Homology with coefficients

$H_\bullet(X) = H_\bullet^{sing}(X)$ was defined by

$$C_n(X) = \{ \underbrace{\sum n_\sigma \cdot \sigma}_{\text{finite sum}} \ : \ \sigma : \Delta^n \to X \ , \ n_\sigma \in \mathbb{Z} \}$$

and $\partial : C_n(X) \to C_{n-1}(X)$ alt. sum of faces

$\rightsquigarrow$ we can choose a different system of coefficients as long as the intermediate formulas make sense

obs. we only use sums & differences

$\Rightarrow$ any commutative group will do

G : commutative group $\quad \mathbb{Z}_{p^k}, \ \mathbb{Q}, \ \mathbb{T} = S^1, \ldots$

X : topological space, $\quad A \subset X$: subspace

Def $\quad C_n(X;G) = \{ \underbrace{\sum a_\sigma \cdot \sigma}_{\text{finite sum}} : \sigma : \Delta^n \to X, \ a_\sigma \in G \}$

$\partial_n : C_n(X;G) \to C_{n-1}(X;G)$

$\sigma \mapsto \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$ $\quad$ (same formula)

$H_n^{(\text{sing})}(X;G) = \ker \partial_n / \operatorname{img} \partial_{n+1}$

<u>singular</u> <u>homology</u> <u>of</u> X <u>with</u> <u>coefficients</u> <u>in</u> G

$C_n(X, A; G) = C_n(X;G) / C_n(A;G)$

$H_n(X, A; G) =$ n-th homology of $(C_\bullet(X, A; G), \partial)$

relative singular homology with coeff. in G

We have similar properties:

- $f: X \to Y$ induces $f_*: H_n(X;G) \to H_n(Y;G)$

  induced by $\sigma \longmapsto f \circ \sigma$

  f_* only depends on the homotopy type of f

  $f_0 \simeq f_1 \Rightarrow$ induced maps on $C_\bullet(X)$ are

  <u>chain homotopic</u> $\exists\, h: C_n(X) \to C_{n+1}(Y)$

  $(f_0)_\# - (f_1)_\# = \partial h + h \partial$

  $\Rightarrow$ same after $\sim \otimes G$

- $H_n(S^n;G) \cong G$ for $n = 1, 2, \ldots$ (allow $n=0$ with $\tilde{H}_0$)
- $A \subset X \rightsquigarrow$ long exact seq.

$$\cdots \to H_n(A;G) \to H_n(X;G) \to H_n(X,A;G) \to H_{n-1}(A;G) \to$$

$$\cdots \to H_0(A;G) \to H_0(X;G) \to H_0(X,A;G) \to 0$$

We can define analogues

- X Δ-complex $\Rightarrow$ $H_\bullet^{\Delta}(X;G)$ from $(\Delta_\bullet(X)\otimes G, \partial)$, where $\Delta_\bullet(X)\otimes G = \{\sum a_\sigma\cdot\sigma : \sigma : \text{simplex},\ a_\sigma \in G\}$

- X cell complex $\Rightarrow$ $H_\bullet^{CW}(X;G)$ from

$$\Big(\underbrace{H_n(X^n, X^{n-1};G)}_{\cong\ \bigoplus_{n\text{-cells}} G}, \text{ boundary map}\Big)$$

these are isomorphic to $H_\bullet^{\text{sing}}(X;G)$

- $(\Delta_\bullet(X), \partial) \underset{\text{chain homotopy}}{\simeq} (C_\bullet(X), \partial)$ $\Rightarrow$ same for $-\otimes G$

$\Rightarrow$ $H_\bullet^{\Delta}(X;G) \cong H_\bullet^{\text{sing}}(X;G)$

- $H_\bullet^{CW}(X;G) \cong H_\bullet^{\text{sing}}(X;G)$: redo the proof of Thm 2.35 (April 20)

"Universal coefficient theorem"

if $x = \sum n_\sigma \cdot \sigma \in C_n(X)$ is in $\ker \partial_n$ and $a \in G$

then "$x \otimes a$" $= \sum \underbrace{(n_\sigma a)}_{\text{in } G} \cdot \sigma \in C_n(X;G)$ is also in $\ker \partial_n$

formally $C_\bullet(X;G) \cong C_\bullet(X) \otimes G \qquad \partial \leftrightarrow \partial \otimes \mathrm{id}_G$

similarly $x \in C_n(X)$ is in $\operatorname{img} \partial_{n+1} \Rightarrow x \otimes a \in \operatorname{img} \partial_{n+1}$

so $H_n(X) \otimes G \to H_n(X;G)$, $[x] \otimes a \mapsto [x \otimes a]$

is well defined

(§3.A) $\quad H_n(X;G) \cong H_n(X) \otimes G \oplus \operatorname{Tor}_1(H_{n-1}(X), G)$

