$K(G,1)$-spaces (first Eilenberg-MacLane spaces)

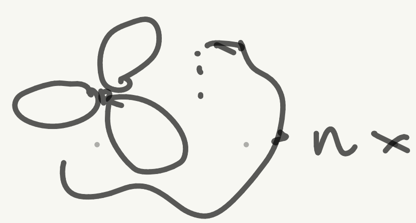
G: group (with discrete topology)

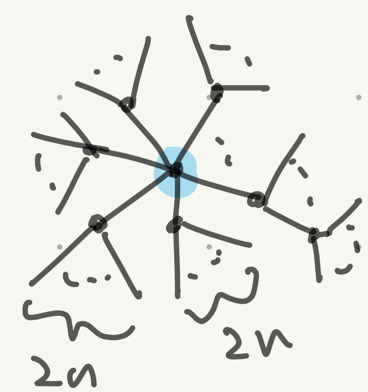
Def X is a <u>$K(G,1)$-space</u> if

- path-connected
- $\pi_1(X,*) \cong G$
- the universal covering $\tilde{X} \ (\xrightarrow{p} X)$ is contractible

Examples 1. S^1 is a $K(\mathbb{Z},1)$-space $\quad \tilde{S}^1 = \mathbb{R}$

2. $X = \underbrace{S^1 \vee \cdots \vee S^1}_{n \times}$ is a $K(\mathbb{F}_n, 1)$-space



$\tilde{X} =$ 

2n-regular tree

Prop 1 Y contractible, $G \curvearrowright Y$ free and properly discontinuous action

i.e., • $gy = y$ for some $y \Rightarrow g = e$

• any y has a neighborhood U s.t. $gU \in U \Rightarrow g = e$

then $X = G \backslash Y$ is a $K(G,1)$-space

Proof (outline)

• $Y \xrightarrow{p} X$ is a covering (Section 1.3 exercise 23 March 10)

$p^{-1}(U) \cong G \times U \to {}^{\exists}U \ni {}^{\forall}x$

• Y (locally) path connected $\Rightarrow X$ (locally) path-conn.

same with locally-simply-connectedness

$\forall x \in X$, $\underset{\text{open}}{V} \ni x$ $\exists U \supset \underset{\text{open}}{U} \ni x$ $\pi_1(U, x) = \{e\}$

$\Rightarrow Y \cong \tilde{X}$ by classification of coverings (March 2) (cont.) and $\pi_1(X, x) \cong G$ = deck transform. group of $Y = \tilde{X}$ ☒

Example $G = \mathbb{Z}_2$

$Y = S^\infty$ = union of S^n for inclusions $S^n \hookrightarrow S^{n+1}$

$$\begin{array}{ccc} S^n & \hookrightarrow & S^{n+1} \\ \cap & & \cap \\ \mathbb{R}^{n+1} & \hookrightarrow & \mathbb{R}^{n+2} \\ x & \mapsto & (x, 0) \end{array}$$

- S^∞ is contractible by

$x = (x_0, x_1, \dots) \rightsquigarrow$ normalization of $(1-t)x + t(0, x)$

$\rightsquigarrow (0, x) \rightsquigarrow (\sin t, (\cos t) x) \rightsquigarrow (1, 0, \dots)$

- $\mathbb{Z}_2 = \{e, g\}$ acts freely (and prop. discont.) on S^∞ by

$g x = -x$

$\Rightarrow \mathbb{R}P^\infty = \mathbb{Z}_2 \backslash S^\infty$ is a $K(\mathbb{Z}_2, 1)$-space

Example (1B.7) a canonical model of $G \curvearrowright Y$ contractible

EG : Δ-complex with

- n-simplices labelled by $[g_0, \dots, g_n]$ $\quad g_i \in G$
- i-th face of $[g_0, \dots, g_n] = [g_0, \dots, \hat{g_i}, \dots, g_n]$ $\quad (0 \le i \le n)$

($[g_0, \dots, g_n]$: formal convex combination of $g_0, \dots, g_n$)

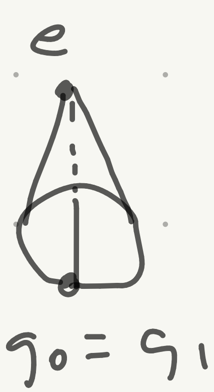
homotopy equivalence $EG \simeq \{e\}$ from

$$x = \sum_{i=0}^{n} s_i g_i \in [g_0, \dots, g_n] \rightsquigarrow te + \sum_i (1-t) s_i g_i$$

