Lecture Notes on Algebraic Topology II

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Contents

Foreword

These are notes intended for the author's Algebraic Topology II lectures at the University of Oslo in the fall term of 2012. The main reference for the course will be:

- Allen Hatcher's book "Algebraic Topology" [1], drawing on chapter 3 on cohomology and chapter 4 on homotopy theory.

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Chapter 1

Introduction

1.1 Integration, chains and cochains

One of the origins of (co-)homology lies in multivariable calculus, where one might consider an open domain $\Omega \subseteq \mathbb{R}^2$, a continuously differentiable path $\gamma\colon [a,b] \to \Omega$ with $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, and a pair of continuous functions $P, Q\colon \Omega \to \mathbb{R}$. The path integral

$$\int_\gamma P\,dx + Q\,dy$$

is then defined to be equal to the integral

$$\int_a^b \big(\gamma_1'(t)P(\gamma(t)) + \gamma_2'(t)Q(\gamma(t))\big)\,dt\,.$$

In more advanced calculus, one can consider a smooth manifold M, a smooth map $\gamma\colon [0,1]^k \to M$ parametrizing a singular k-dimensional cube in M, and a differential k-form ω on M, and define the integral

$$\int_\gamma \omega$$

to be the k-fold iterated integral

$$\int_{[0,1]^k} \gamma^*\omega = \int_0^1 \cdots \int_0^1 f(t_1, \dots, t_k)\,dt_1 \cdots dt_k$$

where $\gamma^*\omega = f(t_1, \dots, t_k)\,dt_1 \cdots dt_k$ in the standard coordinates on $[0,1]^k$.

If $\alpha\colon [a,b] \to \Omega$ and $\beta\colon [b,c] \to \Omega$ are differentiable paths that can be joined to a single differentiable path $\gamma\colon [a,c] \to \Omega$, meaning that $\alpha(b) = \beta(b)$ and $\alpha'(b) = \beta'(b)$, then we have the relation (for the 1-form $\omega = P\,dx + Q\,dy$)

$$\int_\alpha \omega + \int_\beta \omega = \int_\gamma \omega\,,$$

and it is convenient to define the integral

$$\int_{\alpha+\beta} \omega$$

to be equal to the left hand side. Note that the left hand side also makes sense, even if α and β cannot be joined to a single differentiable path, either because they do not meet up (so that $\alpha(b) \neq \beta(b)$, or because there is a corner at the meeting point (so that $\alpha'(b) \neq \beta'(b)$). The expression $\alpha + \beta$ is then no longer a path in Ω, so we instead call it a *chain* (of paths) in Ω.

More generally, if $\sum_i n_i \gamma_i$ is a formal sum of k-cubes $\gamma_i \colon [0,1]^k \to M$, where i runs over some finite indexing set, and ω is a k-form, we can define the integral

$$\int_{\sum_i n_i \gamma_i} \omega$$

to be equal to the linear combination

$$\sum_i n_i \int_{\gamma_i} \omega \, .$$

We call the finite formal sum $\sum_i n_i \gamma_i$ a *k-chain* (of singular cells) in M. We would usually assume that the coefficients n_i are integers, but the definition also makes sense if they are real numbers. In this way we extend the generality of the definition of an integral, by allowing chains instead of cells as the geometric integration domains.

This added generality can be useful in the context of Green's theorem (for $\Omega \subseteq \mathbb{R}^2$) and the more general Stokes' theorem (for general M). Recall that if $R \subseteq \Omega$ is a closed region bounded by a simple closed curve $\gamma = \partial R$, positively oriented, then Green's theorem asserts that there is an identity

$$\int_{\partial R} P\,dx + Q\,dy = \iint_R \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dxdy \, .$$

However, the theorem also holds true if ∂R is not a simple closed curve, but a finite union

$$\partial R = \bigcup_i \gamma_i$$

of simple closed curves. If not all of the curves γ_i are oriented in the same way as ∂R, we can compensate for this by working with formal negatives, so as to write

$$\partial R = \sum_i n_i \gamma_i$$

where each coefficient n_i is $+1$ if γ_i is oriented like ∂R, and it is -1 otherwise. In this case ∂R is no longer a (simple closed) curve by a chain in Ω. The point of working with integrals over chains is that now the identity of Green's theorem also works for regions R bounded by a chain of simple closed curves, not just for a single curve.

In the same way Stokes' theorem gives an identity

$$\int_{\partial R} \omega = \int_R d\omega$$

for $(k+1)$-cells R and k-forms ω on a smooth manifold M. Here ∂R can usually be parametrized as a k-cell, but it is more natural to think of it as a k-chain (with one summand for each k-dimensional face of the $(k+1)$-cube parametrizing

R), and now it is straightforward to deduce that Stokes' theorem also holds for $(k+1)$-chains R and k-forms ω, as above.

To compare Green's and Stokes' theorems, note that for the 1-form $\omega = P\,dx + Q\,dy$ the exterior derivative is

$$d\omega = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,.$$

An interesting special case of Stokes' theorem arises for *closed k-forms* ω, i.e., those with $d\omega = 0$. In the case of Green's theorem, this means that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, so that the vector field (P, Q) has no curl. Then Stokes' theorem tells us that

$$\int_{\partial R} \omega = \int_R d\omega = \int_R 0 = 0$$

vanishes, so that the integral of a closed form along any k-chain that is a boundary is zero. We call such k-chains *k-boundaries*. Hence the integrals over two k-chains α and β that differ by a k-boundary

$$\beta - \alpha = \partial R$$

will be equal

$$\int_\alpha \omega = \int_\beta \omega$$

for closed ω, since their difference is $\int_{\partial R} \omega = 0$. We say that such k-chains α and β are *homologous*. This defines an equivalence relation on the set of k-chains, and the equivalence class $[\alpha]$ of a k-chain is its homology class. The conclusion is that for closed ω the integral $\int_\alpha \omega$ only depends on the homology class of α, not on the particular k-chain α itself.

Another interesting special case of Stokes' theorem arises for *exact k-forms*, i.e., those of the form $\omega = d\eta$. In the case of Green's theorem, this means that $P = \frac{\partial g}{\partial x}$ and $Q = \frac{\partial g}{\partial y}$ for some $g\colon \Omega \to \mathbb{R}$, so that (P, Q) is a gradient field. Then Stokes' theorem tells us that

$$\int_\gamma \omega = \int_\gamma d\eta = \int_{\partial\gamma} \eta = \int_0 \eta = 0$$

vanishes for *k-cycles* γ, i.e., those k-chains with $\partial\gamma = 0$. This means that each $(k-1)$-cell in the boundary of γ occurs algebraically zero times, i.e., as often with one orientation as with the opposite orientation.

It is known that each k-boundary $\gamma = \partial R$ is a k-cycle, since $\partial\gamma = \partial\partial R = 0$. It is an interesting geometric question about chains in the manifold M whether the converse holds true, and if not, to what extent it fails. This leads to the study of the quotient of the abelian group of k-cycles by the subgroup of k-boundaries, which is called the *(cubical) k-th homology group* of M, denoted $H_k(M)$.

It is also known that each exact k-form $\omega = d\eta$ is closed, since $d\omega = dd\eta = 0$. Again it is an interesting analytic question about forms on the manifold M whether the converse holds true, and if not, how to measure its failure. This leads to the study of the quotient of the real vector space of closed k-forms by the subspace of exact k-forms, which is called the *k-th (de Rham) cohomology* *group* of M, here denoted $H^k_{dR}(M)$. We say that two closed k-forms that differ by an exact k-form are *cohomologous*, and write $[\omega]$ for the equivalence class in $H^k_{dR}(M)$ of a closed k-form, i.e., its cohomology class.

These questions are closely related. Integration defines a rule

$$(\omega, \gamma) \longmapsto \int_\gamma \omega$$

that takes a closed k-form ω and a k-cycle γ to the integral of the latter over the former. By the first special case of Stokes' theorem, for closed ω, we see that the value of $\int_\gamma \omega$ only depends on the cubical homology class $[\gamma]$ of γ. By the second special case of Stokes' theorem, for cycles γ, the value of $\int_\gamma \omega$ only depends on the de Rham cohomology class $[\omega]$ of ω. Hence integration does in fact define a rule

$$\begin{aligned} H^k_{dR}(M) \times H_k(M) &\longrightarrow \mathbb{R} \\ ([\omega], [\gamma]) &\longmapsto \int_\gamma \omega \end{aligned}$$

which is a bilinear pairing. Keeping $[\omega]$ momentarily fixed we get a homomorphism

$$h\colon H^k_{dR}(M) \longrightarrow \mathrm{Hom}(H_k(M), \mathbb{R})$$

taking $[\omega]$ to the homomorphism $H_k(M) \to \mathbb{R}$ that takes $[\gamma]$ to $\int_\gamma \omega$.

By combining two theorems, known as *de Rham's theorem* and the *universal coefficient theorem*, we can state:

Theorem 1.1.1. *The homomorphism* $h\colon H^k_{dR}(M) \to \mathrm{Hom}(H_k(M), \mathbb{R})$ *is an isomorphism for every smooth manifold* M.

Hence if every k-cycle in M is a k-boundary, so that $H_k(M) = 0$, we deduce that $H^k_{dR}(M) = 0$, so that every closed k-form on M is exact. This is, for instance, the case when $M = \Omega \subseteq \mathbb{R}^2$ is simply-connected and $k = 1$. There is also a converse, but one needs to take into account that $\mathrm{Hom}(H_k(M), \mathbb{R}) = 0$ does not in general imply that $H_k(M) = 0$.

The theorem is even more interesting in the case when it establishes an isomorphism between two nontrivial groups, since it then provides a correspondence between the cohomology classes of closed k-forms and functionals on the homology classes of k-cycles.

The point of this discussion has been to emphasize the role of chains, cycles and boundaries on the geometric side, and forms, closed forms and exact forms on the analytical side, and to introduce homology and cohomology as related invariants of smooth manifolds.

1.2 Homology and cohomology theories

There are various ways of associating to each topological object, like a topological space or a differentiable manifold, an algebraic object, like a group or a graded commutative ring. This can be interesting because of what the resulting algebraic object tells us about the topological object, or because known topological examples can produce novel algebraic examples. This is often the general framework of algebraic topology.

Usually the algebraic objects are constructed by comparing the given topological object, say a topological space X, with familiar topological objects, like the standard simplices Δ^n or the complex plane/line $\mathbb{C}$, or specially designed topological spaces, like the Eilenberg–Mac Lane spaces $K(G, n)$.

For example, to study singular homology, one considers the continuous maps $\sigma\colon \Delta^n \to X$ for all $n \geq 0$, assembles these into the singular chain complex $(C_*(X), \partial)$, and passes to homology, to obtain the singular homology groups $H_n(X)$ for $n \geq 0$. This is a standard approach in algebraic topology. The construction involves maps into X, and is covariant in X, in the sense that for a map $f\colon X \to Y$ there is an induced homomorphism $f_*\colon H_n(X) \to H_n(Y)$ in the same direction.

As another example, one may consider the commutative ring $C(X)$ of continuous maps $\varphi\colon X \to \mathbb{C}$ under pointwise addition and multiplication. These are the global sections in a sheaf of rings that to each open subset $U \subseteq X$ associates the ring $C(U)$ of continuous functions on U. Under suitable assumptions on X one may consider refined versions of this: if X is a complex variety one can consider the ring $\mathscr{O}(X)$ of holomorphic maps $\varphi\colon X \to \mathbb{C}$. Using sheaf cohomology one can associate cohomology groups $H^n(X)$ to these ringed spaces. This is a standard approach in algebraic geometry. The construction involves maps out of X, and is contravariant in X, in the sense that for a (regular) map $f\colon X \to Y$ there is an induced homomorphism $f^*\colon H^n(Y) \to H^n(X)$ in the opposite direction.

If X is a smooth (infinitely differentiable) manifold one can consider the ring $C^\infty(X)$ of smooth maps $\varphi\colon X \to \mathbb{R}$. Each point $p \in X$ determines a maximal ideal $\mathfrak{m}_p$, and the rule $\varphi \mapsto d\varphi(p)$ induces an isomorphism $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong T_p^*X$ to the cotangent space at p, dual to the tangent space. Gluing these vector spaces together one can view each differential n-form ω on X as a section in a vector bundle over X. These can be assembled into the de Rham complex (Ω^*X, d), whose cohomology defines the de Rham cohomology $H^*_{dR}(X)$. This is a standard approach in differential topology. The differential forms on X are again maps out of X, and the construction is contravariant in X.

There is a variant of singular homology, called singular cohomology, which is also contravariant. Its construction is of somewhat mixed variance, since it is given in terms of functions out of things given by maps into X. More precisely, one considers functions $\varphi\colon \{n\text{-simplices in } X\} \to G$ from the set of singular n-simplices $\sigma\colon \Delta^n \to X$ to a fixed abelian group G. This is equivalent to considering homomorphisms $\varphi\colon C_n(X) \to G$ from the free abelian group of singular n-chains on X. From these functions or homomorphisms one forms a cochain complex $C^*(X; G)$, whose cohomology groups are the singular cohomology groups $H^n(X; G)$.

There is also a more directly contravariant construction, valid for all topological spaces X. For each abelian group G and each $n \geq 0$ there exists a topological space $K(G, n)$, well-defined up to homotopy equivalence, such that the group $\pi_i K(G, n) = [S^i, K(G, n)]$ of homotopy classes of maps $S^i \to K(G, n)$ is trivial for $i \neq n$, and is identified with G for $i = n$. Such a space is called an Eilenberg–Mac Lane complex of type (G, n). The group $[X, K(G, n)]$ of homotopy classes of maps $X \to K(G, n)$ defines a cohomology theory in X, which is isomorphic to the singular cohomology group $H^n(X; G)$ for a large class of spaces X.

Given this wealth of possible constructions, the good news is that there are interesting uniqueness theorems: For large classes of “reasonable” topological spaces the various constructions agree. The formulation and proof of these theorems is best done in the language of category theory, in terms of functors and natural transformations, which was originally developed by Eilenberg and Mac Lane, largely for this purpose. The result is in some sense surprising, since it is not so clear that an abelian group built out of the continuous maps $\Delta^n \to X$ should have much to do with another abelian group built out of the continuous maps $X \to \mathbb{C}$ or $X \to K(G, n)$.

Consider for example the space $X = \mathbb{Q}$ of rational numbers, with the subspace topology from $\mathbb{R}$. Any continuous map $\Delta^n \to \mathbb{Q}$ is constant, so to the eyes of singular homology and cohomology, $\mathbb{Q}$ could equally well have had the discrete topology. On the other hand, not every map $\mathbb{Q} \to \mathbb{C}$ is continuous, so to the eyes of sheaf cohomology, the choice of topology on $\mathbb{Q}$ makes an essential difference. This example shows that for general (“unreasonable”) topological spaces, the various constructions of cohomology do not agree.

The standard techniques of singular (co-)homology, like homotopy invariance, the long exact sequence of a pair, excision, behavior on sums, and the dimension axiom, suffice to prove uniqueness results for the homology and cohomology of CW spaces, i.e., spaces that can be given the structure of a CW complex, and more generally for all spaces that are of the homotopy type of a CW complex. These are then the “reasonable” spaces in the sense above. Any manifold or complex variety is a CW space, so for geometric purposes, this class of spaces is usually fully adequate for topological work. On the other hand, the space of rational numbers mentioned above is not of the homotopy type of a CW complex. When going outside of this class of spaces, there are many variant (co-)homology theories, often with special properties that may be useful in particular settings.

These constructions are discussed in Section 3.1 and 3.A of Hatcher [1].

1.3 Cup product and the Künneth theorem

The (co-)homology groups of a topological space are useful in classification of general classes of spaces, and in answering questions about special classes of spaces. The classification problem concerns questions like: “what are the possible spaces of this type?” and “given a space, which one is it?” Since the (co-)homology groups of a space are usually quite easy to compute, and two abelian groups can usually quite easily be compared to each other, it is useful to try to answer these questions in terms of the (co-)homology groups of the space.

As a first step towards determining what possibilities there are for a class of topological objects, one should then determine what possibilities there are for the corresponding class of algebraic objects. Here it turns out to be fruitful to consider the (co-)homology groups as examples of a richer algebraic structure than just a sequence of abelian groups.

One extra structure comes from the same source as the commutative ring structure on the set $C(X)$ of continuous functions on a space X. This was given by the pointwise sum and product of functions, so given two maps $\varphi, \psi \colon X \to \mathbb{C}$, we can form the sum given by $(\varphi + \psi)(p) = \varphi(p) + \psi(p)$ and the product given by $(\varphi \cdot \psi)(p) = \varphi(p)\psi(p)$. To make it clearer what structures are involved, we might express these formulas in terms of diagrams. Since the right hand sides in these expressions involve evaluation at p two times, we need to make two copies of that point. This is done using the diagonal map

$$\Delta\colon X \to X \times X$$

that takes $p \in X$ to $(p, p) \in X$. The sum of φ and ψ is then given by the composite map

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\varphi\times\psi} \mathbb{C} \times \mathbb{C} \xrightarrow{+} \mathbb{C}\,,$$

where the last map is the addition in $\mathbb{C}$, and similarly for the product. The commutativity of the product is derived from the fact that the composite

$$\tau\Delta\colon X \xrightarrow{\Delta} X \times X \xrightarrow{\tau} X \times X$$

is equal to Δ, where $\tau\colon X \times X \to X \times X$ is the twist homeomorphism that takes (p, q) to (q, p).

