# Exercise set 7 (§2.2)

## Problem 12

(1) consider $S^1 \wedge S^1 = (S^1 \times S^1) / \underbrace{S^1 \vee S^1}_{(S^1 \times \{*\}) \cup (\{*\} \times S^1)} \cong S^2$

the associated quotient map $S^1 \times S^1 \xrightarrow{p} S^2$ is not homotopic to a constant map

(look at $p_* : H_2(S^1 \times S^1) \to H_2(S^2)$ )

(2) any map $S^2 \to S^1 \times S^1$ is homotopic to a constant map (consider covering spaces)

(1) we have $H_2(S^1 \times S^1) \cong \mathbb{Z} \cong H_2(S^2)$

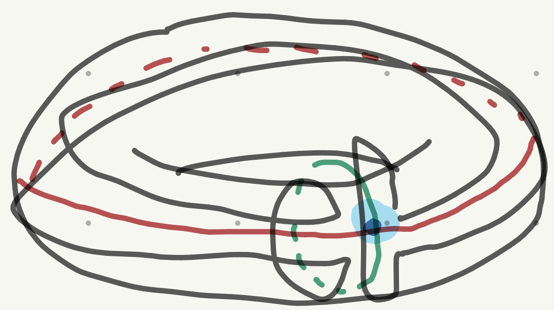
we want to show that $p_*$ corresp. to $\mathrm{id}_{\mathbb{Z}}$

$\rightsquigarrow$ look at cellular homology

consider the cell complex structures

$X = S^1 \times S^1$ ; $X^0 = \{*\}$, $X^1 = S^1 \vee S^1$, $X^2 = X$

one 0-cell, two 1-cells, one 2-cell



$Y = S^2$ ; $Y^0 = Y^1 = \{*\}$, $Y^2 = S^2$

one 0-cell (img of $X^0$ = img $X^1$), no 1-cell

one 2-cell

then we have $p(X^k) \subset Y^k$

↑ actually equality

(cont.) by naturality of relative homology

$$p_* : H_k(X^k, X^{k-1}) \to H_k(Y^k, Y^{k-1})$$

compatible with boundary maps:

$$\begin{tikzcd}
H_k(X^k, X^{k-1}) \arrow[rr] \arrow[dr] & & H_{k-1}(X^{k-1}, X^{k-2}) \\
& H_{k-1}(X^{k-1}) \arrow[ur] &
\end{tikzcd}$$

from long exact seq. for $(X^k, X^{k-1})$

from long exact seq. for $(X^{k-1}, X^{k-2})$

(and same for $Y$; will work this out in Prob. 17)

we know this boundary map is triv. (see Apr. 12)

$$H_2(X^2, X^1) \cong H_2(X^2/X^1) \cong H_2(Y^2) \cong H_2(Y^2, Y^1) \cong \mathbb{Z}$$

and this identification is given by $p$

(cont.) so $p_*$ induces isom

$$H_2^{CW}(X) \cong H_2(X^2, X^1) \to H_2^{CW}(Y) \cong H_2(Y^2, Y^1)$$

$$\Rightarrow p_* : H_2^{sing}(S^1 \times S^1) \longrightarrow H_2^{sing}(S^2) \text{ is also isom.}$$

If $p$ is homotopic to a const. map we would have

$p_* = 0$ on $H_k^{sing}(S^1 \times S^1)$ $(k > 0)$

(2) $S^1 \times S^1 \cong \mathbb{R}^2 / \mathbb{Z}^2$ for $\mathbb{Z}^2 \underset{\text{translation}}{\curvearrowright} \mathbb{R}^2$

this is a free & properly discontinuous action

$\Rightarrow \mathbb{R}^2 \to S^1 \times S^1$ is a covering space

$\mathbb{R}^2$ is contractible (hence $\pi_1(\mathbb{R}^2, *) = \{e\}$)

$\Rightarrow \mathbb{R}^2$ is the universal covering $\widetilde{S^1 \times S^1}$

lifting criterion says:

$f : X \to S^1 \times S^1$, $X$ path-conn., locally path-conn.