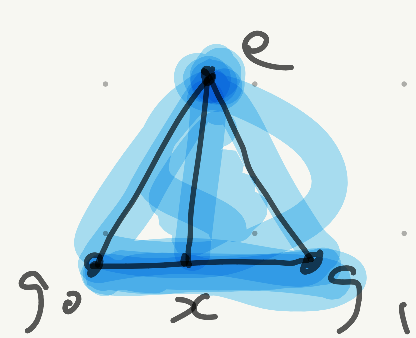
$0 \le s_i, \ \sum s_i = 1$ $\qquad$ in $[e, g_0, \dots, g_n]$

containing $[g_0, \dots, g_n]$ as the 0-th face

e as a vertex



$n = 1$:



$G \curvearrowright EG$: $g \in G$ sends $[g_0, \dots, g_n]$ to $[gg_0, \dots, gg_n]$

this is free and properly discontinuous

$\Rightarrow BG = G \backslash EG$ is a $K(G,1)$-space.

BG : Δ-complex with

- n-simplices : $[g_1 | \cdots | g_n]$ = img of $[e, g_1, g_1g_2, \dots, g_1 \cdots g_n]$
- boundaries of $[g_1 | \cdots | g_n]$

$[g_2 | \cdots | g_n]$, $[g_1g_2, g_3, \dots, g_n]$, $\dots$, $[g_1, \dots, g_{n-1}g_n]$,

$[g_1 | \cdots | g_{n-1}]$

0-th face of $[e, g_1, g_1g_2, \dots] = [g_1, g_1g_2, \dots, g_1 \cdots g_n]$

$= \cancel{g_1} [e, g_2, g_2g_3, \dots, g_2 \cdots g_n]$

Th'm 1B.8 Y, Y' cell complexes which are $K(G,1)$ spaces $\Rightarrow$ $Y \simeq Y'$

Key : Proposition 1B.9 X : conn. cell complex

Y : $K(G,1)$ space $\quad x_0 \in X$, $y_0 \in Y$
0-cell

- any hom $\pi_1(X, x_0) \xrightarrow{\phi} \pi_1(Y, y_0) \cong G$ is of the form $\phi = f_*$ for $f : X \to Y$, $f(x_0) = y_0$
- such f is unique up to homotopy

1B.9 $\Rightarrow$ 1B.8 : use existence of f to get

$Y \xrightarrow{f} Y'$, $Y' \xrightarrow{f'} Y$

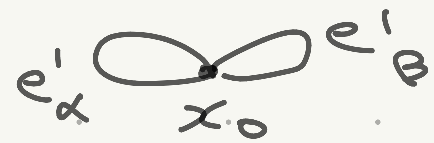
use uniqueness to compare ff' with $\mathrm{id}_{Y'}$

$f'f$ with id_Y

Proof of 1B.9 : first consider the case $X^0 = \{x_0\}$

then each 1-cell e^1_α has endpoint x_0

i.e. $X^1 \cong \bigvee_{\alpha \in I_1} S^1$



Existence of f

Step 1 defining $f : X^1 \to Y$

$[e^1_\alpha] \in \pi_1(X, x_0)$ "image" of e^1_α as $S^1 \xrightarrow[\text{at } e^1_\alpha]{} X^1 \to X$

for each $\alpha \in I_1$ choose $S^1 \xrightarrow{g_\alpha} Y$ representing

$\phi([e^1_\alpha]) \in \pi_1(Y, y_0)$, "$f|_{e^1_\alpha} = g_\alpha$" up to

parametrization $S^1 \to \overline{e^1_\alpha} \subset X^1$

Step 2 extending f to $X^2 = \left(X^1 \amalg \left(\coprod_{\beta \in I_2} D^2\right)\right) / \substack{\text{glue } S^1 = \partial D^2 \\ \text{at } \beta\text{-th comp.}}$

by

fix e^2_β (2-cell of X), $\varphi_\beta : S^1 \to X^1$ gluing map

to extend f on e^2_β we need to know that

$f \circ \varphi_\beta : S^1 \to Y$ is homotopic to the const y_0

φ_β gives $[\varphi_\beta] \in \pi_1(X^1, x_0)$ and incl. $X^1 \hookrightarrow X$

$[f \circ \varphi_\beta] \in \pi_1(Y, y_0)$ is equal to $\varphi(i_*([\varphi_\beta]))$

but $i_*([\varphi_\beta]) = e$ because φ_β bounds e^2_β

$\Rightarrow [f \circ \varphi_\beta] = e$ in $\pi_1(Y, y_0)$

Step n extending f from X^{n-1} to X^n $(n > 2)$

e^n_γ : n-cell, $\varphi_\gamma : S^{n-1} \to X^{n-1}$ gluing map

$\rightsquigarrow$ want $f \circ \varphi_\gamma : S^{n-1} \to Y$ to be homotopic to const y_0

use the lifting criterion (Prop 1.33 March 2) to get

$S^{n-1} \to \tilde{Y}$ lift of $f \circ \varphi_\gamma$ ($\pi_1(S^{n-1}, *) = \{e\}$ by $n > 2$)