so (in principle) $H_n(X)$ determines $H_n(X;G)$

(cont.) this follows from the general fact

$$H_n(C_\bullet \otimes G) \cong H_n(C_\bullet) \otimes G \oplus \mathrm{Tor}_1(H_{n-1}(C_\bullet), G)$$

for any chain complex $C_\bullet = ((C_n)_n, \partial : C_n \to C_{n-1})$

Examples

1. $G = \mathbb{Q}$ then $\mathrm{Tor}_1(M, \mathbb{Q}) = 0$ for any M

so $H_n(X;\mathbb{Q}) \cong H_n(X) \otimes \mathbb{Q}$

2. $G = \mathbb{Z}_m$ then $\mathrm{Tor}_1(M, \mathbb{Z}_m) = \ker(M \to M, x \mapsto mx)$

$$H_n(X;\mathbb{Z}_m) \cong H_n(X)/m\,H_n(X) \oplus (\text{above for } M = H_{n-1}(X))$$

$$H_n(X)/m\,H_n(X) \cong H_n(X) \otimes \mathbb{Z}_m$$

Application : Borsuk-Ulam theorem

Goal (Cor. 2B.7) for any cont. map. $g: S^n \to \mathbb{R}^n$

$\exists\, x \in S^n$ s.t. $g(x) = g(-x)$

Key step (Prop. 2B.6) suppose $f: S^n \to S^n$ satisfies

$f(-x) = -f(x)$ (odd map) Then deg f is odd

2B.6 $\Rightarrow$ 2B.7 given g, put $f(x) = g(x) - g(-x)$

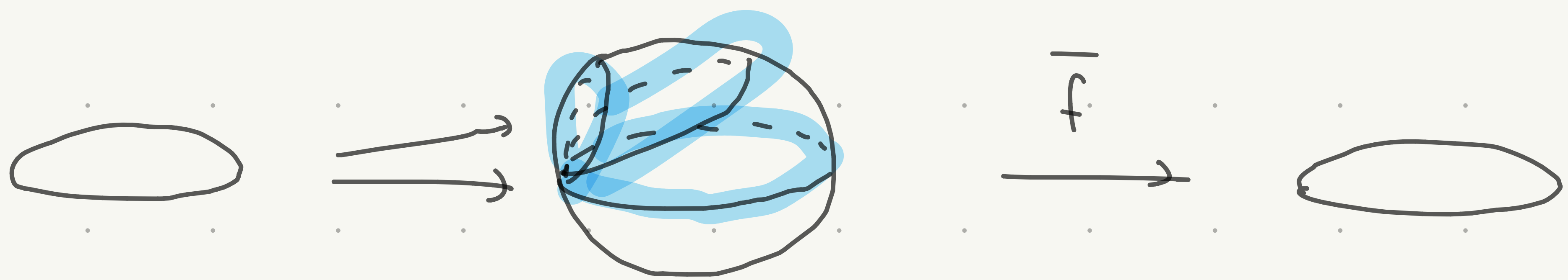
then $f: S^n \to \mathbb{R}^n$, $f(-x) = -f(x)$

we want to find $x \in S^n$ s.t. $f(x) = 0$

if there is no such x, $\bar{f}(x) = \dfrac{f(x)}{|f(x)|}$ is $S^n \xrightarrow[\text{odd}]{} S^{n-1}$

so $\bar{f}|_{S^{n-1}}$ must have odd deg. by 2B.6

(cont.) but $\bar{f}|_{S^{n-1}}$ is homotopic to a const. map



so $\deg \bar{f}|_{S^{n-1}} = 0$ contradiction

Proof of 2B.6

Strategy: show that $f_* : H_n(S^n; \mathbb{Z}_2) \to H_n(S^n; \mathbb{Z}_2)$ is the identity map (or nonzero)

$H_n(S^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and f_* acts by $\deg f$

$\Rightarrow$ $\deg f$ must be odd

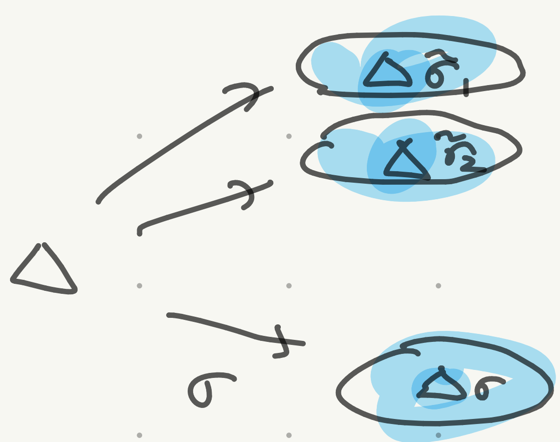
Main tool : "transfer sequence" for double covers

suppose $Y \xrightarrow{p} X$ is a covering space s.t.