What is the associated structure in (co-)homology? The diagonal map induces a homomorphism

$$\Delta_*\colon H_n(X) \longrightarrow H_n(X \times X)\,,$$

but this lands in the homology of $X \times X$, not the homology of X. With a little care it is possible to define a homology cross product map

$$\times\colon H_i(X) \otimes H_j(X) \longrightarrow H_{i+j}(X \times X)$$

for all $i, j \geq 0$, and these assemble to a map

$$\bigoplus_{i+j=n} H_i(X) \otimes H_j(X) \longrightarrow H_n(X \times X)\,.$$

In general this map is not an isomorphism. If it were, we could have composed Δ_* with the inverse isomorphism, and obtained a homomorphism

$$H_n(X) \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(X)$$

for all n. As a convention, the tensor product of two graded abelian groups is defined so that the collection of all of these maps could be written as a *coproduct*

$$H_*(X) \longrightarrow H_*(X) \otimes H_*(X)\,,$$

which would make the homology groups $H_*(X)$ into a *graded coring*.

The combined cross product map

$$H_*(X; F) \otimes_F H_*(X; F) \longrightarrow H_*(X \times X; F)$$

is an isomorphism if we work with homology with coefficients in a field F, and this is one reason to consider homology groups with coefficients. This is a special case of the *Künneth theorem* for homology.

However, the algebraic structures of corings or coalgebras are unfamiliar ones. It is therefore most often more convenient to dualize, and to consider cohomology instead of homology.

Let R be a commutative ring, for instance the ring of integers $\mathbb{Z}$. The diagonal map induces a homomorphism

$$\Delta^*\colon H^*(X \times X; R) \longrightarrow H^*(X; R)$$

and there is a cohomology cross product map

$$\times\colon H^i(X; R) \otimes H^j(X; R) \longrightarrow H^{i+j}(X \times X; R)$$

for all $i, j \geq 0$. The composite is a homomorphism

$$\cup = \times \circ \Delta^*\colon H^i(X; R) \times H^j(X; R) \to H^{i+j}(X; R)$$

for all $i, j \geq 0$, called the cup product. We may assemble these cup product maps to a pairing

$$\cup\colon H^*(X; R) \otimes H^*(X; R) \longrightarrow H^*(X; R)\,,$$

which makes $H^*(X; R)$ a graded ring, or more precisely, a graded R-algebra. In fact, the cohomology cross product map, taking $a \otimes b$ to $a \times b$, is compatible with the twist homeomorphism τ, in the graded sense that $\tau^*(a \times b) = (-1)^{ij} b \times a$, where i and j are the degrees of a and b, respectively, so that the cohomology ring $H^*(X; R)$ becomes graded commutative.

In much work in algebraic topology, it is therefore standard to consider the cohomology $H^*(X; R)$ of a space X, not as a graded abelian group, but as a graded commutative ring or algebra. This enriched algebraic structure is still manageable, but often carries much more useful information than the plain group structure.

These constructions are discussed in Section 3.2 and 3.B of Hatcher [1].

1.4 Poincaré duality

Much geometric work is concerned with manifolds, or smooth varieties, rather than general topological spaces. In an n-dimensional manifold there is a certain duality between k-dimensional subobjects and suitable $(n-k)$-dimensional subobjects. For example, each compact, convex polyhedron in $\mathbb{R}^3$ determines a cell structure on its boundary, a topological 2-sphere, dividing it into vertices (0-cells), edges (1-cells) and faces (2-cells). There is also a dual cell structure, with a 0-cell for each of the old faces, a 1-cell for each of the old edges, and a 2-cell for each of the old vertices. We can superimpose these cell structures, so that each of the old k-cells meets one of the new $(2-k)$-cells, in a single point.

Algebraically, this is reflected in a certain duality in the homology, or cohomology, of a manifold. It says that for suitable n-manifolds X (closed, connected and oriented) there is a preferred isomorphism $H^n(X; R) \cong R$, and the cup product pairing

$$H^k(X; R) \otimes H^{n-k}(X; R) \xrightarrow{\cup} H^n(X; R) \cong R$$

defines a perfect pairing modulo torsion. This is the *Poincaré duality theorem*. If $R = F$ is a field this means that the corresponding homomorphisms

$$H^k(X; F) \longrightarrow \mathrm{Hom}(H^{n-k}(X; F), F) = H^{n-k}(X; F)^*$$

are isomorphisms, for all k. This homomorphism takes $a \in H^k(X; F)$ to the homomorphism $H^{n-k}(X; F) \to F$ that takes b to the image of $a \cup b \in H^n(X; F)$ in F, under the preferred isomorphism. In particular

$$\dim_F H^k(X; F) = \dim_F H^{n-k}(X; F)$$

for all k.

This kind of symmetry, between dimension k and codimension k phenomena in the (co-)homology of an n-manifold, is the key feature taken as the starting point for the classification of manifolds, as a special class of topological objects among all topological spaces.

These constructions are discussed in Section 3.3 and 3.H of Hatcher [1].

1.5 Lie groups, H-spaces and Hopf algebras

A second interesting kind of topological spaces are the *topological groups*, i.e., topological spaces G equipped with a continuous multiplication $m: G \times G \to G$, a unit element $e \in G$ and a continuous group inverse $i: G \to G$ such that the usual group axioms are satisfied. In the case where G is also a manifold we get the notion of a *Lie group*. Examples of these include the matrix groups $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ of $n \times n$ invertible matrices with real and complex entries, and their subgroups $O(n)$ and $U(n)$ of orthogonal and unitary matrices, respectively. These play an important role in Riemannian geometry, representation theory and mathematical physics.

In the cases $n = 1$ we get well-known Lie group structures on the spheres $O(1) = \{\pm 1\} = S^0$ and $U(1) = S^1$. The division algebra structure on $\mathbb{R}^4 = \mathbb{H}$ given by the Hamiltonian quaternions restricts to a (non-commutative) group structure on the unit sphere $S^3 \subset \mathbb{H}$, and the division algebra structure on $\mathbb{R}^8 = \mathbb{O}$ due to Graves and Cayley restricts to a (non-associative) multiplication on the unit sphere $S^7 \subset \mathbb{O}$.

To the eyes of homology with field coefficients, a topological group G gives rise to a graded F-vector space $H_*(G; F)$ that is both a coalgebra and an algebra. The coalgebra structure comes from the diagonal map $\Delta: G \to G \times G$ and the Künneth isomorphism

$$H_*(G; F) \xrightarrow{\Delta_*} H_*(G \times G; F) \cong H_*(G; F) \otimes_F H_*(G; F)$$

while the algebra structure comes from the multiplication map $m: G \times G \to G$ and the Künneth isomorphism

$$H_*(G; F) \otimes_F H_*(G; F) \cong H_*(G \times G; F) \xrightarrow{m_*} H_*(G; F) \,.$$

These are compatible in the sense that the coproduct is an algebra map and the product is a coalgebra map. There is also a compatibility with the homomorphism $i_*: H_*(G; F) \to H_*(G; F)$ induced by the inverse map. The combined structure is called a *Hopf algebra*.

The full strength of the group axioms is not needed to obtain this algebraic structure on $H_*(G; F)$, so Hopf also considered the more flexible notion of an *H-space* G, with comes with a pairing $m: G \times G \to G$ that only needs to satisfy the group axioms up to homotopy, and the role of the inverse map becomes negotiable.

Under suitable finiteness hypotheses, there is also a dual Hopf algebra structure in cohomology, with product

$$\Delta^*: H^*(G; F) \otimes_F H^*(G; F) \longrightarrow H^*(G; F)$$

given by the cup product, and coproduct

$$m^*: H^*(G; F) \longrightarrow H^*(G; F) \otimes_F H^*(G; F)$$

induced by the H-space pairing. (Here we suppress the required cohomology Künneth isomorphisms from the notation.)

There are interesting theorems of Hopf and Borel on the possible algebraic structures that can be realized by Hopf algebras, and a famous theorem of Frank Adams saying that the only spheres S^n that admit H-space structures are those mentioned above, i.e., for $n = 0$, 1, 3 and 7. This is related to the result that the only real vector spaces $\mathbb{R}^n$ that admit division algebra structures are those for $n = 1$, 2, 4 and 8.

These results are discussed in Section 3.C of Hatcher [1].

Chapter 2

Singular homology and cohomology

We first review the definition of singular homology, and then introduce singular homology with coefficients and singular cohomology.

2.1 Chain complexes

A *chain complex* is a diagram

$$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \dots$$

of abelian groups (or R-modules, or objects in a more general abelian category), such that the composite

$$\partial^2 = \partial\partial \colon C_{n+1} \longrightarrow C_{n-1}$$

is the zero homomorphism, for each integer n. We also use the abbreviated notation (C_*, ∂), or just C_*, for the diagram above. The elements of C_n are called *n-chains*. We think of C_* as a *graded abelian group*, with C_n in degree n and ∂ of degree -1. This is the standard convention in algebraic topology.

Let

$$B_n = B_n(C_*, \partial) = \operatorname{im}(\partial \colon C_{n+1} \to C_n)$$

be the group of *n-boundaries*, and let

$$Z_n = Z_n(C_*, \partial) = \ker(\partial \colon C_n \to C_{n-1})\,.$$

be the group of *n-cycles*. Then

$$B_n \subseteq Z_n \subseteq C_n$$

since any element of B_n has the form $x = \partial y$, and then $\partial x = \partial^2 y = 0$. In general, the inclusion $B_n \subseteq Z_n$ may be a proper inclusion. To detect the possible difference, we form the quotient group

$$H_n(C_*, \partial) = Z_n/B_n\,,$$

called the *n-th homology group* of (C_*, ∂). A necessary and sufficient condition for an n-cycle $x \in Z_n$ to be an n-boundary is then that its equivalence class (= coset) $[x] \in Z_n/B_n = H_n(C_*, \partial)$ is zero. We call the equivalence class $[x]$ the *homology class* of the cycle x.

If there is no difference between cycles and boundaries, so that $B_n = \mathrm{im}(\partial)$ is equal to $Z_n = \ker(\partial)$, as subgroups of C_n, then we say that the chain complex is *exact at C_n*. This is equivalent to the vanishing $H_n(C_*, \partial) = 0$ of the n-th homology group. A chain complex is *exact* if it is exact at each object in the diagram.

An exact chain complex is also called a *long exact sequence*. An exact chain complex of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

(extended by 0's in both directions) is called a *short exact sequence*. Exactness at A means that i is injective, exactness at B means that $\mathrm{im}(i) = \ker(j)$, and exactness at C means that j is surjective.

Let (C_*, ∂) and (D_*, ∂) be two chain complexes. A *chain map* $f_\#\colon C_* \to D_*$ is a commutative diagram

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow & \dots \\
 & & \big\downarrow{\scriptstyle f_{n+1}} & & \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle f_{n-1}} & & \\
\dots & \longrightarrow & D_{n+1} & \xrightarrow{\partial} & D_n & \xrightarrow{\partial} & D_{n-1} & \longrightarrow & \dots
\end{array}$$

of abelian groups. In other words, it is a sequence of group homomorphisms $f_n\colon C_n \to D_n$ such that $\partial f_n = f_{n-1}\partial\colon C_n \to D_{n-1}$, for all n. A chain map $f_\#\colon C_* \to D_*$ restricts to homomorphisms $B_n(C_*, \partial) \to B_n(D_*, \partial)$ and $Z_n(C_*, \partial) \to Z_n(D_*, \partial)$, hence induces a homomorphism of quotient groups:

$$f_*\colon H_n(C_*, \partial) \to H_n(D_*, \partial)$$

for each n. If $g_\#\colon D_* \to E_*$ is another chain map, then we have the relation

$$(gf)_* = g_* f_*\colon H_n(C_*, \partial) \to H_n(E_*, \partial)$$

for each n, saying that the homology groups $H_n(C_*, \partial)$ are (covariant) functors of the chain complex (C_*, ∂). (We omit to mention the identity condition.)

If the groups are reindexed by superscripts:

$$C^m = C_{-m}$$

we obtain a diagram

$$\dots \to C^{m-1} \xrightarrow{\delta} C^m \xrightarrow{\delta} C^{m+1} \to \dots$$

such that the composite

$$\delta^2 = \delta\delta\colon C^{m-1} \longrightarrow C^{m+1}$$

is the zero homomorphism, for each m. This is called a *cochain complex*. We abbreviate this to (C^*, δ), or just C^*. The elements of C^m are called *m-cochains*.

Again C^* is a graded abelian group, with C^m in degree m and δ of degree $+1$. This is the standard convention in algebraic geometry.

Let $B^m = \mathrm{im}(\delta\colon C^{m-1} \to C^m)$ and $Z^m = \ker(\delta\colon C^m \to C^{m+1})$ be the groups of *m-coboundaries* and *m-cocycles*, respectively. Then

$$B^m \subseteq Z^m \subseteq C^m$$

as before, and the quotient group

$$H^m(C^*, \delta) = \frac{Z^m}{B^m}$$

is called the *m-th cohomology group* of (C^*, δ). If C^* is obtained from C_* by the reindexing $C^m = C_{-m}$, then $B^m = B_{-m}$, $Z^m = Z_{-m}$ and $H^m(C^*, \delta) = H_{-m}(C_*, \partial)$.

Exercise 2.1.1. (a) Let R be a ring (associative, with unit), and let

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

be a *short exact sequence* of left R-modules, meaning that i and j are R-module homomorphisms, i is injective, the image of i equals the kernel of j, and j is surjective. We may think of i as an *inclusion* and j as a *projection.* Prove that j induces an isomorphism of R-modules from $B/i(A)$ to C.

(b) By a *section* to j we mean an R-module homomorphism $s\colon C \to B$ such that $js = 1_C$ is the identity homomorphism of C. By a *retraction* of i we mean an R-module homomorphism $r\colon B \to A$ such that $ri = 1_A$ is the identity homomorphism of A. Prove that j admits a section s if and only if i admits a retraction r. In this case we say that the short exact sequence is *split.*

(c) Prove that if the short exact sequence is split, then the homomorphisms $i + s\colon A \oplus C \to B$ and $(r, j)\colon B \to A \times C$ are isomorphisms of R-modules.

(d) Give an example of a short exact sequence that is not split.

(e) Let M be another left R-module, and let the Hom-*group* $\mathrm{Hom}_R(A, M)$ denote the abelian group of R-module homomorphisms $f\colon A \to M$. Let

$$i^*\colon \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A, M)$$

be the group homomorphism that takes $g\colon B \to M$ to the composite $gi\colon A \to M$. Prove that

$$0 \to \mathrm{Hom}_R(C, M) \xrightarrow{j^*} \mathrm{Hom}_R(B, M) \xrightarrow{i^*} \mathrm{Hom}_R(A, M)$$

is an exact sequence of abelian groups, meaning that j^* is injective, and the image of j^* equals the kernel of i^*.

(f) Provide an example of a case where i^* is not surjective.

(g) Prove that if the short exact sequence $0 \to A \to B \to C \to 0$ is split, then i^* is surjective, so that the exact sequence in (e) can be extended by a 0 at the right hand side to become a short exact sequence.

(h) Let N be a right R-module, and let the *tensor product* $N \otimes_R A$ denote the abelian group generated by symbols $n \otimes a$, with $n \in N$ and $a \in A$, subject to the bilinearity relations

$$(n_1 + n_2) \otimes a = n_1 \otimes a + n_2 \otimes a$$
$$n \otimes (a_1 + a_2) = n \otimes a_1 + n \otimes a_2$$
$$nr \otimes a = n \otimes ra$$

for $n, n_1, n_2 \in N$, $r \in R$ and $a, a_1, a_2 \in A$. Let $i_* \colon N \otimes_R A \to N \otimes_R B$ be the group homomorphism that takes $n \otimes a$ to $n \otimes i(a)$. Prove that

$$N \otimes_R A \xrightarrow{i_*} N \otimes_R B \xrightarrow{j_*} N \otimes_R C \to 0$$

is exact, meaning that the image of i_* equals the kernel of j_*, and j_* is surjective. [Hint: See Lemma 3A.1 of Hatcher [1].]

(i) Give an example where i_* is not injective.

(j) Prove that if $0 \to A \to B \to C \to 0$ splits, then i_* is injective, so that the exact sequence in (h) can be extended by a 0 at the left hand side to become a short exact sequence.

2.2 Some homological algebra

A *short exact sequence of chain complexes* is a diagram

$$0 \to A_* \xrightarrow{i_\#} B_* \xrightarrow{j_\#} C_* \to 0\,,$$

where (A_*, ∂), (B_*, ∂) and (C_*, ∂) are chain complexes, $i_\# \colon A_* \to B_*$ and $j_\# \colon B_* \to C_*$ are chain maps, and

$$0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$$

is a short exact sequence (of abelian groups, or R-modules) in each degree n.

Alternatively, such a diagram can be drawn as a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\partial"'] & B_{n+1} \arrow[r, "j_{n+1}"] \arrow[d, "\partial"'] & C_{n+1} \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "\partial"'] & B_n \arrow[r, "j_n"] \arrow[d, "\partial"'] & C_n \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "\partial"'] & B_{n-1} \arrow[r, "j_{n-1}"] \arrow[d, "\partial"'] & C_{n-1} \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "i_{n-2}"] \arrow[d] & B_{n-2} \arrow[r, "j_{n-2}"] \arrow[d] & C_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

of abelian groups (or R-modules), where each column is a chain complex and each row is a short exact sequence.

The chain maps $i_\# \colon A_* \to B_*$ and $j_\# \colon B_* \to C_*$ induce homomorphisms of homology groups $i_* \colon H_n(A_*, \partial) \to H_n(B_*, \partial)$ and $j_* \colon H_n(B_*, \partial) \to H_n(C_*, \partial)$

in all degrees n, as discussed above. We usually omit the boundary map from the notation, and write $i_*\colon H_n(A_*) \to H_n(B_*)$ and $j_*\colon H_n(B_*) \to H_n(C_*)$.