$\Rightarrow$ $f \circ \varphi_\gamma \simeq$ const by the contractibility of $\tilde{Y}$

Uniqueness up to homotopy of $f : X \to Y$, $x_0 \mapsto y_0$ $(f_* = \phi)$

suppose f_0 and f_1 are such maps

- $f_0 \simeq f_1$ on X^1 : follows from $f_{0*} = f_{1*}$

(cont.) we get $Z^{(1)} = \underbrace{X^1 \times I}_{\substack{\text{use } h \text{ impl.} \\ f_0 \simeq f_1}} \cup \underbrace{X \times \{0,1\}}_{\text{use } f_0 \,\&\, f_1} \xrightarrow{F} Y$

inductively construct

$Z^{(n)} = X^n \times I \cup X \times \{0,1\}$: glue $e^n_\gamma \times (0,1)$ $(\gamma \in I_n)$ to $Z^{(n-1)}$

so $X \times I = \bigcup_n Z^{(n)}$,

we want $Z^{(n)} \xrightarrow{F} Y$ extending $Z^{(n-1)} \xrightarrow{F} Y$

to get $F : X \times I \to Y$ implementing $f_0 \simeq f_1$

gluing of $e^n_\gamma \times (0,1)$: from $\hat{f}_\gamma : S^n \to Z^{(n-1)}$

so we can use lifting criterion to get $S^n \to \tilde{Y}$

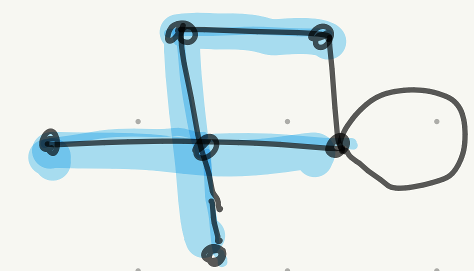
lifting $F \circ \hat{f}_\gamma$ then get $F \circ \hat{f}_\gamma \simeq$ const.

then extension of F to $e^n_\gamma \times (0,1)$

Reduction of general case to $X^0 = \{x_0\}$

choose a tree subgraph $T \subset X^1$ with $X^0 \subset T$
(possible by the connectedness of X)

X/T (collapse T to a point)
is a cell complex with single 0-cell



$X \to X/T$ is a homotopy equivalence

$\leadsto$ we can work with X/T $\square$

Consequence: we can talk about "the" $K(G,1)$ space (written as $K(G,1)$) as a well defined (-up-to-homotopy) cell complex

$K(G,n)$ for G commutative $(n>1)$

$$(\#)\left(\begin{array}{ll} \pi_n(K(G,n),*) \cong G & \\ \pi_m(K(G,n),*) \cong \{e\} & m \neq n \end{array}\right.$$

$$[X, K(G,n)] \cong \underline{H^n(X;G)}$$

$$X = S^m \Rightarrow (\#)$$

$K(G,1)$ as a classifying space

Def principal G-bundle over Z is given by

$Y \xrightarrow{p} Z$ surjective, local homeo. , $G \curvearrowright Y$

covering space free

fiberwise action s.t. $G \backslash Y \cong Z$ induced by p

Prop 2 Z : locally-path-connected

$$\{\text{principal } G\text{-bundles } Y \xrightarrow{p} Z\} \text{ up to isom}$$

$$\xleftrightarrow{1:1} [Z, BG] \quad \text{set of homotopy classes of maps}$$

$f : Z \to BG$ corresponds to $Y_f = f^* EG = \{(z, a) : f(z) = p(a)\}$

$$\begin{tikzcd} & & EG \arrow[d] \\ Y \arrow[r, "p"'] \arrow[rru, bend left] & Z \arrow[r] & BG \end{tikzcd}$$

for $EG \xrightarrow{p} BG$

Relation to group homology

Notation $H_n(G) = H_n(K(G,1)) = H_n(BG)$

n-th group homology of G

Recall the standard Δ-complex structure of BG:

- n-simplices : $[g_1|\cdots|g_n]$ = img of $[e, g_1, g_1g_2, \cdots, g_1\cdots g_n]$
- boundaries of $[g_1|\cdots|g_n]$

