§2.1 Equivalence of simplicial and singular homology

X : Δ-complex $\rightsquigarrow$ we want $\underset{\text{from simplexes of } X.}{H_n^{\Delta}(X)} \cong \underset{\text{in terms of cont maps } \Delta^n \to X}{H_n(X)}$

(Thm 2.27)

Intermediate tool: rel. homology for simplicial case $H_n^{\Delta}(X, A)$ and long ex. seq.

$A \subset X$ subcomplex. (closed subset, union of simplexes)

$\Delta_n(X, A) = \Delta_n(X) / \Delta_n(A)$: lin. span of n-simplexes <u>not</u> in A

$H_n^{\Delta}(X, A)$: homology of $\Delta_\bullet(X, A)$

$\rightsquigarrow$ long ex. $\to H_n^{\Delta}(A) \to H_n^{\Delta}(X) \to H_n^{\Delta}(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$

Example. $H_n^{\Delta}(\Delta^k, \partial\Delta^k) \cong \begin{cases} \mathbb{Z} & (n=k) \\ 0 & (n \neq k) \end{cases}$

from $\Delta_n(\Delta^k, \partial\Delta^k) \cong \begin{cases} \mathbb{Z} \\ 0 \end{cases}$ same cond.

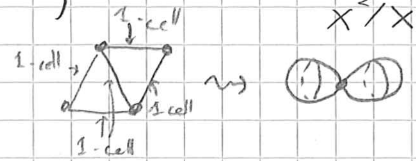
Similarly X Δ-cplx $\rightsquigarrow$ k-skeleton X^k (union of k-simpls)

$\rightsquigarrow H_n^{\Delta}(X^k, X^{k-1}) \cong \begin{cases} \Delta_k(X) \cong \mathbb{Z}^{\#(k\text{-simpls})} & (n=k) \\ 0 \end{cases}$

On the other hand:

$H_n(X^k, X^{k-1}) \underset{\text{Prop. 2.22}}{\cong} H_n(X^k / X^{k-1})$

$X^k / X^{k-1} \cong \underbrace{S^k \vee \cdots \vee S^k}_{\#\text{ of } k\text{-cells in } X}$



$\rightsquigarrow H_n(X^k, X^{k-1}) \cong \tilde{H}_n(S^k \vee \cdots \vee S^k) \cong \left.\begin{cases} \Delta_k(X) & n = k \\ 0 & n \neq k \end{cases}\right\}$

$\cong H_n^{\Delta}(X^k, X^{k-1})$

as Lem

Proof of Th'm 2.27

Strategy : by induction on k, get

$$H_n(X^k) \cong H_n^{\Delta}(X^k)$$

- $k > n \Rightarrow H_n^{\Delta}(X) \cong H_n^{\Delta}(X^k)$ by const.
- singular homol. : any cpt subset $Z \subset X$ is contained in some X^j (j dep. on Z.)

$\rightsquigarrow$ $H_n(X) \cong \varinjlim_{j\to\infty} H_n(X^j)$

we also know $H_n(X^j) \cong H_n(X^{j+1})$ for $j > n$ from long ex. seq.

$(H_n(X^{j+1}, X^j) \cong 0$ for $j > n)$

$\Rightarrow$ we can conclude $H_n(X) \cong H_n^{\Delta}(X)$.

To get $H_n(X^k) \cong H_n^{\Delta}(X^k)$;

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, "h'"] & H_n^{\Delta}(X^{k-1}) \arrow[r] \arrow[d, "f"] & H_n^{\Delta}(X^k) \arrow[r] \arrow[d, "g"] & H_n^{\Delta}(X^k, X^{k-1}) \arrow[r] \arrow[d, "h"] & H_{n-1}^{\Delta}(X^{k-1}) \arrow[d, "f'"] \\
\cdots \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1})
\end{tikzcd}$$

h & h' isom by Lem.

f & f' isom by induction hypothesis.

$\underset{\text{five lemma}}{\Rightarrow}$ g is isom. ∎

Cor. "the Euler char." of surface S can be computed as $\chi(S) = \#(\text{vert}) - \#(\text{edges}) + \#(\text{faces})$ for any polyhedron structure on S.

Similarly X Δ cplx with fin. num of cells ($\dim \leq N$).

$$\chi(X) = \sum_{k=0}^{N} (-1)^k \#(k\text{-dim simplexes})$$

only dep. on $H_\bullet(X)$.