Furthermore, to each short exact sequence of chain complexes as above, we can associate a *connecting homomorphism*

$$\partial\colon H_n(C_*, \partial) \longrightarrow H_{n-1}(A_*, \partial)\,.$$

Again, we usually omit the boundary map from the notation.

Definition 2.2.1. The connecting homomorphism $\partial\colon H_n(C_*) \to H_{n-1}(A_*)$ takes the homology class $[c]$ of an n-cycle $c \in Z_n(C_*, \partial)$ to the homology class $[a]$ of the $(n-1)$-cycle $a \in Z_{n-1}(A_*, \partial)$ that satisfies $i_{n-1}(a) = \partial(b)$ in B_{n-1}, where $b \in B_n$ is any choice of n-chain satisfying $j_n(b) = c$.

$$\begin{tikzcd} & b \arrow[r, mapsto, "j_n"] \arrow[d, mapsto, "\partial"'] & c \\ a \arrow[r, mapsto, "i_{n-1}"] & \partial(b) & \end{tikzcd}$$

Lemma 2.2.2. *The connecting homomorphism $\partial\colon H_n(C_*) \to H_{n-1}(A_*)$ is a well-defined homomorphism.*

Proof. Given c, a choice of b exists because j_n is surjective. Then $\partial(b)$ satisfies $j_{n-1}\partial(b) = \partial j_n(b) = \partial(c) = 0$, since $j_\#$ is a chain map and c is an n-cycle. Thus $\partial(b)$ is in $\ker(j_{n-1})$, which equals $\mathrm{im}(i_{n-1})$ by horizontal exactness at B_{n-1}, so there exists an a with $i_{n-1}(a) = \partial(b)$. This a is unique, given b, since i_{n-1} is injective. Now $i_{n-2}\partial(a) = \partial i_{n-1}(a) = \partial\partial(b) = 0$, since $i_\#$ is a chain map and (B_*, ∂) is a chain complex. Using that i_{n-2} is injective we deduce that $\partial(a) = 0$, so that a is an $(n-1)$-cycle.

A second choice b' of lift of a differs from b by an element in $\ker(j_n)$, hence can be written $b' = b + i_n(a')$ for some n-chain $a' \in A_n$, by horizontal exactness at B_n. Then $\partial(b') = \partial(b) + \partial i_n(a') = i_{n-1}(a) + i_{n-1}\partial(a') = i_{n-1}(a + \partial(a'))$, so that the $(n-1)$-cycle a is replaced by the $(n-1)$-cycle $a + \partial(a')$. This does not alter the homology class of a in $H_{n-1}(A_*)$.

Another choice c' of n-cycle representing the homology class of c has the form $c' = c + \partial(c'')$ for some $(n+1)$-chain c'' in C_{n+1}. Let b'' be a lift of c'', with $j_{n+1}(b'') = c''$. Then $b' = b + \partial(b'')$ is a lift of c', since j_* is a chain map, and $\partial(b') = \partial(b)$, since B_* is a chain complex, so the same $(n-1)$-cycle a as before satisfies $i_{n-1} = \partial(b')$.

Hence the choices involved in representing a homology class x in $H_n(C_*)$ by an n-cycle in C_*, or in lifting that n-cycle to an n-chain in B_*, do not affect the value of the homology class $\partial(x)$, so ∂ is well-defined.

To see that it is a homomorphism, consider a sum $x + x'$ in $H_n(C_*)$. If c and c' are n-cycles representing x and x', respectively, then $c + c'$ represents $x + x'$. If b and b' are n-chains lifting c and c', respectively, then $b + b'$ is an n-chain lifting $c + c'$. If a and a' map to $\partial(b)$ and $\partial(b')$, respectively, then $a + a'$ maps to $\partial(b + b')$. Hence $\partial(x + x')$ is the homology class of $a + a'$, which equals $\partial(x) + \partial(x')$. (Similarly for R-linearity.) □

Theorem 2.2.3 (Long exact sequence in homology)**.** *To each short exact sequence*

$$0 \to A_* \xrightarrow{i_\#} B_* \xrightarrow{j_\#} C_* \to 0$$

of chain complexes there is associated a long exact sequence

$$\cdots \to H_{n+1}(C_*) \xrightarrow{\partial} H_n(A_*) \xrightarrow{i_*} H_n(B_*) \xrightarrow{j_*} H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \to \dots$$

of homology groups, where i_* and j_* are the induced homomorphisms, and ∂ is the connective homomorphism.

By a *map of short exact sequences of chain complexes*, we mean a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "i_\#"] \arrow[d, "\alpha_\#"'] & B_* \arrow[r, "j_\#"] \arrow[d, "\beta_\#"'] & C_* \arrow[r] \arrow[d, "\gamma_\#"'] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "i_\#"] & B'_* \arrow[r, "j_\#"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

of chain complexes and chain maps, such that each row is a short exact sequence of chain complexes.

Theorem 2.2.4 (Naturality)**.** *The connecting homomorphism and long exact sequence in homology associated to a short exact sequence of chain complexes is natural, in the sense that the diagram*

$$\begin{tikzcd}
H_{n+1}(C_*) \arrow[r, "\partial"] \arrow[d, "\gamma_*"'] & H_n(A_*) \arrow[r, "i_*"] \arrow[d, "\alpha_*"'] & H_n(B_*) \arrow[r, "j_*"] \arrow[d, "\beta_*"'] & H_n(C_*) \arrow[r, "\partial"] \arrow[d, "\gamma_*"'] & H_{n-1}(A_*) \arrow[d, "\alpha_*"'] \\
H_{n+1}(C'_*) \arrow[r, "\partial"] & H_n(A'_*) \arrow[r, "i_*"] & H_n(B'_*) \arrow[r, "j_*"] & H_n(C'_*) \arrow[r, "\partial"] & H_{n-1}(A'_*)
\end{tikzcd}$$

commutes in all degrees n.

Proof. Commutation of the two central squares follows from the commutativity of the two squares in the map of short exact sequences of chain complexes, combined with naturality of the homology of a chain complex: $\beta_* i_* = H_n(\beta_\# i_\#) = H_n(i_\# \alpha_\#) = i_* \alpha_*$ and $\gamma_* j_* = H_n(\gamma_\# j_\#) = H_n(j_\# \beta_\#) = j_* \beta_*$.

The commutation of the outer squares amounts to the naturality of the connecting homomorphism. Consider a class $x = [c] \in H_n(C_*)$ with $c \in Z_n(C_*)$. Choose a lift $b \in B_n$ with $j_n(b) = c$, and let $a \in Z_{n-1}(A_*)$ be given by $i_{n-1}(a) = \partial(b)$. Then $\alpha_{n-1}(a) \in Z_{n-1}(A'_*)$ represents $\alpha_* \partial(x)$.

On the other hand, $\gamma_n(c) \in Z_n(C'_*)$ represents $\gamma_*(x) \in H_n(C'_*)$, and $\beta_n(b) \in B'_n$ is a lift of $\gamma_n(c)$, while $\alpha_{n-1}(a)$ satisfies $i_{n-1}(\alpha_{n-1}(a)) = \partial \beta_n(b)$. Hence $\partial \gamma_*(x)$ is also represented by $\alpha_{n-1}(a) \in Z_{n-1}(A'_*)$. $\square$

The following lemma is useful for arguing about maps of (quite) long exact sequences.

Lemma 2.2.5 (Five-lemma)**.** *Let*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\beta"'] & C \arrow[r, "k"] \arrow[d, "\gamma"'] & D \arrow[r, "\ell"] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"'] \\
A' \arrow[r, "i"] & B' \arrow[r, "j"] & C' \arrow[r, "k"] & D' \arrow[r, "\ell"] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows.

(a) If α is surjective, β is injective and δ is injective, then γ is injective.

(b) If β is surjective, δ is surjective and ϵ is injective, then γ is surjective.

(c) If α is surjective, β and δ are isomorphisms and ϵ is injective, then γ is an isomorphism. In particular, if α, β, δ and ϵ are isomorphisms, then so is γ.

Proof. ((Diagram chases. Case (c) follows from (a) and (b).)) □

Example 2.2.6. Consider a map of short exact sequences of chain complexes as above. If α_* and γ_* are isomorphisms in all degrees, then β_* is an isomorphism in all degrees, by the five-lemma applied to the diagram displayed in the naturality theorem above.

Similarly, if α_* and β_* are isomorphisms is all degrees, then so is γ_*, by the five-lemma applied to the diagram shifted one step to the right.

Finally, if β_* and γ_* are isomorphisms in all degrees, then so is α_*, by the five-lemma applied to the diagram shifted one step to the left.

Definition 2.2.7. A *chain homotopy* $P_\#$ between two chain maps $f_\# \colon A_* \to B_*$ and $g_\# \colon A_* \to B_*$ is a sequence of homomorphisms $P_n \colon A_n \to B_{n+1}$ in all degrees n, satisfying the condition that

$$\partial P_n + P_{n-1}\partial = g_n - f_n$$

for all n. In this case we write $P_\# \colon f_\# \simeq g_\#$ for this chain homotopy.

$$\begin{tikzcd}
 & \cdots \arrow[r, "\partial"] & A_n \arrow[r, "\partial"] \arrow[dl, "P_n"'] \arrow[d, "g_n - f_n"] & A_{n+2} \arrow[dl, "P_{n-1}"] \\
 & B_{n-1} \arrow[r, "\partial"'] & B_n \arrow[r, "\partial"'] & \cdots
\end{tikzcd}$$

Lemma 2.2.8. *Any two chain homotopic chain maps $f_\#, g_\# \colon A_* \to B_*$ induce the same homomorphism $f_* = g_* \colon H_n(A_*) \to H_n(B_*)$ in homology, in all degrees n.*

Proof. For each n-cycle $c \in Z_n(A_*)$ representing $x = [c] \in H_n(A_*)$, the relation

$$\partial P_n(c) + P_{n-1}\partial(c) = g_n(c) - f_n(c)$$

tells us that $f_n(c)$ and $g_n(c) = f_n(c) + \partial P_n(c)$ are homologous. Hence $f_*(x) = [f_n(c)]$ equals $[g_n(c)] = g_*(x)$. □

Lemma 2.2.9. *Chain homotopy defines an (additive) equivalence relation on the abelian group of chain maps $A_* \to B_*$.*

((Easy.))

Definition 2.2.10. A chain map $f_\# \colon A_* \to B_*$ is called a *chain homotopy equivalence* if there exists a chain map $g_\# \colon B_* \to A_*$ such that the composite $g_\# f_\# \colon A_* \to A_*$ is chain homotopic to the identity map of A_*, and the composite $f_\# g_\# \colon B_* \to B_*$ is chain homotopic to the identity map of B_*. In this case, $g_\#$ is called a *chain homotopy inverse* to $f_\#$.

Lemma 2.2.11. *A chain homotopy equivalence $f_\# \colon A_* \to B_*$ induces an isomorphism $f_* \colon H_n(A_*) \to H_n(B_*)$ in homology, in all degrees n.*

((Easy.))

2.3 The definition of singular homology

For each $n \geq 0$, let the *standard n-simplex* Δ^n be the subspace

$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \text{each } t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$$

of $\mathbb{R}^{n+1}$ consisting of all convex linear combinations

$$(t_0, t_1, \dots, t_n) = \sum_{i=0}^{n} t_i v_i$$

of the $(n+1)$ unit vectors $v_0, v_1, \dots, v_n$, where

$$v_i = (0, \dots, 0, 1, 0, \dots, 0)$$

has a single 1 in the i-th position, counting from 0. We call t_i the i-th barycentric coordinate of the point $(t_0, t_1, \dots, t_n)$. We call v_i the *i-th vertex* of Δ^n. Note that Δ^n has $(n+1)$ vertices.

For each $0 \leq i \leq n$, with $n \geq 1$, there is an affine linear embedding

$$\delta^i_n \colon \Delta^{n-1} \to \Delta^n ,$$

called the *i-th face map*, that takes $(t_0, \dots, t_{n-1}) \in \Delta^{n-1}$ to

$$\delta^i_n(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) .$$

In other words, it takes the j-th vertex of Δ^{n-1} to the j-th vertex of Δ^n for $0 \leq j < i$, and to the $(j+1)$-th vertex of Δ^n for $i \leq j \leq n-1$. In this way it omits the i-th vertex of Δ^n, and induces the unique order-preserving correspondence between the n vertices of Δ^{n-1} with the remaining n vertices of Δ^{n+1}.

The image of δ^i_n is the subspace of Δ^n where the i-th barycentric coordinate t_i is zero:

$$\delta^i_n(\Delta^{n-1}) = \{(t_0, \dots, t_n) \in \Delta \mid t_i = 0\}$$

We call this part of the boundary of Δ^n the *i-th face*. The topological boundary of Δ^n, as a subspace of the hyperplane in $\mathbb{R}^{n+1}$ where $\sum_{i=0}^{n} t_i = 1$, is the union of these faces:

$$\partial\Delta^n = \bigcup_{i=0}^{n} \delta^i_n(\Delta^{n-1}) .$$

Let X be any topological space. A map (= a continuous function) $\sigma \colon \Delta^n \to X$ is called a *singular n-simplex* in X. Let the *singular n-chains*

$$C_n(X) = \mathbb{Z}\{\sigma \colon \Delta^n \to X\}$$

be the free abelian group generated by the set of singular n-simplices in X. Its elements are finite formal sums

$$\sum_{\sigma} n_\sigma \, \sigma ,$$

where σ ranges over the maps $\Delta^n \to X$, each n_σ is an integer, and only finitely many of the n_σ are different from zero. This abelian group can also be written as the direct sum

$$C_n(X) = \bigoplus_{\sigma\colon \Delta^n \to X} \mathbb{Z}$$

of one copy of the integers for each singular n-simplex.

For each singular n-simplex $\sigma\colon \Delta^n \to X$, and each face map $\delta^i_n\colon \Delta^{n-1} \to \Delta^n$, the composite map

$$\sigma\delta^i_n = \sigma \circ \delta^i_n\colon \Delta^{n-1} \to X$$

is a singular $(n-1)$-simplex in X. Under the identification of Δ^{n-1} with the i-th face in the boundary of Δ^n, we can think of $\sigma\delta^i_n$ as the restriction of σ to that subspace. We call this $(n-1)$-simplex the *i-th face of* σ, and use one of the notations

$$\sigma|[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n] = \sigma|[v_0, \dots, \hat{v}_i, \dots, v_n]\,,$$

where the "hat" indicates a term to be omitted.

The restriction of σ to the boundary of Δ^n is not itself a simplex, but $\partial\Delta^n$ is covered by the $(n+1)$ faces $\delta^i_n(\Delta^{n-1})$, and we define the boundary of σ as a sum of the corresponding faces $\sigma\delta^i_n$. For reasons having to do with the ordering of the vertices of a simplex, or more precisely, with the orientation of a simplex, it turns out to be best to make this an alternating sum, with the i-th face taken with the sign $(-1)^i$.

For each singular n-simplex $\sigma\colon \Delta^n \to X$ in X, with $n \ge 1$, let the *boundary* $\partial\sigma$ be the singular $(n-1)$-chain

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma\delta^i_n = \sum_{i=0}^{n} (-1)^i \sigma|[v_0, \dots, \hat{v}_i, \dots, v_n]\,.$$

More generally, define the boundary homomorphism

$$\partial\colon C_n(X) \to C_{n-1}(X)$$

to be the additive extension of this rule, so that

$$\partial(\sum_\sigma n_\sigma\, \sigma) = \sum_\sigma n_\sigma\, \partial\sigma\,.$$

It is then a consequence of the relation

$$\delta^j_{n+1} \circ \delta^i_n = \delta^i_{n+1} \circ \delta^{j-1}_n\colon \Delta^{n-1} \to \Delta^{n+1}$$

for $0 \le i < j \le n+1$ (both maps omit the i-th and j-th vertices), that $\partial^2 = 0\colon C_{n+1}(X) \to C_{n-1}(X)$. Hence the diagram

$$\dots \to C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \dots \to C_0(X) \to 0 \to \dots$$

is a chain complex, called the *singular chain complex of X*. By convention, $C_n(X) = 0$ for $n < 0$.

We consider the group

$$B_n(X) = \operatorname{im}(\partial\colon C_{n+1}(X) \to C_n(X))$$

of *singular n-boundaries*, and the group

$$Z_n(X) = \ker(\partial\colon C_n(X) \to C_{n-1}(X))$$

of *singular n-cycles*, both of which are subgroups of $C_n(X)$, and call the quotient group

$$H_n(X) = \frac{Z_n(X)}{B_n(X)}$$

the *n-th singular homology group of X*.

2.4 Tensor product and Hom-groups

Let G and A be abelian groups. The *tensor product* $G \otimes A$ is the abelian group generated by symbols $g \otimes a$, with $g \in G$ and $a \in A$, subject to the bilinearity relations

$$(g + g') \otimes a = g \otimes a + g' \otimes a$$

and

$$g \otimes (a + a') = g \otimes a + g \otimes a'$$

for $g, g' \in G$ and $a, a' \in A$. The Hom-*group* $\mathrm{Hom}(A, G)$ is the abelian group of group homomorphisms $f\colon A \to G$, with the group operation given by pointwise addition:

$$(\varphi + \varphi')(a) = \varphi(a) + \varphi'(a)$$

for $\varphi, \varphi'\colon A \to G$, $a \in A$. The sum $\varphi + \varphi'$ is a group homomorphism since G is abelian.