$[g_2|\cdots|g_n]$, $[g_1g_2, g_3, \cdots, g_n]$, $\cdots$, $[g_1, \cdots, g_{n-1}g_n]$,

$[g_1|\cdots|g_{n-1}]$

$\rightsquigarrow$ simplicial homology is given by

$$\Delta_n(BG) = \{ \sum a_{[g_1|\cdots|g_n]} \cdot [g_1|\cdots|g_n] : a_\bullet \in \mathbb{Z} \}$$

$$\partial : \Delta_n(BG) \to \Delta_{n-1}(BG)$$

$$\partial [g_1|\cdots|g_n] = [g_2|\cdots|g_n] - [g_1 g_2 | g_3 | \cdots] + [g_1 | g_2 g_3 | \cdots] - \cdots + (-1)^n [g_1|\cdots|g_{n-1}]$$

this is the <u>bar complex</u> of G

and we get $H_n(G) \cong \mathrm{Tor}_n^G(\mathbb{Z}, \mathbb{Z})$ etc.

Fact $H_n(\mathbb{Z}_m) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_m & n \text{ odd} \\ 0 & n \text{ even}, n > 0 \end{cases}$

in particular, there is no finite-dimensional realization of $K(\mathbb{Z}_m, 1)$

Mayer - Vietoris sequence

Thm X top. space, $A, B \subset X$ subspaces s.t. $\mathring{A} \cup \mathring{B} = X$ then we have an exact seq.

$$\cdots \to H_n^{sing}(A \cap B) \xrightarrow{\Phi} H_n^{sing}(A) \oplus H_n^{sing}(B) \xrightarrow{\Psi} H_n^{sing}(X)$$

$$\xrightarrow{\partial} H_{n-1}^{sing}(A \cap B) \to \cdots \quad \to H_0^{sing}(X) \to 0$$

where $\Phi(x) = (i_*(x), -i_*(x))$, $\Psi(x, y) = i_*(x) + i_*(y)$

for incl. maps $i : A \cap B \to A, B \to X$

Ex. $S^2 = D^2 \underset{S^1}{\cup} D^2$

$$\cdots \to \underbrace{H_2(D^2) \oplus H_2(D^2)}_{0} \to \underbrace{H_2(S^2)}_{\mathbb{Z}} \xrightarrow{\partial} \underbrace{H_1(S^1)}_{\mathbb{Z}} \to \underbrace{H_1(D^2) \oplus H_1(D^2)}_{0}$$

Key point (see April 7 note):

$\mathcal{U} = (U_i)_{i \in I}$ collection of subspaces w/ $\bigcup_{i \in I} \mathring{U}_i = X$

(we use $\mathcal{U} = \{A, B\}$)

$C_n^{\mathcal{U}}(X) < C_n(X)$: span of singular n-simplices

$\sigma : \Delta^n \to U_i$ for $i \in I$

$\rightsquigarrow$ subcomplex $(C_\bullet^{\mathcal{U}}(X), \partial) \overset{i}{\subset} (C_\bullet(X), \partial)$

Prop 2.21 this inclusion is a chain homotopy equiv.

i.e. $\exists\ p : C_\bullet(X) \to C_\bullet^{\mathcal{U}}(X)$ s.t. $p i \simeq \mathrm{id}_{C_\bullet^{\mathcal{U}}}$, $i p \simeq \mathrm{id}_{C_\bullet}$

in particular $H_n^{\mathrm{sing}}(X) \cong H_n^{\mathcal{U}}(X)$

n-th homology of $C_\bullet^{\mathcal{U}}(X)$

with $\mathcal{U} = \{A, B\}$, write $C_n(A+B) = C_n^{\mathcal{U}}(X)$

we have $C_n(A) \oplus C_n(B) \xrightarrow{\Phi} C_n(A+B)$

$(\alpha, \beta) \mapsto i_*(\alpha) + i_*(\beta)$

- surjective by definition

- kernel is equal to the span of

$((i_A)_*(\sigma), -(i_B)_*(\sigma))$ for $\Delta^n \xrightarrow{\sigma} A \cap B$, $i_A : A \cap B \to A$, $i_B : A \cap B \to B$

i.e. $0 \to C_\bullet(A \cap B) \xrightarrow{\Psi} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{\Phi} C_\bullet(A+B) \to 0$

is exact

$\rightsquigarrow$ exact seq.

$$\cdots \to H_n^{sing}(A \cap B) \to H_n^{sing}(A) \oplus H_n^{sing}(B) \to H_n(A+B) \xrightarrow{\partial} H_{n-1}^{sing}(A \cap B) \to \cdots$$

$$H_n(A+B) \cong H_n^{sing}(X)$$