$\forall\, x \in X \quad p^{-1}(x)$ has two elements

Lem 1 any $\sigma : \Delta^n \to X$ lifts to Y in exactly two ways $\tilde{\sigma}_1, \tilde{\sigma}_2$

Proof use lifting criterion and $\Delta^n \simeq pt$.



$p^{-1}(U)$

U

(or use $\Delta^n \cong I^n$ and homotopy lifting property n times)

$\square$

$\rightsquigarrow$ we get $\tau : C_n(X; \mathbb{Z}_2) \to C_n(Y; \mathbb{Z}_2)$, $\sigma \mapsto \tilde{\sigma}_1 + \tilde{\sigma}_2$

well defined

Lem 2 $\quad \operatorname{img} \tau = \ker p_\#$ $\quad \big(p_\# : C_n(Y;\mathbb{Z}_2) \to C_n(X;\mathbb{Z}_2)$ induced by p $\big)$

$\sigma \longmapsto p \circ \sigma$

We have $\tilde{\sigma}_1 + \tilde{\sigma}_2 \overset{p_\#}{\longmapsto} \sigma + \sigma = 2\sigma = 0$ in $C_n(X;\mathbb{Z}_2)$

this is the "only cancellation" after $p_\#$ ☐

$\rightsquigarrow$ short ex. seq. of chain complexes

$$0 \to C_\bullet(X;\mathbb{Z}_2) \xrightarrow{\tau} C_\bullet(Y;\mathbb{Z}_2) \xrightarrow{p_\#} C_\bullet(X;\mathbb{Z}_2) \to 0$$

$\rightsquigarrow$ long ex. seq. of homology

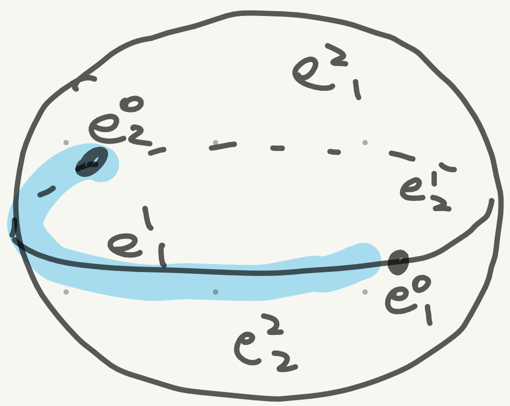
$$\cdots \to H_n(X;\mathbb{Z}_2) \xrightarrow{\tau} H_n(Y;\mathbb{Z}_2) \xrightarrow{p_*} H_n(X;\mathbb{Z}_2) \xrightarrow{\partial} H_{n-1}(X;\mathbb{Z}_2) \to \cdots$$

we will use this for $Y = S^n$, $X = \mathbb{R}P^n = \mathbb{Z}_2 \backslash S^n$

Lem 3 $\quad H_k(\mathbb{RP}^n; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & 0 \le k \le n \\ 0 & k > n \end{cases}$

Proof: $X = \mathbb{RP}^n$ has a cell complex structure with: one cell for each dim $0, 1, \dots, n$; no cells beyond this

(S^n has cell complex str. with two cells for these dims, $\mathbb{Z}_2$ permutes cells of same dim)



gluing map $S^{k-1} \to X^{k-1} = \mathbb{RP}^{k-1}$ is the natural projection (cont.) cellular homology complex is given by

$$\cdots \to H^k(X^k, X^{k-1}) \xrightarrow{\partial} H^{k-1}(X^{k-1}, X^{k-2}) \to \cdots$$

$$\begin{array}{ccccccc} \cong & & \cong & & \cong & & \cong \\ \mathbb{Z} & \xrightarrow[0]{} & \mathbb{Z} & \xrightarrow[2]{\hspace{3em}} & \mathbb{Z} & \xrightarrow[0]{\hspace{3em}} & \mathbb{Z} \end{array}$$

(Example 2.42) $\Rightarrow H_k(\mathbb{R}P^n) \cong \mathbb{Z}$ or $\mathbb{Z}_2$ or 0

with $\mathbb{Z}_2$ coefficient, boundary maps are the "same"

$$\cdots \to H^k(X^k, X^{k-1}; \mathbb{Z}_2) \xrightarrow{\partial} H^{k-1}(X^{k-1}, X^{k-2}; \mathbb{Z}_2) \to \cdots$$

$$\begin{array}{ccccccc} \cong & & \cong & & \cong & & \cong \\ \mathbb{Z}_2 & \xrightarrow[0]{} & \mathbb{Z}_2 & \xrightarrow[2=0]{\hspace{3em}} & \mathbb{Z}_2 & \xrightarrow[0]{\hspace{3em}} & \mathbb{Z}_2 \end{array}$$

$$\Rightarrow H_k(\mathbb{R}P^n; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