If $f\colon A \to B$ is a homomorphism of abelian groups, then there are induced homomorphisms

$$f_* = 1 \otimes f\colon G \otimes A \to G \otimes A$$

given by $f_*(g \otimes a) = g \otimes f(a)$ for $g \in G$, $a \in A$, and

$$f^* = \mathrm{Hom}(f, 1)\colon \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$$

given by $f^*(\psi)(a) = \psi(f(a))$ for $\psi\colon B \to G$, $a \in A$. Note how the direction of the map f^* is reversed, compared to that of f and f_*. If $g\colon B \to C$ is a second homomorphism, then we have the relations

$$(gf)_* = g_* f_*\colon G \otimes A \to G \otimes A$$

and

$$(gf)^* = f^* g^*\colon \mathrm{Hom}(C, G) \to \mathrm{Hom}(A, G)\,,$$

saying that $G \otimes (-)$ is a covariant functor and $\mathrm{Hom}(-, G)$ is a contravariant functor (in the indicated variable).

If $A = \mathbb{Z}$, then there is a natural isomorphism $G \otimes \mathbb{Z} \cong G$, taking $g \otimes n$ to the multiple ng formed in the group G. (This is the sum of n copies of g if $n \geq 0$, and of $-n$ copies of $-g$ if $n \leq 0$.) More generally, if $A = \mathbb{Z}\{S\} = \bigoplus_S \mathbb{Z}$ is the free abelian group generated by a set S, then

$$G \times \mathbb{Z}\{S\} \cong G\{S\} \cong \bigoplus_S G$$

is the direct sum of one copy of G for each element of S.

If $A = \mathbb{Z}$, then there is a natural isomorphism $\mathrm{Hom}(\mathbb{Z}, G) \cong G$, taking $\varphi\colon \mathbb{Z} \to G$ to the value $\varphi(1)$ at $1 \in \mathbb{Z}$. If $A = \mathbb{Z}\{S\} = \bigoplus_S \mathbb{Z}$ then

$$\mathrm{Hom}(\mathbb{Z}\{S\}, G) \cong \prod_S G$$

is the product of one copy of G for each element of S. A homomorphism $\varphi\colon \mathbb{Z}\{S\} \to G$ corresponds to the sequence $(\varphi(s))_{s\in S}$ in G, of values of φ at the generators $s \in S$ viewed as elements of $\mathbb{Z}\{S\}$.

2.5 Homology with coefficients

Let X be any topological space and G any abelian group. The *singular chain complex of X with coefficients in G* is the diagram

$$\dots \to G \otimes C_{n+1}(X) \xrightarrow{1\otimes\partial} G \otimes C_n(X) \xrightarrow{1\otimes\partial} G \otimes C_{n-1}(X) \to \dots .$$

Here $(1 \otimes \partial)(1 \otimes \partial) = 1 \otimes \partial^2 = 0$, by functoriality, so this is indeed a chain complex. Note that

$$G \otimes C_n(X) \cong G\{\sigma\colon \Delta^n \to X\} \cong \bigoplus_{\sigma\colon \Delta^n \to X} G$$

is the direct sum of one copy of the group G for each singular n-simplex. Its elements are finite formal sums

$$\sum_\sigma g_\sigma\, \sigma\,,$$

where σ ranges over the singular n-simplices in X, each g_σ is an element of G, and only finitely many of them are nonzero.

We also use the notations $C_n(X; G) = G \otimes C_n(X)$,

$$B_n(X; G) = \mathrm{im}(1 \otimes \partial\colon C_{n+1}(X; G) \to C_n(X; G))$$

and

$$Z_n(X; G) = \ker(1 \otimes \partial\colon C_n(X; G) \to C_{n-1}(X; G))$$

for the *singular n-chains, n-boundaries and n-cycles in X with coefficients in G*, respectively. We often abbreviate $1 \otimes \partial$ to ∂. By definition, the *n-th singular homology group of X with coefficients in G* is the quotient group

$$H_n(X; G) = \frac{Z_n(X; G)}{B_n(X; G)} = H_n(C_*(X; G), \partial)\,.$$

For example, let $X = \star$ be a single point. Then there is a unique singular n-simplex $\sigma_n\colon \Delta^n \to \star$ for each $n \geq 0$, so $C_n(\star) = \mathbb{Z}\{\sigma_n\}$ and $C_n(\star; G) = G\{\sigma_n\}$ for each $n \geq 0$. We have $\sigma_n \delta^i_n = \sigma_{n-1}$ for each $0 \leq i \leq n$, $n \geq 1$, so $\partial\sigma_n = \sum_{i=0}^n (-1)^i \sigma_{n-1}$ equals σ_{n-1} for $n \geq 2$ even, and equals 0 for $n \geq 1$ odd. Hence $C_*(\star; G)$ appears as follows:

$$\dots \xrightarrow{1} G\{\sigma_3\} \xrightarrow{0} G\{\sigma_2\} \xrightarrow{1} G\{\sigma_1\} \xrightarrow{0} G\{\sigma_0\} \to 0$$

The boundary homomorphisms labeled 1 are isomorphisms and the ones labeled 0 are trivial. Hence $B_n(\star; G)$ equals $G\{\sigma_n\}$ for $n \ge 1$ odd, and is zero otherwise, while $Z_n(\star; G)$ equals $G\{\sigma_n\}$ for $n \ge 1$ odd, or for $n = 0$, and is zero otherwise. Thus for $n \ne 0$ we have $B_n(\star; G) = Z_n(\star; G)$ and $H_n(\star; G) = 0$. In the case $n = 0$ we have

$$H_0(\star; G) = Z_0(\star; G)/B_0(\star; G) = G\{\sigma_0\}/0 \cong G\,.$$

Let $f\colon X \to Y$ be any map of topological spaces. There is an induced chain map

$$f_\# = C_*(f; G)\colon C_*(X; G) \to C_*(Y; G)$$

given by the formula

$$f_\#(\sum_\sigma g_\sigma\,\sigma) = \sum_\sigma g_\sigma\, f\sigma\,.$$

Here $\sigma\colon \Delta^n \to X$ ranges over the singular n-simplices of X, and the composite

$$f\sigma\colon \Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$$

is an n-simplex of Y. This is a chain map because the associativity of composition, $(f\sigma)\delta^i_n = f(\sigma\delta^i_n)$, implies that $\partial(f\sigma) = f(\partial\sigma)$. Hence there is an induced homomorphism of homology groups,

$$f_* = H_n(f; G)\colon H_n(X; G) \to H_n(Y; G)$$

for all n. If $g\colon Y \to Z$ is a second map, then the relation

$$(gf)_* = g_* f_*\colon H_n(X; G) \to H_n(Z; G)$$

holds.

((Discuss augmentations and reduced homology.))

Remark 2.5.1. When $G = \mathbb{Z}$ is the group of all integers, we usually omit it from the notation. Hence $C_*(X) = C_*(X; \mathbb{Z})$ and $H_*(X) = H_*(X; \mathbb{Z})$. We may refer to these as *integral* chains and homology groups. Notice that $C_*(X; G) = G \otimes C_*(X)$. It is not true in general that $H_*(X; G) = G \otimes H_*(X)$. We shall return to this in the context of the universal coefficient theorem for homology.

Remark 2.5.2. It is more common to define $C_n(X; G)$ as the tensor product $C_n(X) \otimes G$, with G on the right hand side. The two tensor products are isomorphic, so this is only a matter of conventions. If we think of $C_n(X)$ as a left $\mathbb{Z}$-module, it is a little more natural to form the tensor product $G \otimes C_n(X)$, where G is thought of as a right $\mathbb{Z}$-module, than to convert the left module structure on $C_n(X)$ into a right module structure. Since $\mathbb{Z}$ is commutative, there is no real difference, but our convention generalizes more easily to the case of modules over non-commutative rings R, where we can make an identification $C_n(X; N) \cong N \otimes_R C_n(X; R)$ for right R-modules N. The homology group $\tilde{H}_n(X; G)$ can also be presented as the homotopy group $\pi_n(HG \wedge X)$ of the Eilenberg–Mac Lane spectrum HG smashed with the space X, and here it is common to place HG on the left hand side.

2.6 Relative homology

Let $A \subseteq X$ be any subspace. Write $i\colon A \to X$ for the inclusion map. The chain map $i_\#\colon C_*(A;G) \to C_*(X;G)$ is injective in each degree, identifying each simplex $\sigma\colon \Delta^n \to A$ with the composite $i\sigma\colon \Delta^n \to X$. Let the group of *relative n-chains in (X,A) with coefficients in G* be the quotient group

$$C_n(X,A;G) = \frac{C_n(X;G)}{C_n(A;G)}$$

of n-chains in X modulo the n-chains in A. Since $i_\#$ is a chain map, there is an induced boundary homomorphism

$$\partial\colon C_n(X,A;G) \to C_{n-1}(X,A;G)$$

given by taking the equivalence class of an n-chain x in X modulo n-chains in A to the equivalence class of the $(n-1)$-chain ∂x in X modulo $(n-1)$-chains in A. Since $\partial^2 = 0$ in $C_*(X;G)$, we must have $\partial^2 = 0$ in $C_*(X,A;G)$, so $(C_*(X,A;G),\partial)$ is a chain complex. We write

$$B_n(X,A;G) = \mathrm{im}(\partial\colon C_{n+1}(X,A;G) \to C_n(X,A;G))$$

and

$$Z_n(X,A;G) = \ker(\partial\colon C_n(X,A;G) \to C_{n-1}(X,A;G))$$

like before, and define the *n-th singular homology group of the pair (X,A) with coefficients in G* to be the quotient group

$$H_n(X,A;G) = \frac{Z_n(X,A;G)}{B_n(X,A;G)} = H_n(C_*(X,A;G),\partial)\,.$$

Let $j_\#\colon C_n(X;G) \to C_n(X,A;G)$ be the canonical quotient homomorphism. Then $j_\#$ is a chain map. Drawing the chain complexes vertically and the chain maps horizontally, we have a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial"'] & \vdots \arrow[d, "\partial"'] & \vdots \arrow[d, "\partial"'] & \\
0 \arrow[r] & C_n(A;G) \arrow[r, "i_\#"] \arrow[d, "\partial"'] & C_n(X;G) \arrow[r, "j_\#"] \arrow[d, "\partial"'] & C_n(X,A;G) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{n-1}(A;G) \arrow[r, "i_\#"] \arrow[d, "\partial"'] & C_{n-1}(X;G) \arrow[r, "j_\#"] \arrow[d, "\partial"'] & C_{n-1}(X,A;G) \arrow[r] \arrow[d, "\partial"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with exact rows. We usually draw this more compactly as the following short exact sequence of chain complexes:

$$0 \to C_*(A;G) \xrightarrow{i_\#} C_*(X;G) \xrightarrow{j_\#} C_*(X,A;G) \to 0$$

Note that if $A = \varnothing$ is empty, then $j_\#$ is an isomorphism of chain complexes $C_*(X;G) \cong C_*(X,\varnothing;G)$, and j_* is an isomorphism $H_n(X;G) \cong H_n(X,\varnothing;G)$ for all n, so (absolute) homology is a special case of relative homology.

There is a *connecting homomorphism in homology*

$$\partial\colon H_n(X,A;G) \to H_{n-1}(A;G)$$

defined by taking the homology class $[x]$ of a relative n-cycle $x \in Z_n(X,A;G)$ to the homology class $[\partial\tilde{x}]$ of the unique lift to $C_{n-1}(A;G)$ of the boundary in $C_{n-1}(X;G)$ of a representative $\tilde{x}$ in $C_n(X;G)$ of x. Here $j_\#(\tilde{x}) = x$, so the lift exists because $j_\#(\partial\tilde{x}) = \partial x = 0$ in $C_{n-1}(X,A;G)$. It is an $(n-1)$-cycle, since its boundary in $C_{n-2}(A;G)$ maps to $\partial^2\tilde{x} = 0$ in $C_{n-2}(X;G)$ under the injective homomorphism $i_\#$. As we proved earlier, ∂ is well-defined and additive.

Exercise 2.6.1. Express the group of relative n-cycles $Z_n(X,A;G)$ as a quotient of the group $\{x \in C_n(X;G) \mid \partial x \in C_{n-1}(A;G)\}$ of n-chains in X with boundary in A.

Similarly, express the group of relative n-boundaries $B_n(X,A;G)$ as a quotient of the group $\{\partial y \in C_n(X;G) \mid y \in C_{n+1}(X;G)\}$ of n-chains in X that are boundaries of chains in X.

We refer to the pair (X,A), with A a subspace of X, as a *pair of spaces*. Let $f\colon (X,A) \to (Y,B)$ be any *map of pairs of spaces*. This is a map $f\colon X \to Y$, subject to the condition that $f(A) \subseteq B$, so that we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
B \arrow[r, "i"] & Y
\end{tikzcd}$$

where f' denotes the restriction of f of A, corestricted to B. Then we have a commutative diagram of chain complexes and chain maps

$$\begin{tikzcd}
0 \arrow[r] & C_*(A;G) \arrow[r, "i_\#"] \arrow[d, "f'_\#"'] & C_*(X;G) \arrow[r, "j_\#"] \arrow[d, "f_\#"'] & C_*(X,A;G) \arrow[r] \arrow[d, "f''_\#"'] & 0 \\
0 \arrow[r] & C_*(B;G) \arrow[r, "i_\#"] & C_*(Y;G) \arrow[r, "j_\#"] & C_*(Y,B;G) \arrow[r] & 0
\end{tikzcd}$$

where the left hand square is induced by the square above, and the rows are short exact sequences of chain complexes. The chain map $f''_\#$ on the right hand side is then determined by $f_\#$ by the passage to a quotient.

In particular, we have an induced homomorphism

$$f_* = H_n(f;G)\colon H_n(X,A;G) \to H_n(Y,B;G)$$

for each n. If $g\colon (Y,B) \to (Z,C)$ is a second map of pairs of spaces, then $(gf)_* = g_*f_*$. Under the isomorphism $C_*(X;G) \cong C_*(X,\varnothing;G)$ we can identify $j_\#\colon C_*(X;G) \to C_*(X,A;G)$ with the chain map $j''_\#$ for j equal to the map of pairs $(X,\varnothing) \to (X,A)$ given by the identity on X.

The connecting homomorphism $\partial\colon H_n(X, A; G) \to H_{n-1}(A; G)$ is *natural*, in the sense that for any map of pairs $f\colon (X, A) \to (Y, B)$ the diagram

$$\begin{tikzcd} H_n(X, A; G) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(A; G) \arrow[d, "f'_*"'] \\ H_n(Y, B; G) \arrow[r, "\partial"] & H_{n-1}(B; G) \end{tikzcd}$$

commutes.

Remark 2.6.2. We briefly write $C_*(X, A)$ and $H_*(X, A)$ for the relative chains and homology groups with integral coefficients. Each short exact sequence

$$0 \to C_n(A) \xrightarrow{i_n} C_n(X) \to C_n(X, A) \to 0$$

admits a splitting, since $C_n(X, A)$ is the free abelian group on the set of n-simplices in X that do not lie in A, hence there is a short exact sequence

$$0 \to G \otimes C_n(A) \xrightarrow{1 \otimes i_n} G \otimes C_n(X) \to G \otimes C_n(X, A) \to 0$$

and we have an isomorphism $C_n(X, A; G) \cong G \otimes C_n(X, A)$. It is compatible with the boundary maps, so that $G \otimes C_*(X, A) \cong C_*(X, A; G)$. Again, this does not generally imply that $G \otimes H_*(X, A) \cong H_*(X, A; G)$.

2.7 Some categorical language

Definition 2.7.1. A *category* $\mathscr{C}$ consists of

(a) a class of *objects*, denoted $\mathrm{obj}\,\mathscr{C}$,

(b) for each pair of objects X and Y, a set of *morphisms* from X to Y, denoted $\mathscr{C}(X, Y)$,

(c) for each triple of objects X, Y and Z, a *composition* rule

$$\circ\colon \mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \longrightarrow \mathscr{C}(X, Y)$$

taking (g, f) to $g \circ f$, for each $g \in \mathscr{C}(Y, Z)$ and $f \in \mathscr{C}(X, Y)$, and

(d) for each object X, an *identity* morphism

$$1_X \in \mathscr{C}(X, X)\,.$$

The composition rule is supposed to satisfy the *associative law*

$$h \circ (g \circ f) = (h \circ g) \circ f$$

in $\mathscr{C}(X, W)$, for X, Y, Z, f and g as above, W another object in $\mathscr{C}$, and $h \in \mathscr{C}(Z, W)$ another morphism in $\mathscr{C}$, and the *left and right unital laws*

$$1_Y \circ f = f = f \circ 1_X$$

in $\mathscr{C}(X, Y)$, for X, Y and f as above.

We often write $f\colon X \to Y$ to denote a morphism $f \in \mathscr{C}(X,Y)$ from X to Y, when the category $\mathscr{C}$ is implicitly understood. We often abbreviate $g \circ f$ to gf, and expressions like hgf are meaningful, by the associative law.

A category is a context where it is meaningful to talk about commutative diagrams, like the following:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \arrow[dr, "hg"] & \\ & Z \arrow[r, "h"'] & W \end{tikzcd}$$

Definition 2.7.2. A morphism $f\colon X \to Y$ in a category $\mathscr{C}$ is an *isomorphism* if there exists a morphism $g\colon Y \to X$ in $\mathscr{C}$ with $gf = 1_X$ and $fg = 1_Y$. In this case we say that g is an *inverse* of f, and that X and Y are *isomorphic*. Any morphism has at most one inverse, so we can talk about "the inverse", when one exists. We often label an isomorphism with the symbol $\cong$, as in

$$f\colon X \xrightarrow{\cong} Y .$$

A category in which the class of objects $\operatorname{obj}\mathscr{C}$ is a set (in the interpretation of set theory that is used) is called a *small category*. A category where each morphism is an isomorphism is called a *groupoid*.