$x_0 \in X$, $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\widetilde{S^1 \times S^1}, \tilde{x}_0)) = \text{trivial}$

$\Rightarrow \exists$ lift $\tilde{f} : X \to \widetilde{S^1 \times S^1}$

now we have $\pi_1(S^2, *) = \{e\}$, so this applies

for any $f : S^2 \to S^1 \times S^1$

(cont.) take a lift $\tilde{f} : S^2 \to \mathbb{R}^2$

then $\tilde{f}$ is homotopic to a const. map

$\rightsquigarrow$ compose with $\mathbb{R}^2 \xrightarrow{p} S^1 \times S^1$ and get homotopy

between $f = p \circ \tilde{f}$ and a const. map. $\boxed{5}$

( same with $M_g$ instead of $S^1 \times S^1$ :

$g \geq 1$

$\tilde{M}_g \cong \mathbb{H}$ hyperbolic plane

Problem 17

$X, Y$ : cell complexes

$f : X \to Y$ cont. map s.t. $\forall k$ $f(X^k) \subset Y^k$

( cellular map )

$\rightsquigarrow$ we get a map of chain complexes between

$$\underbrace{\left(H_k(X^k, X^{k-1})\right)_{k=0}^{\infty}}_{\text{computes } H_\bullet^{CW}(X)} \quad \text{and} \quad \underbrace{\left(H_k(Y^k, Y^{k-1})\right)_{k=0}^{\infty}}_{\text{computes } H_\bullet^{CW}(Y)}$$

and the induced map $H_\bullet^{CW}(X) \to H_\bullet^{CW}(Y)$

agrees with $f_*$ up to the isoms. $H_k^{sing}(X) \cong H_k^{CW}(X)$

etc.

Construction of chain map

from $f(X^k) \subset Y^k$ $(k=0,1,\cdots)$ we get a map of

long exact seqs.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X^{k-1}) \arrow[r] \arrow[d] & H_n(X^k) \arrow[r] \arrow[d] & H_n(X^k, X^{k-1}) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(X^{k-1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n(Y^{k-1}) \arrow[r] & H_n(Y^k) \arrow[r] & H_n(Y^k, Y^{k-1}) \arrow[r, "\partial"] & H_{n-1}(Y^{k-1}) \arrow[r] & \cdots
\end{tikzcd}$$

checking consistency with cellular boundary maps →

$$\begin{tikzcd}
H_k(X^k, X^{k-1}) \arrow[rr, "d_k"] \arrow[dr, "\partial"] \arrow[ddd] & & H_{k-1}(X^{k-1}, X^{k-2}) \arrow[ddd] \\
& H_{k-1}(X^{k-1}) \arrow[ur] \arrow[ddd] & \\
& & \\
H_k(Y^k, Y^{k-1}) \arrow[rr, "d_k"] \arrow[dr, "\partial"] & & H_{k-1}(Y^{k-1}, Y^{k-2}) \\
& H_{k-1}(Y^{k-1}) \arrow[ur] &
\end{tikzcd}$$

squares with → commute

Comparison with $f_* : H_n(X) \to H_n(Y)$

recall (April 20)

$$H_n(X^{n+1}) \cong \underbrace{\underbrace{H_n(X^n)}_{\cong\ \ker \partial_n} / \underbrace{\partial_{n+1}(H_{n+1}(X^{n+1}, X^n))}_{\operatorname{img} \partial_{n+1}}}_{\cong\ H_n^{CW}(X)}$$

$H_n(X^{n+1}) \cong H_n(X)$

$f_*$ gives maps

$$\boxed{\begin{array}{c} H_n(X^{n+1}) \\ \downarrow \\ H_n(Y^{n+1}) \end{array}} \qquad \boxed{\begin{array}{ccc} H_n(X^n) & / & \partial_{n+1}(H_{n+1}(X^{n+1}, X^n)) \\ \downarrow & & \downarrow \\ H_n(Y^n) & / & \partial_{n+1}(H_{n+1}(Y^{n+1}, Y^n)) \end{array}}$$

same as

$f_* : H_n(X) \to H_n(Y)$

inducing the map $H_n^{CW}(X) \to H_n^{CW}(Y)$

$\boxtimes$

Problem 22 $X$ finite cell complex (finite number of cells)