$\boxtimes$

Now transfer seq. for $Y = S^n \xrightarrow{p} X = \mathbb{R}P^n$ becomes

$$\underset{H_{n+1}(X;\mathbb{Z}_2)}{0} \to \underset{H_n(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\tau} \underset{H_n(Y;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{p_*} \underset{H_n(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\partial} \underset{H_{n-1}(X;\mathbb{Z}_2)}{\mathbb{Z}_2}$$

$$\xrightarrow{\tau} \underset{H_{n-1}(Y;\mathbb{Z}_2)}{0} \xrightarrow{p_*} \underset{H_{n-1}(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\partial} \underset{H_{n-2}(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\tau} \underset{H_{n-2}(Y;\mathbb{Z}_2)}{0} \xrightarrow{p_*} \cdots$$

$$\cdots \xrightarrow{\partial} \underset{H_0(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\tau} \underset{H_0(Y;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{p_*} \underset{H_0(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \to 0$$

Claim 1 $H_n(X;\mathbb{Z}_2) \xrightarrow{\partial} H_{n-1}(X;\mathbb{Z}_2)$ is surjective

Claim 2 $H_n(Y;\mathbb{Z}_2) \xrightarrow{p_*} H_n(X;\mathbb{Z}_2)$ is trivial

$$\Rightarrow \begin{cases} p_* : H_k(Y;\mathbb{Z}_2) \to H_k(X;\mathbb{Z}_2) \text{ is triv. except for } k=0 \\ \tau : H_k(X;\mathbb{Z}_2) \to H_k(Y;\mathbb{Z}_2) \text{ is triv. except for } k=n \end{cases}$$

Recap: $X = \mathbb{RP}^n$, $Y = S^n$

$$\underset{H_{n+1}(X;\mathbb{Z}_2)}{0} \to \underset{H_n(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\tau} \underset{H_n(Y;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{p_* = 0} \underset{H_n(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\partial} \underset{H_{n-1}(X;\mathbb{Z}_2)}{\mathbb{Z}_2}$$

$$\xrightarrow{\tau} \underset{H_{n-1}(Y;\mathbb{Z}_2)}{0} \xrightarrow{p_*} \underset{H_{n-1}(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\partial} \underset{H_{n-2}(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\tau} \underset{H_{n-2}(Y;\mathbb{Z}_2)}{0} \xrightarrow{p_*} \cdots$$

$$\cdots \xrightarrow{\partial} \underset{H_0(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{\tau = 0} \underset{H_0(Y;\mathbb{Z}_2)}{\mathbb{Z}_2} \xrightarrow{p_*} \underset{H_0(X;\mathbb{Z}_2)}{\mathbb{Z}_2} \to 0$$

Suppose $f: S^n \to S^n$ is odd; then it induces

$$\bar{f}: \mathbb{RP}^n \to \mathbb{RP}^n, \quad [x] \mapsto [f(x)] \qquad ([x] = [-x] \mapsto [-f(x)] = [f(x)])$$

f and $\bar{f}$ induce a self map of above long exact seq.

Claim 3 $\bar{f}_* : H_k(\mathbb{R}P^n; \mathbb{Z}_2) \to H_k(\mathbb{R}P^n; \mathbb{Z}_2)$ is identity

Proof we have

$$\begin{tikzcd}
0 = H_k(S^n;\mathbb{Z}_2) \arrow[r, "p_*"] & H_k(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[r, "\partial"] \arrow[d, "\bar{f}_*"] & H_{k-1}(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[r, "\tau"] \arrow[d, "\bar{f}_*"] & 0 \\
0 = H_k(S^n;\mathbb{Z}_2) \arrow[r, "p_*"] & H_k(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[r, "\partial"] & H_{k-1}(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[r, "\tau"] & 0
\end{tikzcd}$$

$\Rightarrow$ $\bar{f}_*$ is invertible on $H_k(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ ⊠

Claim 4 (what we wanted) $f_* : H_n(S^n; \mathbb{Z}_2) \to H_n(S^n; \mathbb{Z}_2)$ is identity

Proof

$$\begin{tikzcd}
H_n(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[r, "\tau : \text{iso.}"] \arrow[d, "\bar{f}_* : \text{id}"'] & H_n(S^n;\mathbb{Z}_2) \arrow[d, "f_*"] \\
H_n(\mathbb{R}P^n;\mathbb{Z}_2) \arrow[r, "\tau"'] & H_n(S^n;\mathbb{Z}_2)
\end{tikzcd}$$

commutes