Example 2.7.3. The category **Ab** has the collection of all abelian groups as its class of objects, and the set $\mathbf{Ab}(A,B)$ of group homomorphisms $f\colon A \to B$ as the set of morphisms from A to B. Composition and identity morphisms are defined as for sets and functions. An isomorphism $f\colon A \to B$ in this category is the same as a group isomorphism.

Example 2.7.4. The category **GrAb** has the collection of all graded abelian groups as its class of objects, and the set $\mathbf{GrAb}(A_*, B_*)$ of degree-preserving group homomorphisms $f_\#\colon A_* \to B_*$ as the set of morphisms from A_* to B_*. More explicitly, a graded abelian group A_* is a sequence $(A_n)_{n\in\mathbb{Z}}$ of abelian groups, and a degree-preserving group homomorphism $f_\#\colon A_* \to B_*$ is a sequence $(f_n\colon A_n \to B_n)_{n\in\mathbb{Z}}$ of group homomorphisms.

Example 2.7.5. The category **Top** has the collection of all topological spaces as its class of objects, and the set $\mathbf{Top}(X,Y)$ of continuous functions $f\colon X \to Y$ as the set of morphisms from X to Y. Composition and identity morphisms are defined as for sets and functions.

Example 2.7.6. The category **TopPair** has the collection of all pairs of topological spaces as its class of objects, and the set $\mathbf{TopPair}((X,A),(Y,B))$ of maps of pairs $f\colon (X,A) \to (Y,B)$ as the set of morphisms from (X,A) to (Y,B). An isomorphism in this category is the same as a homeomorphism $f\colon X \to Y$ that restricts to a homeomorphism $f'\colon A \to B$.

Definition 2.7.7. A (covariant) *functor* F from a category $\mathscr{C}$ to a category $\mathscr{D}$ consists of

(a) a rule that to each object X of $\mathscr{C}$ assigns an object $F(X)$ of $\mathscr{D}$, and

(b) a rule that to each morphism $f\colon X \to Y$ in $\mathscr{C}$ assigns a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathscr{D}$.

The rule on morphisms is supposed to satisfy the covariant composition law

$$F(g \circ f) = F(g) \circ F(f)$$

in $\mathscr{D}(F(X), F(Z))$, for each composable pair of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathscr{C}$, and the identity law

$$F(1_X) = 1_{F(X)}$$

in $\mathscr{D}(F(X), F(X))$, for each object X in $\mathscr{C}$.

We often write $F\colon \mathscr{C} \to \mathscr{D}$ to denote a functor F from $\mathscr{C}$ to $\mathscr{D}$. When the functor is understood, we often abbreviate $F(f)$ to f_*, in which case the composition law appears as follows:

$$(gf)_* = g_* f_*$$

A functor is a rule that can be applied to commutative diagrams, producing new commutative diagrams. ((Elaborate?))

Example 2.7.8. Let G be an abelian group. There is a (covariant) functor $G \otimes (-)$ from **Ab** to **Ab**, denoted

$$G \otimes (-)\colon \mathbf{Ab} \longrightarrow \mathbf{Ab}\,,$$

that takes an abelian group A to the tensor product $G \otimes A$, and takes a homomorphism $f\colon A \to B$ of abelian groups to the homomorphism $f_* = 1 \otimes f\colon G \otimes A \to G \otimes B$.

Example 2.7.9. Let G be an abelian group. There is a (covariant) functor $H_*(-,-;G)$ from **TopPair** to **GrAb**, denoted

$$H_*(-,-;G)\colon \mathbf{TopPair} \longrightarrow \mathbf{GrAb}$$

or

$$(X, A) \longmapsto H_*(X, A; G)\,,$$

that takes a pair of spaces (X, A) to the graded abelian group $H_*(X, A; G)$, with $H_n(X, A; G)$ in degree n for each integer n, and takes a map $f\colon (X, A) \to (Y, B)$ of pairs of spaces to the induced homomorphism $f_* = H_*(f; G)\colon H_*(X, A; G) \to H_*(Y, B; G)$ of graded abelian groups.

Example 2.7.10. Let G be an abelian group. There is a second (covariant) functor from **TopPair** to **GrAb**, denoted

$$(X, A) \longmapsto H_{*-1}(A; G)\,,$$

that takes a pair of spaces (X, A) to the graded abelian group $H_{*-1}(A; G)$, with $H_{n-1}(A; G)$ in degree n for each integer n, and takes a map $f\colon (X, A) \to (Y, B)$ of pairs of spaces to the induced homomorphism

$$f_* = H_{*-1}(f'; G)\colon H_{*-1}(A; G) \to H_{*-1}(B; G)$$

of graded abelian groups. Here $f'\colon A \to B$ denotes the restriction of f to A, as before.

Definition 2.7.11. A *contravariant functor* F from a category $\mathscr{C}$ to a category $\mathscr{D}$ consists of

(a) a rule that to each object X of $\mathscr{C}$ assigns an object $F(X)$ of $\mathscr{D}$, and

(b) a rule that to each morphism $f\colon X \to Y$ in $\mathscr{C}$ assigns a morphism $F(f)\colon F(Y) \to F(X)$ in $\mathscr{D}$.

The rule on morphisms is supposed to satisfy the contravariant composition law

$$F(g \circ f) = F(f) \circ F(g)$$

in $\mathscr{D}(F(Z), F(X))$, for each composable pair of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathscr{C}$, and the identity law

$$F(1_X) = 1_{F(X)}$$

in $\mathscr{D}(F(X), F(X))$, for each object X in $\mathscr{C}$.

Note that F changes the direction of the arrows. When the contravariant functor is understood, we often abbreviate $F(f)$ to f^*, in which case the composition law appears as follows:

$$(gf)^* = f^* g^*$$

Definition 2.7.12. To each category $\mathscr{C}$, we can associate the *opposite category* $\mathscr{C}^{op}$, that consists of

(a) the same class of objects as in $\mathscr{C}$, so that $\operatorname{obj}\mathscr{C}^{op} = \operatorname{obj}\mathscr{C}$,

(b) for each pair of objects X and Y, a set of morphisms in $\mathscr{C}^{op}$ from X to Y that equals the set of morphisms $\mathscr{C}(Y, X)$ from Y to X in $\mathscr{C}$, so $\mathscr{C}^{op}(X, Y) = \mathscr{C}(Y, X)$,

(c) a composition rule in $\mathscr{C}^{op}$

$$\circ\colon \mathscr{C}^{op}(Y, Z) \times \mathscr{C}^{op}(X, Y) \longrightarrow \mathscr{C}^{op}(X, Y)$$

that equals the composite

$$\mathscr{C}(Z, Y) \times \mathscr{C}(Y, X) \cong \mathscr{C}(Y, X) \times \mathscr{C}(Z, Y) \longrightarrow \mathscr{C}(Z, X)\,,$$

where the first bijection interchanges the two factors, and the second function is given by the composition rule in $\mathscr{C}$, and

(d) identity morphisms

$$1_X \in \mathscr{C}^{op}(X, X)$$

that equal the identity morphisms $1_X \in \mathscr{C}(X, X)$ in $\mathscr{C}$.

With this definition, there is then a one-to-one correspondence between contravariant functors F from $\mathscr{C}$ to $\mathscr{D}$ and covariant functors G from $\mathscr{C}^{op}$ to $\mathscr{D}$. At the level of objects we have $G(X) = F(X)$ for all objects X of $\mathscr{C}$ and $\mathscr{C}^{op}$, while at the level of morphisms we have $G(f) = F(f)$ in $\mathscr{C}(X, Y) = \mathscr{C}^{op}(Y, X)$ for all morphisms $f\colon X \to Y$ in $\mathscr{C}$, which are morphisms from Y to X in $\mathscr{C}^{op}$. Hence we usually display a contravariant functor F from $\mathscr{C}$ to $\mathscr{D}$ as a (covariant) functor $F\colon \mathscr{C}^{op} \to \mathscr{D}$.

Example 2.7.13. Let G be an abelian group. There is a contravariant functor $\mathrm{Hom}(-, G)$ from **Ab** to **Ab**, denoted

$$\mathrm{Hom}(-, G) \colon \mathbf{Ab}^{op} \longrightarrow \mathbf{Ab}\,,$$

that takes an abelian group A to the Hom-group $\mathrm{Hom}(A, G)$, and takes a homomorphism $f \colon A \to B$ of abelian groups to the homomorphism $f^* = \mathrm{Hom}(f, G) \colon \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$.

Example 2.7.14. Let G be an abelian group. There is a contravariant functor $H^*(-, -; G)$ from **TopPair** to **GrAb**, denoted

$$H^*(-, -; G) \colon \mathbf{TopPair}^{op} \longrightarrow \mathbf{GrAb}$$

or

$$(X, A) \longmapsto H^*(X, A; G)\,,$$

that takes a pair of spaces (X, A) to the graded abelian group $H^*(X, A; G)$, with $H^n(X, A; G)$ in degree n for each integer n, and takes a map $f \colon (X, A) \to (Y, B)$ of pairs of spaces to the induced homomorphism $f^* = H^*(f; G) \colon H^*(Y, B; G) \to H^*(X, A; G)$ of graded abelian groups.

Example 2.7.15. Let G be an abelian group. There is a second contravariant functor from **TopPair** to **GrAb**, denoted

$$(X, A) \longmapsto H^{*+1}(A; G)\,,$$

that takes a pair of spaces (X, A) to the graded abelian group $H^{*+1}(A; G)$, with $H^{n+1}(A; G)$ in degree n for each integer n, and takes a map $f \colon (X, A) \to (Y, B)$ of pairs of spaces to the induced homomorphism

$$f^* = H^{*+1}(f'; G) \colon H^{*+1}(B; G) \to H^{*+1}(A; G)$$

of graded abelian groups. Here $f' \colon A \to B$ denotes the restriction of f to A, as before.

((Discuss product categories and bifunctors.))

Definition 2.7.16. Let $\mathscr{C}, \mathscr{D}$ be categories, and let $F, G \colon \mathscr{C} \to \mathscr{D}$ be functors from $\mathscr{C}$ to $\mathscr{D}$. A *natural transformation* $\theta \colon F \to G$ consists of a rule that to each object X of $\mathscr{C}$ assigns a morphism

$$\theta_X \colon F(X) \to G(X)$$

in $\mathscr{D}$. This rule is supposed to satisfy the naturality law, that the square

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"] & G(Y) \end{tikzcd}$$

commutes for each morphism $f \colon X \to Y$ in $\mathscr{C}$.

Sometimes a natural transformation is denoted $\theta\colon F \dot{\to} G$ or $\theta\colon F \Rightarrow G$. In abbreviated notation, the naturality law can appear as $f_*\theta_X = \theta_Y f_*$ for $f\colon X \to Y$. A natural transformation is a rule that can compare two functors. ((Elaborate?))

Example 2.7.17. The connecting homomorphism

$$\partial = \partial_{(X,A)}\colon H_*(X, A; G) \longrightarrow H_{*-1}(A; G)$$

defines a natural transformation between the functors $(X, A) \mapsto H_*(X, A; G)$ and $(X, A) \mapsto H_{*-1}(A; G)$ from **TopPair** to **GrAb**.

Example 2.7.18. The connecting homomorphism

$$\delta = \delta_{(X,A)}\colon H^*(X, A; G) \longrightarrow H^{*+1}(A; G)$$

defines a natural transformation between the functors $(X, A) \mapsto H^*(X, A; G)$ and $(X, A) \mapsto H^{*+1}(A; G)$ from $\mathbf{TopPair}^{op}$ to **GrAb**.

Definition 2.7.19. Given categories $\mathscr{C}, \mathscr{D}$, functors $F, G, H\colon \mathscr{C} \to \mathscr{D}$ and natural transformations $\theta\colon F \to G$ and $\eta\colon G \to H$, there is a *composite natural transformation* $\eta\theta\colon F \to H$ that takes each object X in $\mathscr{C}$ to the composite morphism

$$(\eta\theta)_X = \eta_X\theta_X\colon F(X) \to G(X) \to H(X)$$

in $\mathscr{D}$.

There is an obvious *identity transformation* $1_F\colon F \to F$ for each functor $F\colon \mathscr{C} \to \mathscr{D}$. A natural transformation θ admits a left and right inverse under composition if and only if each morphism

$$\theta_X\colon F(X) \xrightarrow{\cong} G(X)$$

is an isomorphism in $\mathscr{D}$. Such θ are called *natural equivalences* or *natural isomorphisms.*

Example 2.7.20. There is a contravariant functor $(-)^* = \mathrm{Hom}(-, \mathbb{Z})\colon \mathbf{Ab}^{op} \to \mathbf{Ab}$, taking each abelian group A to $A^* = \mathrm{Hom}(A, \mathbb{Z})$. When A is finitely generated and free, say $A \cong \mathbb{Z}\{S\}$ for some finite set S, there are group isomorphisms

$$A^* = \mathrm{Hom}(A, \mathbb{Z}) \cong \prod_S \mathbb{Z} \cong \bigoplus_S \mathbb{Z} \cong A\,.$$

However, these isomorphisms depend on the choice of basis S for A, and there is no way to extend these isomorphisms to preferred homomorphisms for general abelian groups A. In particular, there is no natural transformation connecting the contravariant functor $\mathrm{Hom}(-, \mathbb{Z})$ to the (covariant) identity functor $1_{\mathbf{Ab}}\colon \mathbf{Ab} \to \mathbf{Ab}$. ((We have not even defined such a notion of natural transformation.)) On the other hand, the composite functor

$$(-)^{**} = \mathrm{Hom}(\mathrm{Hom}((-), \mathbb{Z}), \mathbb{Z})\colon \mathbf{Ab} \to \mathbf{Ab}\,,$$

taking A to $A^{**} = \mathrm{Hom}(\mathrm{Hom}(A, \mathbb{Z}), \mathbb{Z})$, is a (covariant) functor. In this case there is a natural transformation

$$\rho\colon 1_{\mathbf{Ab}} \to (-)^{**}$$

of functors $\mathbf{Ab} \to \mathbf{Ab}$, taking A to $\rho_A \colon A \to A^{**}$, where ρ_A takes $x \in A$ to the homomorphism $\rho(x) \colon \operatorname{Hom}(A, \mathbb{Z}) \to \mathbb{Z}$ in A^{**} that maps $\varphi \colon A \to \mathbb{Z}$ to $\varphi(x) \in \mathbb{Z}$. When restricted to the subcategory of abelian groups that are finitely generated and free, ρ becomes a natural isomorphism.

Definition 2.7.21. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $F \colon \mathscr{C} \to \mathscr{D}$ is called an *isomorphism of categories* if there exists a functor $G \colon \mathscr{D} \to \mathscr{C}$ such that $GF = 1_{\mathscr{C}}$ and $FG = 1_{\mathscr{D}}$. We then call G an *inverse isomorphism* of categories, and say that F and G are *isomorphic categories*.

An isomorphism of categories $F \colon \mathscr{C} \xrightarrow{\cong} G$ induces, in particular, a one-to-one correspondence between the class of objects of $\mathscr{C}$ and the class of objects of $\mathscr{D}$. This is a rare occurrence. The following more flexible notion of equivalence occurs more commonly.

Definition 2.7.22. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A functor $F \colon \mathscr{C} \to \mathscr{D}$ is called an *equivalence of categories* if there exists a $1_{\mathscr{C}} \xrightarrow{\cong} GF$ of functors $\mathscr{C} \to \mathscr{C}$, and a natural isomorphism $1_{\mathscr{D}} \xrightarrow{\cong} FG$ of functors $\mathscr{D} \to \mathscr{D}$. We then call G an *inverse equivalence* of categories, and say that F and G are *equivalent categories*.

An equivalence of categories $F \colon \mathscr{C} \xrightarrow{\simeq} \mathscr{D}$ induces bijections of morphism sets $\mathscr{C}(X, Y) \xrightarrow{\cong} \mathscr{D}(F(X), F(Y))$, for all objects X, Y in $\mathscr{C}$. Furthermore, each object of $\mathscr{D}$ is isomorphic to one of the form $F(X)$, for some object X in $\mathscr{C}$. We say that F is *faithfully full and essentially surjective*, and this is equivalent to F being an equivalence of categories.

((Relevant example?))

2.8 The Eilenberg–Steenrod axioms for homology

We take the following theorem as known, at least in the case $G = \mathbb{Z}$, from the first course in algebraic topology.

Theorem 2.8.1 (Eilenberg–Steenrod axioms)**.** *Let G be a fixed abelian group and let n range over all integers. We abbreviate $H_n(X, \varnothing; G)$ to $H_n(X; G)$.*

(Functoriality) The rule that takes a pair of spaces (X, A) to $H_n(X, A; G)$, and a map $f \colon (X, A) \to (Y, B)$ to the homomorphism $f_* \colon H_n(X, A; G) \to H_n(Y, B; G)$, defines a covariant functor from pairs of spaces to graded abelian groups.

(Naturality) The rule that takes a pair of spaces (X, A) to the connecting homomorphism $\partial \colon H_n(X, A; G) \to H_{n-1}(A; G)$ is a natural transformation.

(Long exact sequence) The natural diagram

$$\dots \xrightarrow{\partial} H_n(A; G) \xrightarrow{i_*} H_n(X; G) \xrightarrow{j_*} H_n(X, A; G) \xrightarrow{\partial} H_{n-1}(A; G) \xrightarrow{i_*} \dots$$

is a long exact sequence, where i_* is induced by the inclusion $i \colon A \to X$ and j_* is induced by the inclusion $j \colon (X, \varnothing) \to (X, A)$.

