Categorical summary of homology (§2.3, and else)

Observation: many constructions in this course can be regarded as <u>functors</u>:

$\pi_1(X, x_0)$, $H_n^{sing}(X)$ .. admit induced maps

$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$, $H_n^{sing}(X) \to H_n^{sing}(Y)$

for continuous maps $f: X \to Y \xrightarrow{g} Z$, $(gf)_* = g_* f_*$ ..

⇝ Questions

- what are other interesting functors?

  (and which category should we consider?

  cell complexes, general topological spaces, ...)

- what extra property does homology have?

Typical domain categories in algebraic topology

to capture "absolute" invariants $\pi_1(X, x)$, $H_n(X)$, ...

- objects : "spaces" either topological spaces, or cell complexes, Δ-complexes, ...
  or pointed spaces (X, x) ($x \in X$ "basepoint")
  with X any of above

- morphisms : "maps" $X \xrightarrow{f} Y$
  - if objs are general top. spaces
    take arbitrary continuous maps, or homotopy classes of cont. maps.
  - for pointed spaces : $(X, x) \xrightarrow{f} (Y, y)$
    should satisfy $f(x) = y$

- if objs are cell complexes

  take arbitrary continuous maps, or

  cellular maps $f(X^k) \subset Y^k$ $\forall k$, or homotopy types, ..
  (as cont. maps)

  $\rightsquigarrow$ how restrictive is this ?

Th'm (4.8; cellular approximation th'm)

  X, Y : cell complexes

  $f : X \to Y$ continuous map

  then $\exists\, g : X \to Y$ s.t. $f \simeq g$ and $g(X^k) \subset Y^k$
  for all k

So up to homotopy any cont. map can be replaced

  by a cellular map

to capture "relative" invariants $H_n(X,A)$, ...

- objects : pairs (X,A) of spaces, either

  X general top. sp.,

  $A \subset X$ arbitrary subset with induced topology

  (good enough to define $H_n^{sing}(X,A)$)

  or X cell complex, $A \subset X$ subcomplex

  (allows computation using excision, ...)

- morphisms : map of pairs : $(X,A) \xrightarrow{f} (Y,B)$

  map $X \to Y$ s.t. $f(A) \subset B$

  ↖ continuous, cellular, .. depending on context

$\rightsquigarrow$ map of pointed spaces $\equiv$ map of pairs

$(X,x) \to (Y,y)$ $\qquad$ $(X,\{x\}) \to (Y,\{y\})$

(cont.) i.e. category of pointed spaces embeds

into category of pairs by $(X, x) \mapsto (X, \{x\})$

$(X, A) \mapsto (X/A, \underbrace{[A]}_{\text{img of } A \text{ in } X/A})$ can be regarded as one-sided inverse

map of spaces $X \to Y$ can be also interpreted as

map of pairs $(X, \phi) \to (Y, \phi)$ or

map of pointed spaces $(X_+, *) \xrightarrow{f} (Y_+, *)$

with $X_+ = X \amalg \{*\}$ s.t. $f(X) \subset Y$

so it's often more sensible to arrange $X/\phi \cong X_+$

e.g. $X/A = (X \amalg \{*\})/\sim \ : \ \forall a \in A \quad a \sim *$

Other functors

Def (X, x_0) pointed topological space

the <u>n-th homotopy group</u> of (X, x_0) is

$$\pi_n(X, x) = [(S^n, *), (X, x_0)]$$

$$= \{\text{homotopy classes of maps } (S^n, *) \xrightarrow{f} (X, x_0)\}$$

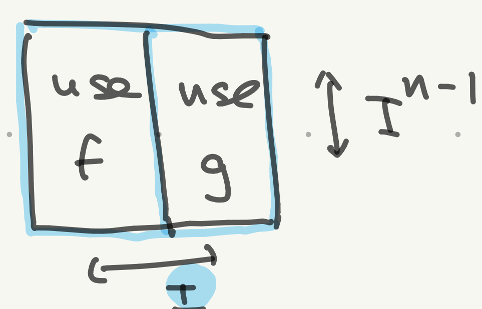
Rec. $f_0 \simeq f_1 \iff \exists\, H : S^n \times I \to X$ $\quad H(x, i) = f_i(x) \quad i = 0, 1$

$f_t : (S^n, *) \to (X, x_0)$ $\quad H(*, t) = x_0$

Prop. $\pi_n(X, x_0)$ is a commutative group for $n \geq 2$

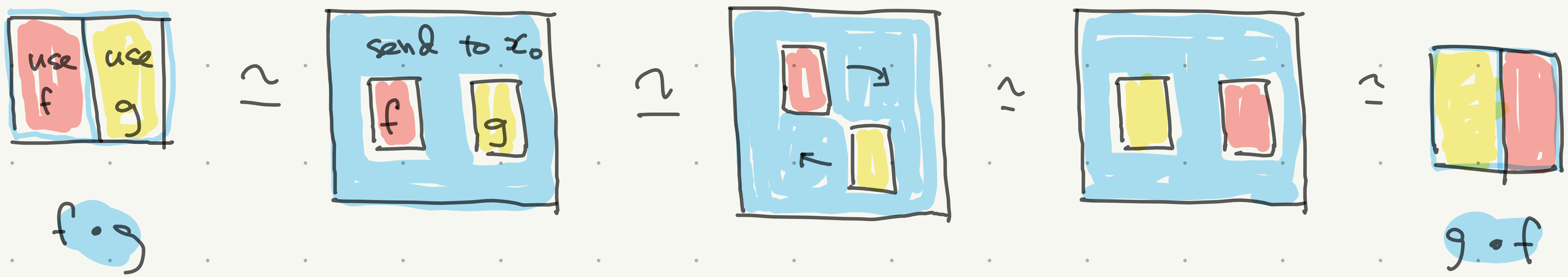
Idea: product in π_n : think $S^n \cong I^n / \partial I^n$ and

