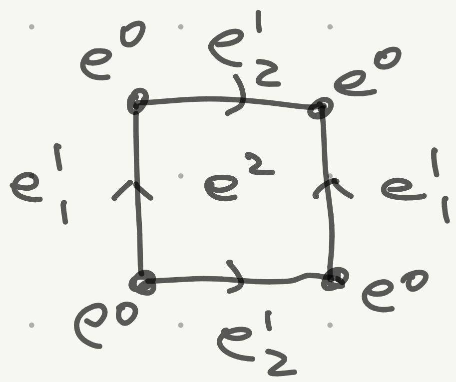
$$H_n(S^2, A) \cong \begin{cases} \mathbb{Z}^{|A|-1} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

For $X = S^1 \times S^1$; claim $\begin{cases} H_2(S^1 \times S^1) \cong H_0(S^1 \times S^1) \cong \mathbb{Z} \\ H_1(S^1 \times S^1) \cong \mathbb{Z}^2 \\ H_n(S^1 \times S^1) = 0 \quad \text{otherwise} \end{cases}$

Proof of the claim ($S^1 \times S^1 = M_1$)

strategy 1) use cellular homology for cell cplx str.

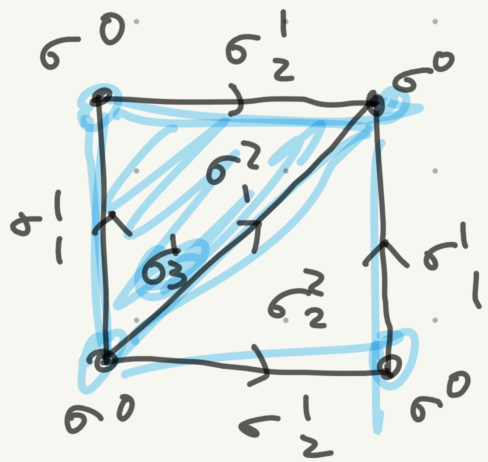
with one 2-cell, two 1-cells, one 0-cell



and see $\partial = 0$ on $C^{cw}_\bullet(S^1 \times S^1)$

strategy 2) use simplicial homology for Δ-cplx str.

with two 2-cells, three 1-cells, one 0-cell



$\sigma^2_1 - \sigma^2_2 \in \Delta_2(S^1 \times S^1)$ generates $\ker \partial$

σ^1_1, σ^1_2 basis of $H_1(S^1 \times S^1)$

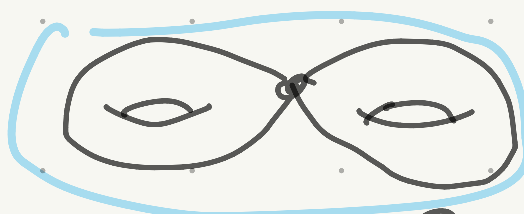
consequently we get

$$H_n(S^1 \times S^1, A) \cong \begin{cases} \mathbb{Z}^{|A|+1} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{otherwise} \end{cases}$$

(b) $A, B \subset M_2$ have neighborhoods U, V s.t.

$A \subset U$, $B \subset V$ are deformation retracts

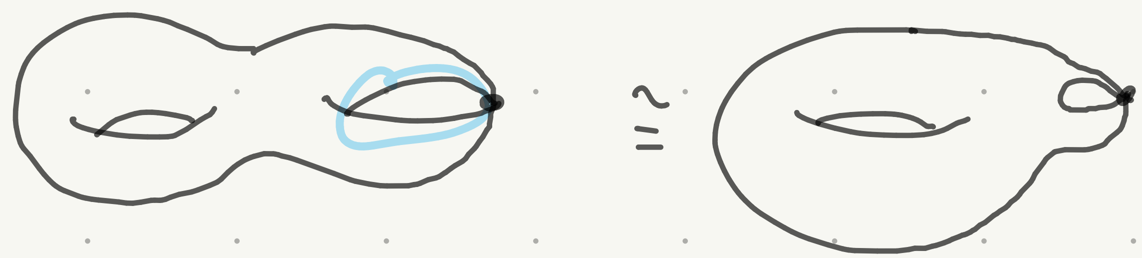
$\Rightarrow H_n(M_2, A) \cong \tilde{H}_n(M_2/A)$ etc.