(Homotopy invariance) If $f \simeq g \colon (X, A) \to (Y, B)$ are homotopic as maps of pairs, then $f_* = g_* \colon H_n(X, A; G) \to H_n(Y, B; G)$.

(Excision) If $Z \subseteq A \subseteq X$ are subspaces, such that the closure of Z is contained in the interior of A, then the inclusion $(X - Z, A - Z) \to (X, A)$ induces isomorphisms

$$H_n(X - Z, A - Z; G) \xrightarrow{\cong} H_n(X, A; G)\,.$$

(Sum) If $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$ is a disjoint union of pairs of subspaces, then the inclusion maps $(X_\alpha, A_\alpha) \to (X, A)$ induce isomorphisms

$$\bigoplus_\alpha H_n(X_\alpha, A_\alpha; G) \xrightarrow{\cong} H_n(X, A; G)\,.$$

(Dimension) Let $\star$ be a one-point space. Then $H_0(\star; G) = G$ and $H_n(\star; G) = 0$ for $n \in \mathbb{Z}$.

The sum axiom is only interesting for infinite indexing sets, since the case of finite disjoint unions follows from the long exact sequence and excision. The dimension axiom implies that the homology of an n-dimensional disc, relative to its boundary, is concentrated in degree n. Hence for $(X, A) = (D^n, \partial D^n)$ the dimension n can be recovered from the homology groups $H_*(X, A; G)$ (for $G \neq 0$!).

((Motivate the definition of chain homotopy by a comparison with space level homotopies $I \times X \to Y$ and chain maps $C_*(I) \otimes C_*(X) \to C_*(I \times X) \to C_*(Y)$; maybe here, but probably better later.))

Definition 2.8.2. A functor $(X, A) \mapsto h_*(X, A)$ and natural transformation $\partial \colon h_*(X, A) \to h_{*-1}(A)$ satisfying all of the Eilenberg–Steenrod axioms for homology, except the dimension axiom, is called a *generalized homology theory*.

Remark 2.8.3. A (generalized) homology theory is determined for all pairs of spaces (X, A) of the homotopy type of a CW pair, in a sense to be made precise, by these axioms and its values $h_n(\star) = h_n(\star, \varnothing)$ for $n \in \mathbb{Z}$ at the one-point space. If one adds the additional axiom that each weak homotopy equivalence induces an isomorphism of homology groups, as is true for the singular homology groups, then the axioms and the value at a point determine the value at all (pairs of) topological spaces.

((Examples: Bordism, oriented bordism, framed bordism and unreduced stable homotopy, etc.))

2.9 The definition of singular cohomology

Let X be any topological space and G any abelian group. The *singular cochain complex of X with coefficients in G* is the diagram

$$\dots \to \mathrm{Hom}(C_{n-1}(X), G) \xrightarrow{\delta} \mathrm{Hom}(C_n(X), G) \xrightarrow{\delta} \mathrm{Hom}(C_{n+1}(X), G) \to \dots$$

where $\delta = \mathrm{Hom}(\partial, 1)$ is called the *coboundary homomorphism*. Here $\delta^2 = \mathrm{Hom}(\partial^2, 1) = 0$, by contravariant functoriality, so this is indeed a cochain complex. Note that

$$\mathrm{Hom}(C_n(X), G) \cong \prod_{\sigma \colon \Delta^n \to X} G$$

is the product of one copy of the group G for each singular n-simplex. Its elements are functions

$$\varphi\colon \{\sigma\colon \Delta^n \to X\} \longrightarrow G$$

where σ ranges over the singular n-simplices in X, and each value $\varphi(\sigma)$ lies in G. Note that for $\varphi \in C^{n-1}(X;G)$, $\delta\varphi \in C^n(X;G)$ corresponds to the function given by the alternating sum

$$(\delta\varphi)(\sigma) = \varphi(\delta\sigma) = \sum_{i=0}^{n}(-1)^i\varphi(\sigma\delta_n^i) = \sum_{i=0}^{n}(-1)^i\varphi(\sigma|[v_0,\dots,\hat{v}_i,\dots,v_n])\,.$$

We also use the notations $C^n(X;G) = \mathrm{Hom}(C_n(X),G)$,

$$B^n(X;G) = \mathrm{im}(\delta\colon C^{n-1}(X;G) \to C^n(X;G))$$

and

$$Z^n(X;G) = \ker(\delta\colon C^n(X;G) \to C^{n+1}(X;G))$$

for the *singular n-cochains, n-coboundaries and n-cocycles in X with coefficients in G*, respectively. By definition, the *n-th singular cohomology group of X with coefficients in G* is the quotient group

$$H^n(X;G) = \frac{Z^n(X;G)}{B^n(X;G)} = H^n(C^*(X;G),\delta)\,.$$

For example, let $X = \star$ be a single point. Then there is a unique singular n-simplex $\sigma_n\colon \Delta^n \to \star$ for each $n \ge 0$, so $C_n(\star) = \mathbb{Z}\{\sigma_n\}$ and $C^n(\star;G) = G\{\varphi^n\}$ for each $n \ge 0$, where $(g\varphi^n)(\sigma_n) = g$. We have $\sigma_n\delta_n^i = \sigma_{n-1}$ for each $0 \le i \le n$, $n \ge 1$, so $\partial\sigma_n = \sum_{i=0}^n(-1)^i\sigma_{n-1}$ equals σ_{n-1} for $n \ge 2$ even, and equals 0 for $n \ge 1$ odd. Hence $\delta\varphi^n$ equals φ^{n+1} for $n \ge 1$ odd, and equals 0 for $n \ge 0$ even. Hence $C^*(\star;G)$ appears as follows:

$$0 \to G\{\varphi^0\} \xrightarrow{0} G\{\varphi^1\} \xrightarrow{1} G\{\varphi^2\} \xrightarrow{0} G\{\varphi^3\} \to \dots$$

The boundary homomorphisms labeled 1 are isomorphisms and the ones labeled 0 are trivial. Hence $B^n(\star;G)$ equals $G\{\varphi^n\}$ for $n \ge 2$ even, and is zero otherwise, while $Z^n(\star;G)$ equals $G\{\varphi^n\}$ for $n \ge 0$ even, and is zero otherwise. Thus for $n \ne 0$ we have $B^n(\star;G) = Z^n(\star;G)$ and $H^n(\star;G) = 0$. In the case $n = 0$ we have

$$H^0(\star;G) = Z^0(\star;G)/B^0(\star;G) = G\{\varphi^0\}/0 \cong G\,.$$

Let $f\colon X \to Y$ be any map of topological spaces. There is an induced cochain map

$$f^\# = C^*(f;G)\colon C^*(Y;G) \to C^*(X;G)$$

given by the formula

$$(f^\#\varphi)(\sigma) = \varphi(f\sigma)\,.$$

Here $\sigma\colon \Delta^n \to X$ ranges over the singular n-simplices of X, and the composite

$$f\sigma\colon \Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$$

is an n-simplex of Y, so that $\varphi(f\sigma)$ takes values in G. This is a cochain map because the associativity of composition, $(f\sigma)\delta_n^i = f(\sigma\delta_n^i)$, implies that

$\delta(f^\#\sigma) = f^\#(\delta\sigma)$. Hence there is an induced homomorphism of cohomology groups,

$$f^* = H^n(f;G)\colon H^n(Y;G) \to H^n(X;G)$$

for all n. If $g\colon Y \to Z$ is a second map, then the relation

$$(gf)^* = f^*g^*\colon H^n(Z;G) \to H^n(X;G)$$

holds, showing that $H^*(X;G)$ is a contravariant functor of X.

((Discuss coaugmentations and reduced cohomology.))

Remark 2.9.1. When $G = \mathbb{Z}$ is the group of all integers, we usually omit it from the notation. Hence $C^*(X) = C^*(X;\mathbb{Z})$ and $H^*(X) = H^*(X;\mathbb{Z})$. We may refer to these as *integral* cochains and cohomology groups. Notice that $C^*(X;G) = \mathrm{Hom}(C_*(X), G)$. It is not true in general that $H^*(X;G) = \mathrm{Hom}(H_*(X), G)$. We shall return to this in the context of the universal coefficient theorem for cohomology.

2.10 Relative cohomology

Let $A \subseteq X$ be any subspace. Write $i\colon A \to X$ for the inclusion map. The cochain map $i^\#\colon C^*(X;G) \to C^*(A;G)$ is surjective in each degree, restricting each homomorphism $\varphi\colon C_n(X) \to G$ on n-chains in X to the n-chains that happen to lie in A. Let the group of *relative n-cochains in (X, A) with coefficients in G* be the subgroup

$$C^n(X, A; G) = \ker(i^\#\colon C^n(X;G) \to C^n(A;G)) \subseteq C^n(X;G)$$

of n-cochains in X that are zero on all n-chains that lie in A. Since $i^\#$ is a cochain map, there is an induced coboundary homomorphism

$$\delta\colon C^{n-1}(X, A; G) \to C^n(X, A; G)$$

given by taking an $(n-1)$-cochain φ on X that vanishes on A to the n-cochain $\delta\varphi$, which vanishes on A since the boundary of a chain in A still lies in A. Since $\delta^2 = 0$ in $C^*(X;G)$, we must have $\delta^2 = 0$ in $C^*(X, A; G)$, so $(C^*(X, A; G), \delta)$ is a cochain complex. We write

$$B^n(X, A; G) = \mathrm{im}(\delta\colon C^{n-1}(X, A; G) \to C^n(X, A; G))$$

and

$$Z^n(X, A; G) = \ker(\delta\colon C^n(X, A; G) \to C^{n+1}(X, A; G))$$

like before, and define the *n-th singular cohomology group of the pair (X, A) with coefficients in G* to be the quotient group

$$H^n(X, A; G) = \frac{Z^n(X, A; G)}{B^n(X, A; G)} = H^n(C^*(X, A; G), \delta)\,.$$

Let $j^\#\colon C^n(X, A; G) \to C^n(X;G)$ be the canonical inclusion homomorphism. Then $j^\#$ is a cochain map. Drawing the cochain complexes vertically and the cochain maps horizontally, we have a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "\delta"'] & \vdots \arrow[d, "\delta"'] & \vdots \arrow[d, "\delta"'] & \\
0 \arrow[r] & C^{n-1}(X, A; G) \arrow[r, "j^\#"] \arrow[d, "\delta"'] & C^{n-1}(X; G) \arrow[r, "i^\#"] \arrow[d, "\delta"'] & C^{n-1}(A; G) \arrow[r] \arrow[d, "\delta"'] & 0 \\
0 \arrow[r] & C^{n}(X, A; G) \arrow[r, "j^\#"] \arrow[d, "\delta"'] & C^{n}(X; G) \arrow[r, "i^\#"] \arrow[d, "\delta"'] & C^{n}(A; G) \arrow[r] \arrow[d, "\delta"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with exact rows. We usually draw this more compactly as the following short exact sequence of cochain complexes:

$$0 \to C^*(X, A; G) \xrightarrow{j^\#} C^*(X; G) \xrightarrow{i^\#} C^*(A; G) \to 0$$

Note that if $A = \varnothing$ is empty, then $j^\#$ is an isomorphism of chain complexes $C^*(X, \varnothing; G) \cong C^*(X; G)$, and j^* is an isomorphism $H^n(X, \varnothing; G) \cong H^n(X; G)$ for all n, so (absolute) cohomology is a special case of relative cohomology.

There is a *connecting homomorphism in cohomology*

$$\delta \colon H^{n-1}(A; G) \to H^n(X, A; G)$$

associated, as usual, to the short exact sequence of cochain complexes above. It is defined by taking the cohomology class $[\varphi]$ of an $(n-1)$-cocycle $\varphi \in Z^{n-1}(A; G)$ to the cohomology class $[\delta\tilde{\varphi}]$ of the unique lift to $C^n(X, A; G)$ of the coboundary in $C^n(X; G)$ of an extension $\tilde{\varphi}$ in $C^{n-1}(X; G)$ of φ. Here $i^\#(\tilde{\varphi}) = \varphi$, so the lift exists because $i^\#(\delta\tilde{\varphi}) = \delta\varphi = 0$ in $C^n(A; G)$. It is an n-cocycle, since its coboundary in $C^{n+1}(X, A; G)$ maps to $\delta^2\tilde{\varphi} = 0$ in $C^{n+1}(X; G)$ under the inclusion $j^\#$. As we proved before, δ is well-defined and additive.

Exercise 2.10.1. Express the relative n-cocycles $Z^n(X, A; G)$ as a subgroup of the group $\{\varphi \in C^n(X; G) \mid \delta\varphi = 0 \in C^{n+1}(X; G)\}$ of n-cochains in X that are zero on boundaries from $C_{n+1}(X)$.

Similarly, express the relative n-coboundaries $B^n(X, A; G)$ as a subgroup of the group $\{\delta\psi \in C^n(X; G) \mid \psi \in C^{n-1}(X; G)\}$ of n-cochains in X whose value on an n-chain only depends on its boundary in $C_{n-1}(X)$.

Let (X, A) and (Y, B) be pairs of spaces, and $f \colon (X, A) \to (Y, B)$ a map of pairs of spaces. Write $f' \colon A \to B$ for the restriction of f to A, corestricted to B. Then we have a commutative diagram of cochain complexes and chain maps

$$\begin{tikzcd}
0 \arrow[r] & C^*(Y, B; G) \arrow[r, "j^\#"] \arrow[d, "f''^\#"'] & C^*(Y; G) \arrow[r, "i^\#"] \arrow[d, "f^\#"'] & C^*(B; G) \arrow[r] \arrow[d, "f'^\#"'] & 0 \\
0 \arrow[r] & C^*(X, A; G) \arrow[r, "j^\#"] & C^*(X; G) \arrow[r, "i^\#"] & C^*(A; G) \arrow[r] & 0
\end{tikzcd}$$

where the right hand square is induced by f, f' and the inclusions $A \subseteq X$ and $B \subseteq Y$, and the rows are short exact sequences of chain complexes. The cochain map $f''^{\#}$ on the left hand side is then determined by $f^{\#}$ by passage to subcomplexes.

In particular, we have an induced homomorphism

$$f^* = H^n(f;G)\colon H^n(Y,B;G) \to H^n(X,A;G)$$

for each n. If $g\colon (Y,B) \to (Z,C)$ is a second map of pairs of spaces, then $(gf)^* = f^*g^*$. Under the isomorphism $C^*(X,\varnothing;G) \cong C^*(X;G)$ we can identify $j^{\#}\colon C^*(X,A;G) \to C^*(X;G)$ with the chain map $j''^{\#}$ for j equal to the map of pairs $(X,\varnothing) \to (X,A)$ given by the identity on X.

The connecting homomorphism $\delta\colon H^{n-1}(A;G) \to H^n(X,A;G)$ is *natural*, in the sense that for any map of pairs $f\colon (X,A) \to (Y,B)$ the diagram

$$\begin{array}{ccc} H^{n-1}(B;G) & \xrightarrow{\delta} & H^n(Y,B;G) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f''^*}\downarrow \\ H^{n-1}(A;G) & \xrightarrow{\delta} & H^n(X,A;G) \end{array}$$

commutes.

Remark 2.10.2. We briefly write $C^*(X,A)$ and $H^*(X,A)$ for the relative cochains and cohomology groups with integral coefficients. Each short exact sequence

$$0 \to C_n(A) \xrightarrow{i_n} C_n(X) \to C_n(X,A) \to 0$$

admits a splitting, since $C_n(X,A)$ is the free abelian group on the set of n-simplices in X that do not lie in A, hence there is a short exact sequence

$$0 \to \mathrm{Hom}(C_n(X,A),G) \to \mathrm{Hom}(C_n(X),G) \xrightarrow{\mathrm{Hom}(i_n,1)} \mathrm{Hom}(C_n(A),G) \to 0$$

and we have an isomorphism $C^n(X,A;G) \cong \mathrm{Hom}(C_n(X,A),G)$. It is compatible with the boundary maps, so that $C^*(X,A;G) \cong \mathrm{Hom}(C_*(X,A),G)$. Again, this does not generally imply that $H^*(X,A;G) \cong \mathrm{Hom}(H_*(X,A),G)$.

2.11 The Eilenberg–Steenrod axioms for cohomology

Theorem 2.11.1 (Eilenberg–Steenrod axioms)**.** *Let G be a fixed abelian group and let n range over all integers. We abbreviate $H^n(X,\varnothing;G)$ to $H^n(X;G)$.*

(Functoriality) The rule that takes a pair of spaces (X,A) to $H^n(X,A;G)$, and a map $f\colon (X,A) \to (Y,B)$ to the homomorphism $f^*\colon H^n(Y,B;G) \to H^n(X,A;G)$, defines a contravariant functor from pairs of spaces to graded abelian groups.

(Naturality) The rule that takes a pair of spaces (X,A) to the connecting homomorphism $\delta\colon H^{n-1}(A;G) \to H^n(X,A;G)$ is a natural transformation.

(Long exact sequence) The natural diagram

$$\dots \xrightarrow{i^*} H^{n-1}(A;G) \xrightarrow{\delta} H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta} \dots$$

is a long exact sequence, where i^* *is induced by the inclusion* $i\colon A \to X$ *and* j^* *is induced by the inclusion* $j\colon (X, \varnothing) \to (X, A)$.

(Homotopy invariance) If $f \simeq g\colon (X, A) \to (Y, B)$ *are homotopic as maps of pairs, then* $f^* = g^*\colon H^n(Y, B; G) \to H^n(X, A; G)$.