$X' \xrightarrow{p} X$ n-sheeted covering space

then $\chi(X') = n\,\chi(X)$ for the Euler characteristic

Ex. $G$ finite group $\curvearrowright X'$ freely $\rightsquigarrow X = G \backslash X'$

$X' \to X$ : $|G|$-sheeted covering $\Rightarrow |G| \mid \chi(X')$

Proof recall

$$\chi(X) = \sum_{k=0}^{N} (-1)^k \operatorname{rk}(H_k(X)) \quad (N \geq \dim \text{ of } X)$$

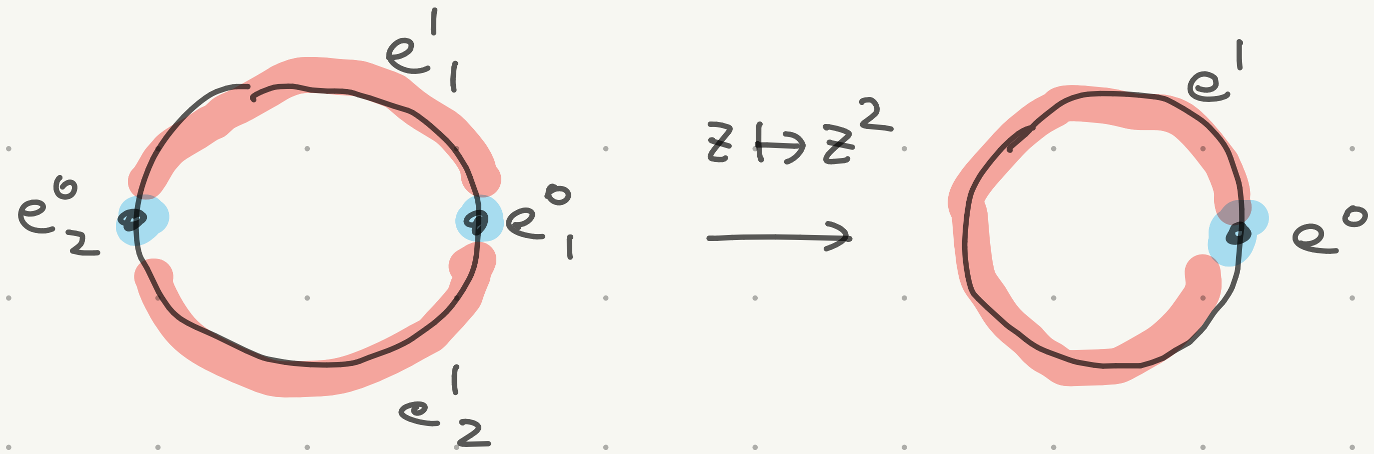
is equal to $\displaystyle\sum_{k=0}^{\infty} (-1)^k \underbrace{\operatorname{rk}(H_k(X^k, X^{k-1}))}_{\text{number of } k\text{-cells}}$ (Apr. 20)

define a cell complex structure on $X'$ by

$k$-cell of $X'$ : conn. comp. of $p^{-1}(e^k_\alpha)$

for some $k$-cell $e^k_\alpha \subset X$

Ex.




$p$ is $n$-to-1 local homeomorphism

$\Rightarrow$ inv. img. of $e^k_\alpha \subset X$ has exactly $n$-components

each homeo. to $D^k$

use lifting criterion to the characteristic map

$\Phi_\alpha : D^k \to X$ and choice of $\tilde{x}_0 \in p^{-1}(\Phi_\alpha(*))$

$\quad\ \ \rotatebox{90}{$\in$}$

$\quad\ \ *$

$\Phi_{\alpha *}(\pi_1(D^k, *)) = \{e\} \Rightarrow \exists !$ lift containing $\tilde{x}_0$

$\tilde{\Phi}_\alpha : D^k \to X'$ char. map

of $k$-cell in $X'$

(cont.) so this const. of cell complex structure on $X'$

is well defined,

$\#(k\text{-cells in } X') = n \times \#(k\text{-cells in } X)$

$\Rightarrow$ we get

$$\chi(X') = \sum_{k=0}^{N} (-1)^k \#(k\text{-cells in } X') = n\,\chi(X)$$

$N \geq \dim X = \dim X'$

Rem. we do not expect $\operatorname{rk} H_k(X') = n \operatorname{rk} H_k(X)$