M_2/A :  $\cong (S^1 \times S^1) \vee (S^1 \times S^1)$

by $\tilde{H}_n(Y \vee Z) \cong \tilde{H}_n(Y) \oplus \tilde{H}_n(Z)$

we get $H_n(M_2, A) \cong \begin{cases} \mathbb{Z}^4 & n=1 \\ \mathbb{Z}^2 & n=2 \\ 0 & \text{otherwise} \end{cases}$

M_2 / B :



$(S^1 \times S^1)/\{x, y\}$

for $x \neq y$

So $\tilde{H}_n(M_2/B) \cong \tilde{H}_n((S^1 \times S^1)/\{x,y\}) \cong H_n(S^1 \times S^1, \{x,y\})$

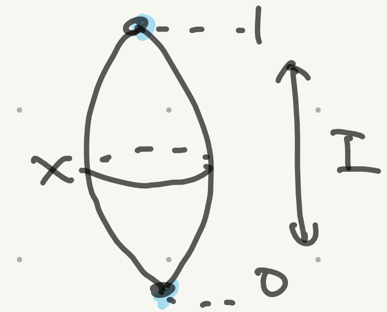
this reduces to (a). we get

$$H_n(M_2, B) \cong \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z}^3 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise 20

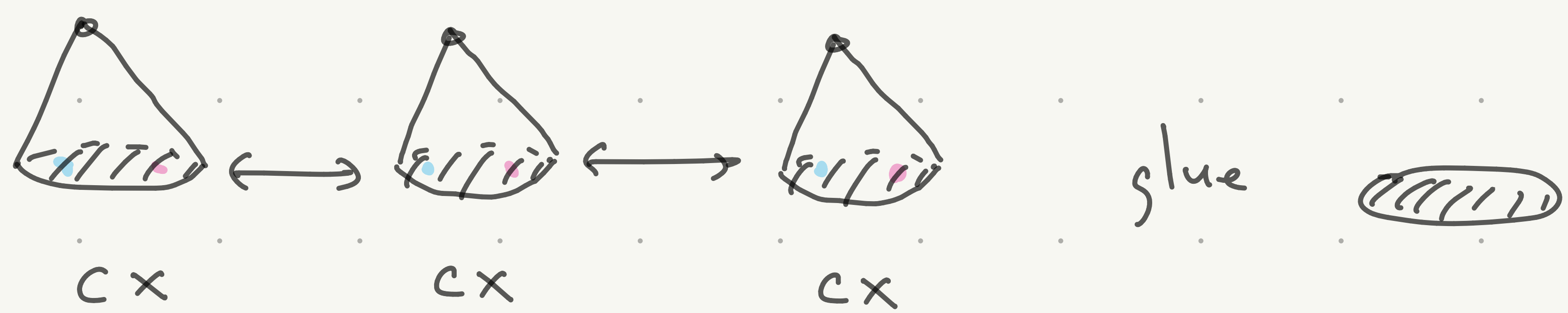
(a) show $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ for the suspension

$$SX = ((I \times X) / \{0\} \times X) / \{1\} \times X$$



(b) compute $\tilde{H}_n(Y)$ for

$$Y = (k \text{ copies of } \underset{\substack{\shortparallel \\ I \times X / \{1\} \times X}}{CX}) / \text{identify copies of } X \text{ at the bases } \{0\} \times X$$



(a) $SX = CX / \{0\} \times X$ and $\{0\} \times X \subset CX$ has

a neighborhood U (say $[0, \frac{1}{2}) \times X$) s.t.

$\{0\} \times X \subset U$ is a deform. retr.

$$\Rightarrow \underline{\tilde{H}_n(SX) \cong H_n(CX, \{0\} \times X)}$$

$\overset{X}{S\text{//}}$

take the long exact seq. for $(CX, \{0\} \times X)$

$$\cdots \to H_n(X) \to \underline{H_n(CX)} \to H_n(CX, \{0\} \times X) \xrightarrow{\partial} H_{n-1}(X) \to \cdots$$

$$\to \tilde{H}_0(X) \to \underline{\tilde{H}_0(CX)} \to H_0(CX, \{0\} \times X) \to 0$$