(Excision) If $Z \subseteq A \subseteq X$ *are subspaces, such that the closure of* Z *is contained in the interior of* A*, then the inclusion* $(X - Z, A - Z) \to (X, A)$ *induces isomorphisms*

$$H^n(X, A; G) \xrightarrow{\cong} H_n(X - Z, A - Z; G)\,.$$

(Product) If $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$ *is a disjoint union of pairs of subspaces, then the inclusion maps* $(X_\alpha, A_\alpha) \to (X, A)$ *induce isomorphisms*

$$H^n(X, A; G) \xrightarrow{\cong} \prod_\alpha H^n(X_\alpha, A_\alpha; G)$$

(Dimension) Let $\star$ *be a one-point space. Then* $H^0(\star; G) = G$ *and* $H^n(\star; G) = 0$ *for* $n \in \mathbb{Z}$.

Definition 2.11.2. A functor $(X, A) \mapsto h^*(X, A)$ and natural transformation $\delta\colon h^{*-1}(A) \to h^*(X, A)$ satisfying all of the Eilenberg–Steenrod axioms for cohomology, except the dimension axiom, is called a *generalized cohomology theory.*

Proof. Contravariant functoriality of the cohomology groups, naturality of the connecting homomorphism and exactness of the long exact sequence are clear from the contravariant functoriality of the short exact sequence

$$0 \to C^*(X, A; G) \xrightarrow{j^\#} C^*(X; G) \xrightarrow{i^\#} C^*(A; G) \to 0$$

of cochain complexes, together with the standard construction of the connecting homomorphism and exactness of the long exact sequence for short exact sequences of chain complexes.

Homotopy invariance for singular cohomology follows from the proof of homotopy invariance for singular homology. Let

$$F\colon (X, A) \times I = (X \times I, A \times I) \to (Y, B)$$

be a homotopy of pairs from $f\colon (X, A) \to (Y, B)$ to $g\colon (X, A) \to (Y, B)$. For each n-simplex Δ^n there is a triangulation of the cylinder $\Delta^n \times I$, where $I = [0, 1]$, which gives rise to a *prism operator*

$$P\colon C_n(X, A) \to C_{n+1}(Y, B)$$

such that

$$\partial P + P\partial = g_\# - f_\#\colon C_n(X, A) \to C_n(Y, B)$$

for each $n \geq 0$. Applying $\mathrm{Hom}(-, G)$, we get the dual prism operator

$$P^* = \mathrm{Hom}(P, 1)\colon C^{n+1}(Y, B; G) \to C^n(X, A; G)$$

such that

$$P^*\delta + \delta P^* = g^\# - f^\#\colon C^n(Y, B; G) \to C^n(X, A; G)\,.$$

For each n-cocycle $\varphi \in Z^n(Y, B; G)$ we have $\delta\varphi = 0$, so $\delta P^*(\varphi) = g^\#(\varphi) - f^\#(\varphi)$, which implies that $f^\#(\varphi)$ and $g^\#(\varphi)$ are *cohomologous*, i.e., they represent the same cohomology class:

$$f^*[\varphi] = [f^\#(\varphi)] = [g^\#(\varphi)] = g^*[\varphi]\,.$$

Thus $f^* = g^* \colon H^n(Y, B; G) \to H^n(X, A; G)$.

Excision for singular cohomology also follows from the proof of excision for singular homology. Let $B = X - Z$, so that $\mathrm{Int}(A)$ and $\mathrm{Int}(B)$ cover X. Let

$$\iota \colon C_*(A + B) \subseteq C_*(X)$$

be the inclusion of the subcomplex of simplices in A or B. Using barycentric subdivision, there is a chain map

$$\rho \colon C_*(X) \longrightarrow C_*(A + B)$$

such that $\rho\iota = 1$ and $1 - \iota\rho = \partial D + D\partial$ for a chain homotopy D. Applying $\mathrm{Hom}(-, G)$, we get dual cochain maps

$$\iota^* \colon C^*(X; G) \longrightarrow C^*(A + B; G) = \mathrm{Hom}(C_*(A + B), G)$$

and

$$\rho^* \colon C^*(A + B; G) \longrightarrow C^*(X; G)$$

such that $\iota^*\rho^* = 1$ and $1 - \rho^*\iota^* = D^*\delta + \delta D^*$. Hence ι^* induces an isomorphism in cohomology. By the five-lemma for the map of long exact sequences induced by the map of short exact sequences of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & C^*(A+B, A; G) \arrow[r] \arrow[d, "\iota''"'] & C^*(A+B; G) \arrow[r] \arrow[d, "\iota^*"'] & C^*(A; G) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & C^*(X, A; G) \arrow[r, "j^\#"] & C^*(X; G) \arrow[r, "i^\#"] & C^*(A; G) \arrow[r] & 0
\end{tikzcd}$$

it follows that also the left hand map ι^* induces an isomorphism in cohomology. There is a natural identification $C^*(A + B, A; G) \cong C^*(B, A \cap B; G)$, and the composite of the induced isomorphism $H^*(A+B, A; G) \cong H^*(B, A\cap B; G)$ with ι^* is the excision isomorphism.

The product axiom is only interesting for infinite indexing sets, since the case of finite disjoint unions follows from the long exact sequence and excision. Since any simplex $\sigma \colon \Delta^n \to \coprod_\alpha X_\alpha$ lands in precisely one of the X_α's, there is a direct sum decomposition $C_*(X, A) \cong \bigoplus_\alpha C_*(X_\alpha, A_\alpha)$. Applying $\mathrm{Hom}(-, G)$ we get a product factorization

$$C^*(X, A; G) \cong \prod_\alpha C^*(X, A; G)\,.$$

Since each coface map δ factors as the product $\prod_\alpha \delta_\alpha$, we also get product factorizations of $B^*(X, A; G)$ and $Z^*(X, A; G)$, which induce the claimed product factorization of $H^*(X, A; G)$.

We discussed the dimension axiom for cohomology above. □

2.12 Triples and triads

By a *triple of spaces* (X, A, B) we mean a topological space X, a subspace $A \subseteq X$, and a further subspace $B \subseteq A$, more briefly expressed as $B \subseteq A \subseteq X$. By the Noether isomorphism theorem

$$\frac{C_n(X, B; G)}{C_n(A, B; G)} \cong C_n(X, A; G)$$

there is a short exact sequence of chain complexes

$$0 \to C_*(A, B; G) \xrightarrow{i_\#} C_*(X, B; G) \xrightarrow{j_\#} C_*(X, A; G) \to 0$$

where $i_\#$ is induced by the map of pairs $i\colon (A, B) \to (X, B)$, and $j_\#$ is induced by the map of pairs $j\colon (X, B) \to (X, A)$.

Theorem 2.12.1. *Let (X, A, B) be a triple and G any abelian group. There is a long exact sequence*

$$\dots \to H_n(A, B; G) \xrightarrow{i_*} H_n(X, B; G) \xrightarrow{j_*} H_n(X, A; G) \xrightarrow{\partial} H_{n-1}(A, B; G) \to \dots$$

where i_* and j_* are induced by $i\colon (A, B) \to (X, B)$ and $j\colon (X, B) \to (X, A)$, respectively, and ∂ is the composite

$$H_n(X, A; G) \longrightarrow H_{n-1}(A; G) \longrightarrow H_{n-1}(A, B; G)$$

of the connecting homomorphism of the pair (X, A) and the homomorphism induced by the map $(A, \varnothing) \to (A, B)$.

Proof. We only need to identify ∂ with the given composite. There is a vertical map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & C_*(A; G) \arrow[r, "i_\#"] \arrow[d] & C_*(X; G) \arrow[r, "j_\#"] \arrow[d] & C_*(X, A; G) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_*(A, B; G) \arrow[r, "i_\#"] & C_*(X, B; G) \arrow[r, "j_\#"] & C_*(X, A; G) \arrow[r] & 0\,,
\end{tikzcd}$$

where the left hand vertical map is induced by the map $(A, \varnothing) \to (A, B)$, and naturality of the connecting homomorphism gives the commutative square

$$\begin{tikzcd}
H_n(X, A; G) \arrow[r] \arrow[d, equal] & H_{n-1}(A; G) \arrow[d] \\
H_n(X, A; G) \arrow[r, "\partial"] & H_{n-1}(A, B; G)
\end{tikzcd}$$

□

Definition 2.12.2. Let D^m_+ and D^m_- be the upper and lower hemispheres, respectively, of the m-sphere $S^m = \partial D^{m+1}$, meeting along the equatorial $(m-1)$-sphere $\partial D^m_+ = \partial D^m_- = S^{m-1}$. Projection to the equatorial hyperplane induces homeomorphisms $D^m_+ \cong D^m \cong D^m_-$.

Theorem 2.12.3.

$$H_n(D^m, S^{m-1}; G) \cong \begin{cases} G & \textit{for } n = m, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This is clear for $m = 0$, when $D^0 \cong \star$ and $\partial D^0 = S^{-1} = \varnothing$. Suppose by induction that the theorem holds for one $m \geq 0$. Consider the long exact sequence

$$\cdots \to H_{n+1}(D^{m+1}, D^m_-; G) \to H_{n+1}(D^{m+1}, S^m; G) \xrightarrow{\partial} H_n(S^m, D^m_-; G) \to H_n(D^{m+1}, D^m_-; G) \,. \to \dots$$

of the triple (D^{m+1}, S^m, D^m_-). The inclusion $D^m_- \subset D^{m+1}$ is a homotopy equivalence, so $H_*(D^{m+1}, D^m_-; G) = 0$ in all degrees, and the connecting homomorphism ∂ is an isomorphism for all n. Furthermore, the inclusion $(D^m_+, S^{m-1}) \subset (S^m, D^m_-)$ induces an isomorphism

$$H_n(D^m_+, S^{m-1}; G) \xrightarrow{\cong} H_n(S^m, D^m_-; G)$$

by the excision theorem, combined with homotopy invariance. Finally, we have the isomorphism $H_n(D^m_+, S^{m-1}; G) \cong H_n(D^m, S^{m-1}; G)$ induced by the homeomorphism $D^m_+ \cong D^m$. Combining these three isomorphisms we deduce that the theorem holds for $m + 1$. □

Corollary 2.12.4.

$$\tilde{H}_n(S^m; G) \cong \begin{cases} G & \textit{for } n = m, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This is clear from the long exact sequence

$$\cdots \to \tilde{H}_{n+1}(D^{m+1}; G) \to H_{n+1}(D^{m+1}, S^m; G) \xrightarrow{\partial} \tilde{H}_n(S^m; G) \to \tilde{H}_n(D^{m+1}; G) \to \dots$$

and the fact that $\tilde{H}_*(D^{m+1}; G) = 0$ since D^{m+1} is contractible, which implies that ∂ is an isomorphism for all n. □

Theorem 2.12.5. *There is a short exact sequence of cochain complexes*

$$0 \to C^*(X, A; G) \xrightarrow{j^\#} C^*(X, B; G) \xrightarrow{i^\#} C^*(A, B; G) \to 0$$

and a long exact sequence in cohomology

$$\dots \xrightarrow{i^*} H^{n-1}(A, B; G) \xrightarrow{\delta} H^n(X, A; G) \xrightarrow{j^*} H^n(X, B; G) \xrightarrow{i^*} H^n(A, B; G) \xrightarrow{\delta} \dots$$

associated to each triple (X, A, B). There are isomorphisms

$$H^n(D^m, S^{m-1}; G) \cong \tilde{H}^n(S^m; G) \cong \begin{cases} G & \textit{for } n = m, \\ 0 & \textit{otherwise,} \end{cases}$$

for all integers n and m and abelian groups G.

Proof. This is dual to the homological proof. □

By a *triad of spaces* $(X; U, V)$ we mean a topological space X together with two subspaces $U \subseteq X$ and $V \subseteq X$. We can then form the subspaces $U \cap V$ and $U \cup V$ of X, which fit together in a diagram of inclusion maps

$$\begin{tikzcd}
U \cap V \arrow[r, "u"] \arrow[d, "v"'] & U \arrow[d, "u'"'] \arrow[ddr, bend left, "u''"] & \\
V \arrow[r, "v'"] \arrow[drr, bend right, "v''"'] & U \cup V \arrow[dr] & \\
 & & X\,.
\end{tikzcd}$$

We are principally interested in the case where U and V are open subsets of X, and $U \cup V = X$, but other cases are also relevant.

Definition 2.12.6. Let $C_*(U + V; G) = C_*(U; G) + C_*(V; G) \subseteq C_*(X)$ be the subcomplex consisting of chains that are sums of chains in (the image of) $C_*(U; G)$ and $C_*(V; G)$. We say that the triad $(X; U, V)$ is G-*excisive* if the chain map $C_*(U + V; G) \to C_*(X; G)$ is a *quasi-isomorphism*, meaning that it induces an isomorphism in homology. By the excision theorem, this holds, for instance, if U and V are open and $X = U \cup V$.

Lemma 2.12.7. *There is a short exact sequence of chain complexes*

$$0 \to C_*(U \cap V; G) \xrightarrow{\varphi} C_*(U; G) \oplus C_*(V; G) \xrightarrow{\psi'} C_*(U + V; G) \to 0$$

where $\varphi(w) = (u_\#(w), -v_\#(w))$ *and* $\psi'(y, z) = u'_\#(y) + v'_\#(z)$.

(The sign conventions in φ and ψ are negotiable.)

Proof. The homomorphism ψ' is surjective by the definition of $C_*(U+V; G)$. Its kernel in degree n consists of pairs (y, z) with $u'_\#(y) + v'_\#(z) = 0$ in $C_n(X; G)$, meaning that y is a chain in U, z is a chain in V, and y is equal to $-z$ as chains in X. This means that y and z are chains in $U \cap V$, so $(y, z) = \varphi(w)$ for a (unique) chain w in $U \cap V$. □

Theorem 2.12.8 (Mayer–Vietoris)**.** *Let* $(X; U, V)$ *be a* G*-excisive pair. Then there is a long exact sequence*

$$\dots \to H_n(U \cap V; G) \xrightarrow{\varphi} H_n(U; G) \oplus H_n(V; G) \xrightarrow{\psi} H_n(X; G) \xrightarrow{\chi} H_{n-1}(U \cap V; G) \to \dots$$

where $\varphi(w) = (u_*(w), -v_*(w))$, $\psi(y, z) = u''_*(y) + v''_*(z)$ *and* χ *is the composite*

$$H_n(X; G) \xrightarrow{j_*} H_n(X, V; G) \xleftarrow{\cong} H_n(U, U \cap V; G) \xrightarrow{\partial} H_{n-1}(U \cap V; G)\,.$$

Proof. It is clear that $H_n(C_*(U; G) \oplus C_*(V; G)) \cong H_n(U; G) \oplus H_n(V; G)$, so we get the following long exact sequence in homology associated to the short exact sequence of the lemma.

$$\dots \to H_n(U \cap V; G) \xrightarrow{\varphi_*} H_n(U; G) \oplus H_n(V; G) \xrightarrow{\psi'_*} H_n(C_*(U + V; G)) \xrightarrow{\partial'} H_{n-1}(U \cap V; G) \to \dots$$

By construction, the connecting homomorphism ∂' takes an n-cycle $y+z$, with y in U and z in V, to the $(n-1)$-cycle $\partial y = -\partial z$ in $U \cap V$.

Using the G-excision hypothesis, we can reroute part of the exact sequence as follows

$$\begin{tikzcd}
H_n(U;G) \oplus H_n(V;G) \arrow[r, "\psi'_*"] \arrow[dr, "\psi_*"'] & H_n(C_*(U+V;G)) \arrow[r, "\partial'"] \arrow[d, "\cong"'] & H_{n-1}(U \cap V;G) \\
& H_n(X;G) \arrow[ur, "\chi"'] &
\end{tikzcd}$$

where ψ_* is as in the statement of the theorem.

It only remains to give the formula for the Mayer–Vietoris connecting homomorphism χ. We have a diagram of vertical maps of short exact sequences of chain complexes.

$$\begin{tikzcd}
0 \arrow[r] & C_*(U \cap V;G) \arrow[r] \arrow[d] & C_*(U;G) \arrow[r] \arrow[d] & C_*(U, U \cap V;G) \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & C_*(V;G) \arrow[r] \arrow[d, equal] & C_*(U+V;G) \arrow[r] \arrow[d, "\simeq"'] & C_*(U+V, V;G) \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & C_*(V;G) \arrow[r] & C_*(X;G) \arrow[r, "j_\#"] & C_*(X, V;G) \arrow[r] & 0
\end{tikzcd}$$

The chain map $C_*(U, U \cap V;G) \to C_*(U+V, V;G)$ is always an isomorphism, essentially by the lemma above. By assumption, the chain map $C_*(U+V;G) \to C_*(X;G)$ is a quasi-isomorphism. By the five-lemma, this is equivalent to the property that the chain map $C_*(U+V, V;G) \to C_*(X, V;G)$ is a quasi-isomorphism.

The connecting map χ now takes an n-cycle x in X to a homologous n-cycle of the form $y+z$, with y in U and z in V, thinks of y as a relative n-cycle in $(U, U \cap V)$, and computes its boundary ∂y as an $(n-1)$-cycle in $U \cap V$. By chasing the diagram above, this is the same as passing to the relative n-cycle $j_n(x)$ in (X, V), lifting this to a homologous relative n-cycle y in $(U, U \cap V)$, and computing its boundary ∂y as an $(n-1)$-cycle in $U \cap V$. This is the asserted description of χ. □

Notice that this long exact sequence only involves absolute homology groups. There is also a Mayer–Vietoris sequence in reduced homology, and in relative homology.