$[f][g] = [f \cdot g]$; $f \cdot g$:



$\to (X, \{x_0\})$

commutativity of this product :



So π_n is a functor

$$(\text{pointed spaces}) \to (\text{commutative groups})$$

- for maps $(X, x_0) \xrightarrow{f} (Y, y_0)$; $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$

$$[g] \mapsto [f \circ g]$$

and $f_0 \simeq f_1 \Rightarrow (f_0)_* = (f_1)_*$

Def. $X \xrightarrow{f} Y$ is a <u>weak homotopy equivalence</u>

if it induces:

- bijection of path-components $\pi_0(X) \xrightarrow{\sim} \pi_0(Y)$
- for any choice of $x_0 \in X$, $n = 1, 2, \ldots$ isomorphism

$$\pi_n(X, x_0) \underset{f_*}{\xrightarrow{\sim}} \pi_n(Y, f(x_0))$$

Rem If f is a homotopy equivalence; $\exists g : Y \to X$

s.t. $f \circ g \simeq id_Y$, $g \circ f \simeq id_X$

Th'm (Whitehead)

X, Y cell complexes

$X \xrightarrow{f} Y$ is a weak homotopy equivalence

iff it is a homotopy equivalence

Axiomatization of homology

We want to capture formal properties of $\tilde{H}_n(X, A)$

Def. a <u>reduced</u> <u>homology</u> <u>theory</u> is a sequence of

functors $\tilde{h}_n : (\underbrace{\text{pairs } (X,A)}) \to (\text{comm. grps.})$

for $n = 0, 1, 2, \dots$ choose convenient cat. depending on context

s.t. (1) $f \simeq g$ for maps $(X,A) \to (Y,B)$

by homotopy of pairs

$\Rightarrow f_* = g_* : \tilde{h}_n(X,A) \to \tilde{h}_n(Y,B)$ for induced homs

(2) writing $\tilde{h}_n(X) = \tilde{h}_n(X, \phi)$ there is a long ex. seq.

$\cdots \to \tilde{h}_n(A) \to \tilde{h}_n(X) \to \tilde{h}_n(X,A) \xrightarrow{\partial} \tilde{h}_{n-1}(A) \to \cdots \to \tilde{h}_0(X) \to \tilde{h}_0(X,A) \to 0$

natural for maps $f : (X,A) \to (Y,B)$

i.e. for $f:(X,A)\to(Y,B)$ then

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{h}_n(A) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{h}_n(X) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{h}_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{h}_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \tilde{h}_n(B) \arrow[r] & \tilde{h}_n(Y) \arrow[r] & \tilde{h}_n(Y,B) \arrow[r, "\partial"] & \tilde{h}_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

(cont.) (3) for pointed spaces (X_α, x_α) $\alpha \in I$

$$\tilde{h}_n\left(\bigvee_{\alpha\in I} X_\alpha, \{*\}\right) \cong \bigoplus_{\alpha\in I} \tilde{h}_n(X_\alpha, \{x_\alpha\})$$

with structure maps $(i_\alpha)_*$; $i_{\alpha_1} : X_{\alpha_1} \to \bigvee_{\alpha\in I} X_\alpha$ emb.

Rem. <u>unreduced</u> homology theory $(h_n)_{n=0}^{\infty}$:

replace (3) by analogue for disj. union.

$\tilde{h}_n \leftrightarrow h_n$ by $h_n(X) = \tilde{h}_n(X_+, \phi)$

$h_n(X, A) = \tilde{h}_n(X/A)$

$\tilde{h}_n(X) = \ker\ h_n(X) \to h_n(pt)$, etc.

for any $pt \in X$

Ex. $\tilde{h}_n(X, A) = H_n(X, A; G)$ for some fixed

comm. grp G, nonempty A

suppose $h_n(pt) = 0$ for $n > 0$

$\left(\Leftrightarrow \tilde{h}_n(pt) = 0 \text{ for all } n \right)$

$$h_n(S^m) = \begin{cases} 0 & m \neq n \\ h_0(pt) & m = n \end{cases}$$

for $S^1 = I/\partial I \rightsquigarrow h_n(S^1) = h_n(I, \partial I)$

look at long ex. seq.

$$\cdots \to \underbrace{h_n(\underbrace{\partial I}_{= \{0,1\}})}_{h_n(pt)^{\oplus 2}} \to \underbrace{h_n(I)}_{h_n(pt)} \to \underbrace{h_n(I, \partial I)}_{h_n(S^1)} \xrightarrow{\partial} \underbrace{h_{n-1}(\partial I)}_{h_{n-1}(pt)^{\oplus 2}} \to \underbrace{h_{n-1}(I)}_{h_{n-1}(pt)} \to \cdots$$

contractible

$(a, b) \mapsto a + b$

$S^m = I^m / \partial I^m \qquad \partial I^m \simeq S^{m-1}$