CX is contractible $\Rightarrow \tilde{H}_n(CX) = 0$ for all n

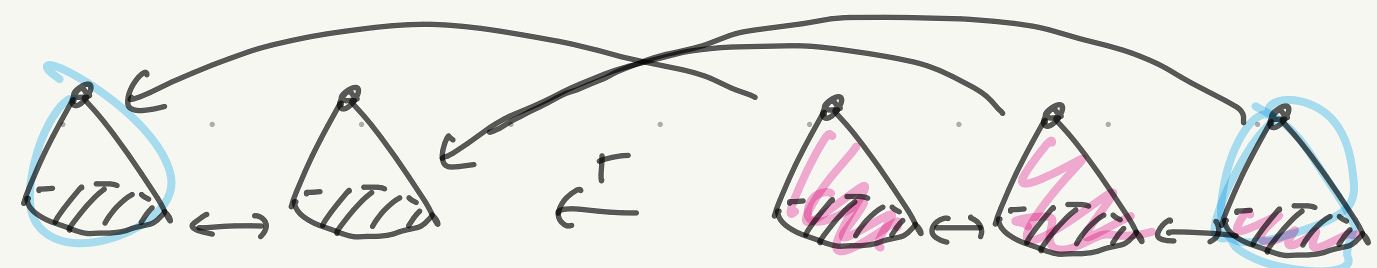
$\Rightarrow H_n(CX, \{0\} \times X) \cong H_{n-1}(X)$ by ∂

(b) write $S^{(k)}X$ for the gluing of $(k+1)$-copies of CX

so $S^{(1)}X \cong SX$, $S^{(k+1)}X = (S^{(k)}X \amalg CX)/$ identify base

$S^{(k)}X \overset{i}{\hookrightarrow} S^{(k+1)}X$ (into the image of first $(k+1)$-copies)

is a retract: send the "last" CX to another copy



$\Rightarrow$ $H_n(S^{(k)}X) \xrightarrow{i_*} H_n(S^{(k+1)}X)$ is embedding as a

direct summand $(H_n(S^{(k+1)}X) \xrightarrow{r_*} H_n(S^{(k)}X)$ proj.)

$\Rightarrow$ the long exact sequence splits,

$$H_n(S^{(k+1)}X) \cong H_n(S^{(k)}X) \oplus H_n(S^{(k+1)}X, S^{(k)}X)$$

$$S^{(k+1)}X / S^{(k)}X \cong CX / \{0\} \times X = SX$$

$$\Rightarrow H_n(S^{(k+1)}X, S^{(k)}X) \cong \tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X)$$

Thus we get

$$H_n(S^{(k)}X) \cong \begin{cases} \mathbb{Z} & n = 0 \quad (\text{and } X \neq \emptyset) \\ H_{n-1}(X)^{\oplus k} & n > 0 \end{cases}$$

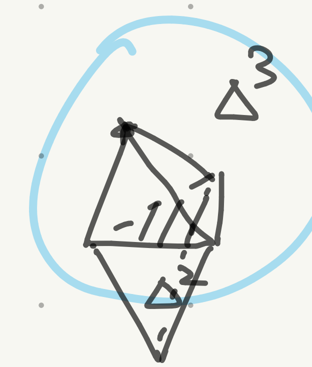
Problem 21

construct explicit chain map $s: C_\bullet(X) \to C_{\bullet+1}(SX)$

implementing $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$

Proof we write $SX = (CX \amalg CX)/$ identify base

take a singular n-simplex $\sigma: \Delta^n \to X$



$\tilde{\sigma}: \Delta^{n+1} \to CX$ compos. of $\Delta^{n+1} \xrightarrow[\text{homeo}]{\simeq} C\Delta^n$

and the induced map $C\Delta^n \to CX$
$[id_I \times \sigma]$

$\tilde{\sigma}^{(1)}, \tilde{\sigma}^{(2)}: \Delta^{n+1} \to SX$; $\tilde{\sigma}$ into the image of first and second copies of CX

Set $s(\sigma) = \tilde{\sigma}^{(1)} - \tilde{\sigma}^{(2)}$

checking $\partial s(\sigma) = s(\partial \sigma)$

$\partial \sigma$ = alternating sum of faces of σ

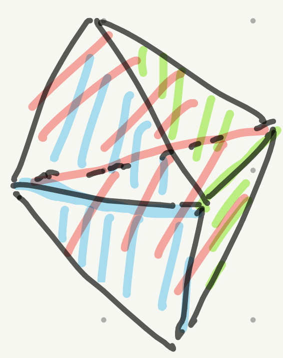
$$\sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$\Rightarrow s(\partial \sigma)$ = alt. sum of:

"cones of faces of σ" at first copy of CX

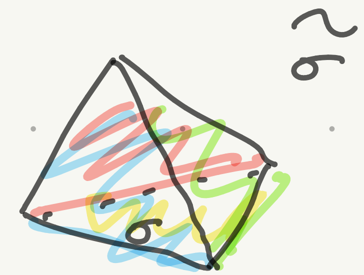
$-$ "cones of faces of σ" at second copy of CX

$$\sum_{i=0}^{n} (-1)^i \left(\widetilde{\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}}^{(1)} - \widetilde{\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}}^{(2)} \right)$$



$\partial(s(\sigma))$ = alt. sum of faces of $\tilde{\sigma}^{(1)} - \tilde{\sigma}^{(2)}$

faces of $\tilde{\sigma}$:



- cones of faces of σ
- "base" face given by σ

$\rightsquigarrow$ cancellation of base faces of $\tilde{\sigma}^{(1)}$ and $\tilde{\sigma}^{(2)}$

get same expression as $s(\partial\sigma)$

Section 2.2

Exercise 8 $f(z)$ complex polynomial

write $S^2 = \mathbb{C} \cup \{\infty\}$, $\hat{f} : S^2 \to S^2$, $z \mapsto f(z)$ $z \in \mathbb{C}$

$\infty \mapsto \infty$

(a) $\deg \hat{f} = \deg f$ (degree as polynomial)

(b) $f(z) = 0 \Rightarrow \deg f|_z$ = multiplicity of z as a root of f

Proof (a) typically for a fixed w; $f(z) = w$ has ($\deg f$)-many solutions in z

$\deg \hat{f} = \sum_{z \in \hat{f}^{-1}(w)} \deg \hat{f}|_z$ $\rightsquigarrow$ we need to show $\deg \hat{f}|_z = 1$

fix z as above. $f(z+h) \sim w + f'(z)h$ when

$|h|$ is small $\Rightarrow$ restriction of $\hat{f}$ to a small

neighborhood U is homotopic to $z+h \mapsto w + ah$

$(a = f'(z) \neq 0)$

changing a continuously to 1,

$\hat{f}_* : H_2(U, U \setminus \{z\}) \to H_2(U', U' \setminus \{w\})$

is the same as the map induced by translation

$\rightsquigarrow$ "id" on $H_2(U, U \setminus \{z\}) \cong \mathbb{Z}$, deg is 1

$H_2(U, U \setminus \{z\}) \overset{\text{excision}}{\cong} H_2(U, A) \cong \tilde{H}_2(U/A)$

$\cong \tilde{H}_2(S^2) \cong \mathbb{Z}$

(b) $f(z)=0$ take small neighborhood U around z

$f(z+h) \sim a\, h^m$ $(h \in U)$ m : multiplicity of z as a root of f

$a = \frac{f^{(m)}(z)}{m!} \neq 0$

again f is homotop. to $z+h \mapsto h^m$ on U

$$\begin{tikzcd}
H_2(U, U\setminus\{z\}) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_1(U\setminus\{z\}) \arrow[r, phantom, "\cong"] \arrow[d, "z+h \mapsto h^m"] & H_1(S^1) \arrow[d, "g_*"] \\
H_2(U', U'\setminus\{0\}) \arrow[r, "\partial"'] & H_1(U'\setminus\{0\}) \arrow[r, phantom, "\cong"] & H_1(S^1)
\end{tikzcd}$$

picks up $\deg f|_z$ →

$\left(\quad g(w) = w^m \quad \text{on} \quad S^1 \subset \mathbb{C} \quad \right)$

$\deg(g) = m$ so $f_* \leftrightarrow (\mathbb{Z} \to \mathbb{Z},\ k \mapsto mk)$

$\Rightarrow$ $\deg f|_z = m$