Another way to proceed is to pass to homology from the diagram above, to get the following diagram of vertical maps of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(U \cap V) \arrow[r, "u_*"] \arrow[d, "v_*"'] & H_n(U) \arrow[r] \arrow[d, "u'_*"'] & H_n(U, U \cap V) \arrow[r, "\partial"] \arrow[d, "\cong"'] & H_{n-1}(U \cap V) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n(V) \arrow[r] \arrow[d, "="'] & H_n(U+V) \arrow[r] \arrow[d, "\cong"'] & H_n(U+V, V) \arrow[r] \arrow[d, "\cong"'] & H_{n-1}(V) \arrow[r] \arrow[d, "="'] & \cdots \\
\cdots \arrow[r] & H_n(V) \arrow[r, "v''_*"] & H_n(X) \arrow[r, "j_*"] & H_n(X, V) \arrow[r] & H_{n-1}(V) \arrow[r] & \cdots
\end{tikzcd}$$

(coefficients in G omitted for space reasons), and to appeal to the following lemma.

Lemma 2.12.9 (Barratt–Whitehead)**.** *Let*

$$\begin{tikzcd} \dots \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\alpha"] & B_n \arrow[r, "j"] \arrow[d, "\beta"] & C_n \arrow[r, "\partial"] \arrow[d, "\gamma", "\cong"'] & A_{n-1} \arrow[r] \arrow[d, "\alpha"] & \dots \\ \dots \arrow[r] & A'_n \arrow[r, "i"] & B'_n \arrow[r, "j"] & C'_n \arrow[r, "\partial"] & A'_{n-1} \arrow[r] & \dots \end{tikzcd}$$

be a vertical map of long exact sequences, where the maps $\gamma\colon C_n \to C'_n$ are isomorphisms for all n. Then the diagram

$$\dots \to A_n \xrightarrow{\varphi} B_n \oplus A'_n \xrightarrow{\psi} B'_n \xrightarrow{\chi} A_{n-1} \to \dots$$

is a long exact sequence, where $\varphi(a) = (i(a), -\alpha(a))$, $\psi(b, a') = \beta(b) + i(a')$ and $\chi(b') = \partial\gamma^{-1}j(b')$.

Exercise 2.12.10. Prove the Barratt–Whitehead lemma.

((A third way to proceed is to form the double mapping cylinder

$$M = U \times \{0\} \cup (U \cap V \times I) \cup V \times \{1\}$$

as a subspace of $X \times I$, which comes equipped with a map $M \to X$. We can embed the disjoint union $U \sqcup V$ as $U \times \{0\} \cup V \times \{1\} \subset M$. The Mayer–Vietoris sequence for $(X; U, V)$ is identified with the long exact sequence of the pair $(M, U \sqcup V)$.))

Theorem 2.12.11.

$$\tilde{H}_n(S^m; G) \cong \begin{cases} G & \textit{for } n = m, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This is clear for $m = 0$, when $S^0 \cong \star_+$. Let $m \geq 1$ and suppose by induction that the theorem holds for $m - 1$. Consider the Mayer–Vietoris sequence

$$\dots \to \tilde{H}_n(D^m_+; G) \oplus \tilde{H}_n(D^m_-; G) \longrightarrow$$
$$\tilde{H}_n(S^m; G) \xrightarrow{\chi} \tilde{H}_{n-1}(S^{m-1}; G) \longrightarrow \tilde{H}_{n-1}(D^m_+; G) \oplus \tilde{H}_{n-1}(D^m_-; G) \to \dots$$

of the triad $(S^m; D^m_+, D^m_-)$, with $D^m_+ \cap D^m_- = S^{m-1}$. This pair is excisive by homotopy invariance and excision. The groups $\tilde{H}_*(D^m_+; G)$ and $\tilde{H}_*(D^m_-; G)$ are zero since D^m_+ and D^m_- are contractible, so χ is an isomorphism for all n. This proves the statement for m. □

Corollary 2.12.12.

$$H_n(D^m, S^{m-1}; G) \cong \begin{cases} G & \textit{for } n = m, \\ 0 & \textit{otherwise.} \end{cases}$$

This follows from the theorem, by the long exact sequence we used in the triple case.

Theorem 2.12.13. *There is a short exact sequence of cochain complexes*

$$0 \to C^*(U+V;G) \longrightarrow C^*(U;G) \times C^*(V;G) \longrightarrow C^*(U \cap V;G) \to 0$$

and a long exact Mayer–Vietoris sequence

$$\cdots \to H^{n-1}(U \cap V;G) \xrightarrow{\chi}$$
$$H^n(X;G) \xrightarrow{\psi^*} H^n(U;G) \times H^n(V;G) \xrightarrow{\varphi^*} H^n(U \cap V;G) \xrightarrow{\chi} \dots$$

in cohomology for excisive triads $(X;U,V)$.

Proof. This is dual to the proof in homology. Note that $C^*(U+V;G)$ is the coimage of $C^*(X;G) \to C^*(U;G) \times C^*(V;G)$, meaning the quotient of $C^*(X;G)$ that is isomorphic to the image of the pair of restriction homomorphisms induced by $U \subseteq X$ and $V \subseteq X$. □

2.13 Cellular homology and cohomology

If X is a CW complex, and $A \subseteq X$ a subcomplex, then the *cellular complexes* $C_*^{CW}(X,A;G)$ and $C^*_{CW}(X,A;G)$ are smaller complexes than the singular ones, which can be used to compute the homology and cohomology groups.

Let

$$\varnothing = X^{(-1)} \subseteq X^{(0)} \subseteq \cdots \subseteq X^{(n-1)} \subseteq X^{(n)} \subseteq \cdots \subseteq X$$

be the skeleton filtration of X, so that there is a pushout square

$$\begin{tikzcd} \coprod_\alpha \partial D^n \arrow[r, tail] \arrow[d, "\varphi"'] & \coprod_\alpha D^n \arrow[d, "\Phi"] \\ X^{(n-1)} \arrow[r, tail] & X^{(n)} \end{tikzcd}$$

for each $n \geq 0$, and $X = \bigcup_{n \geq 0} X^{(n)}$ has the weak (colimit) topology. The index α runs over the set of n-cells in X, and we decompose $\varphi = \coprod_\alpha \varphi_\alpha$ and $\Phi = \coprod_\alpha \Phi_\alpha$, where $\varphi_\alpha \colon \partial D^n \to X^{(n-1)}$ is the *attaching map* and $\Phi_\alpha \colon D^n \to X^{(n)} \subseteq X$ is the *characteristic map* of the α-th n-cell.

Example 2.13.1. A Δ-*complex* in Hatcher's sense is a CW complex where each CW-characteristic map $\Phi_\alpha \colon D^n \to X$ factors as a homeomorphism $D^n \cong \Delta^n$ followed by a Δ-characteristic map

$$\sigma_\alpha \colon \Delta^n \to X\,,$$

and each CW-attaching map $\varphi_\alpha \colon \partial D^n \to X$ factors as the restricted homeomorphism $\partial D^n \cong \partial \Delta^n$ followed by a map

$$\sigma_\alpha| \colon \partial\Delta^n \to X$$

whose restriction $\sigma_\alpha \delta^i_n = \sigma_\alpha|[v_0, \dots, \hat{v}_i, \dots, v_n]$ to the i-th $(n-1)$-face of Δ^n is the Δ-characteristic map σ_β of some $(n-1)$-cell β, for each $0 \leq i \leq n$. (This involves the usual identification of Δ^{n-1} with $\delta^i_n(\Delta^{n-1}) \subset \Delta^n$.)

For each $n \geq 0$, let

$$C_n^{CW}(X) = H_n(X^{(n)}, X^{(n-1)})\,.$$

By excision, homotopy invariance and the sum axiom, there is an isomorphism

$$C_n^{CW}(X) \cong \bigoplus_\alpha H_n(D^n, \partial D^n) \cong \mathbb{Z}\{e_\alpha^n\}\,,$$

where α runs over the set of n-cells in X, and e_α^n denotes the generator of the corresponding copy of $H_n(D^n, \partial D^n) \cong \mathbb{Z}$.

Let

$$d_n \colon C_n^{CW}(X) \longrightarrow C_{n-1}^{CW}(X)$$

be the connecting homomorphism in the long exact sequence

$$\dots \to H_n(X^{(n)}, X^{(n-2)}) \xrightarrow{j_*} H_n(X^{(n)}, X^{(n-1)}) \xrightarrow{d_n} H_{n-1}(X^{(n-1)}, X^{(n-2)}) \\ \xrightarrow{i_*} H_{n-1}(X^{(n)}, X^{(n-2)}) \to \dots$$

of the triple $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$. This equals the composite homomorphism

$$H_n(X^{(n)}, X^{(n-1)}) \xrightarrow{\partial} H_{n-1}(X^{(n-1)}) \xrightarrow{j_*} H_{n-1}(X^{(n-1)}, X^{(n-2)})\,.$$

It follows that $d_n d_{n+1} = j_* \partial j_* \partial = 0$, since the composite ∂j_* is zero.

We call $(C_*^{CW}(X), d)$ the *cellular chain complex* of X, and define the *cellular homology groups* of X to be its homology groups

$$H_n^{CW}(X) = \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})} = H_n(C_*^{CW}(X), d)\,.$$

Proposition 2.13.2. *Under the identifications*

$$C_n^{CW}(X) = H_n(X^{(n)}, X^{(n-1)}) \cong \mathbb{Z}\{e_\alpha^n\}$$
$$C_{n-1}^{CW}(X) = H_{n-1}(X^{(n-1)}, X^{(n-2)}) \cong \mathbb{Z}\{e_\beta^{n-1}\}\,,$$

where α and β range over the sets of n-cells and $(n-1)$-cells of X, respectively, the homomorphism d_n is given by the formula

$$d_n(e_\alpha^n) = \sum_\beta m_{\alpha\beta} e_\beta^{n-1}$$

where the integer $m_{\alpha\beta}$ is the degree of the composite map

$$S^{n-1} \xrightarrow{\varphi_\alpha} X^{(n-1)} \xrightarrow{j} X^{(n-1)}/X^{(n-2)} \xleftarrow{\cong} \bigvee_\beta S^{n-1} \xrightarrow{p_\beta} S^{n-1}\,.$$

Here the homeomorphism is induced by the characteristic map $\Phi \colon \coprod_\beta D^{n-1} \to X^{(n-1)}$ by passage to a quotient, and p_β denotes the projection to the summand indexed by β.

Proof. Consider the commutative diagrams

$$\begin{tikzcd} H_n(D^n, S^{n-1}) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi_{\alpha*}"'] & \tilde{H}_{n-1}(S^{n-1}) \arrow[d, "\varphi_{\alpha*}"'] & \\ H_n(X^{(n)}, X^{(n-1)}) \arrow[r, "\partial"] & \tilde{H}_{n-1}(X^{(n-1)}) \arrow[r, "j_*"] & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \end{tikzcd}$$

and

$$\begin{tikzcd} H_{n-1}(D^{n-1}, S^{n-2}) \arrow[r, "\cong"] \arrow[d, "\Phi_{\beta*}"'] & \tilde{H}_{n-1}(S^{n-1}) \arrow[r, "="] \arrow[d, "\Phi_{\beta*}"'] \arrow[dr, "i_{\beta*}"] & \tilde{H}_{n-1}(S^{n-1}) \\ H_{n-1}(X^{(n-1)}, X^{(n-2)}) \arrow[r, "\cong"'] & \tilde{H}_{n-1}(X^{(n-1)}/X^{(n-2)}) & \tilde{H}_{n-1}(\bigvee_\beta S^{n-1}) \arrow[l, "\Phi_*", "\cong"'] \arrow[u, "p_{\beta*}"'] \end{tikzcd}$$

where i_β and p_β denote inclusion and projection maps. We are looking for the coefficient $m_{\alpha\beta}$ of e^{n-1}_β in $d_n(e^n_\alpha) = j_*\partial(e^n_\alpha)$. Here e^n_α is the image of the preferred generator of $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ under $\Phi_{\alpha*}$. Since that generator maps to the preferred generator of $\tilde{H}_{n-1}(S^{n-1})$ under ∂, we see from the upper diagram that $d_n(e^n_\alpha)$ is the image under $j_*\varphi_{\alpha*}$ of the latter generator. We can recover the coefficient of e^{n-1}_β in the direct sum decomposition

$$H_{n-1}(X^{(n-1)}, X^{(n-2)}) \cong \tilde{H}_{n-1}(\bigvee_\beta S^{n-1}) \cong \bigoplus_\beta \tilde{H}_{n-1}(S^{n-1})$$

by composing with $p_{\beta*}$, as we can see from the lower diagram. □

Corollary 2.13.3. *The cellular complex $C^{CW}_*(X)$ of a Δ-complex X is isomorphic to the* simplicial chain complex $\Delta_*(X)$*, with*

$$\Delta_n(X) = \mathbb{Z}\{e^n_\alpha\}\,,$$

where α runs over the set of n-simplices in X, and $\partial_n \colon \Delta_n(X) \to \Delta_{n-1}(X)$ is given by

$$\partial_n(e^n_\alpha) = \sum_{i=0}^{n} (-1)^i e^{n-1}_{\beta_i}$$

where β_i is the $(n-1)$-simplex of X attached to the i-th face of α, for $0 \le i \le n$. Hence the cellular homology of X is isomorphic to the simplicial homology

$$H^{CW}_n(X) = H_n(C^{CW}_*(X), d) \cong H_n(\Delta_*(X), \partial) = H^\Delta_n(X)$$

in all degrees n.

Proof. ((The sign $(-1)^i$ comes from comparing the orientations of Δ^{n-1} and $\partial\Delta^n$ under δ^i_n. ETC.)) □

If $f\colon X \to Y$ is a cellular map of CW complexes, so that $f(X^{(n)}) \subseteq Y^{(n)}$ for all n, we get a homomorphism

$$f_\# = C^{CW}_n(f) = H_n(f)\colon H_n(X^{(n)}, X^{(n-1)}) \to H_n(Y^{(n)}, Y^{(n-1)})\,,$$

which defines a chain map $f_\#\colon C_*^{CW}(X) \to C_*^{CW}(Y)$ and an induced homomorphism $f_*\colon H_n^{CW}(X) \to H_n^{CW}(Y)$. Hence the cellular complex and cellular homology groups are covariant functors from the category of CW complexes and cellular maps.

Let $A \subseteq X$ be a subcomplex, with skeleton filtration $\{A^{(n)}\}_n$, such that $A^{(n)}$ is built from $A^{(n-1)}$ by attaching a subset of the n-cells of X, with attaching maps landing in $A^{(n-1)} \subseteq X^{(n-1)}$.

The inclusion $i\colon A \to X$ is cellular, and identifies the cellular chain complex $C_*^{CW}(A)$ with a subcomplex of $C_*^{CW}(X)$. Let the *relative cellular n-chains*

$$C_n^{CW}(X, A) = \frac{C_n^{CW}(X)}{C_n^{CW}(A)} \cong H_n(X^{(n)}, X^{(n-1)} \cup A^{(n)})$$

be the quotient group. It is the free abelian group generated by the n-cells of X that are not cells in A. There is a relative boundary homomorphism

$$d_n\colon C_n^{CW}(X, A) \longrightarrow C_{n-1}^{CW}(X, A)$$

and a short exact sequence of cellular chain complexes

$$0 \to C_*^{CW}(A) \xrightarrow{i_\#} C_*^{CW}(X) \xrightarrow{j_\#} C_*^{CW}(X, A) \to 0\,.$$

Exercise 2.13.4. Give an explicit description of the relative boundary homomorphism d_n in terms of the maps i_*, j_* and/or ∂ of various pairs.

Introducing coefficients, let

$$C_n^{CW}(X, A; G) = G \otimes C_n^{CW}(X, A) \cong \bigoplus_\alpha G$$

and

$$C^n_{CW}(X, A; G) = \operatorname{Hom}(C_n^{CW}(X, A), G) \cong \prod_\alpha G\,,$$

where α runs over the set of n-cells in X that are not cells in A. Using the boundary homomorphisms $\partial = 1 \otimes \partial$ and $\delta = \operatorname{Hom}(\partial, 1)$ we get the *cellular homology and cohomology groups*

$$H_n^{CW}(X, A; G) = H_n(C_*^{CW}(X, A; G), \partial)$$

and

$$H^n_{CW}(X, A; G) = H^n(C^*_{CW}(X, A; G), \delta)\,.$$

Lemma 2.13.5. *There are isomorphisms*

$$C_n^{CW}(X, A; G) \cong H_n(X^{(n)}, X^{(n-1)} \cup A^{(n)}; G)$$

and

$$C^n_{CW}(X, A; G) \cong H^n(X^{(n)}, X^{(n-1)} \cup A^{(n)}; G)\,,$$

compatible with the boundary and coboundary homomorphisms, respectively.

Proof. This follows by excision, homotopy invariance and the sum axiom, since $X^{(n)}$ is built from $X^{(n-1)} \cup A^{(n)}$ by attaching one copy of D^n along ∂D^n for each n-cell of X that is not contained in A. □

The following exercises illustrate these definitions. The simplicial chain complex is sometimes easier to compute, while the passage to homology is sometimes easier for the cellular chain complex. See also Exercise 6 in Section 3.1 of Hatcher [1].

Exercise 2.13.6. Give the circle S^1 a Δ-complex structure with one 0-simplex v and one 1-simplex a. (This a special case of a CW-complex structure.) Determine the simplicial chain complex $\Delta_*(S^1)$ and compute the simplicial homology groups $H_*^\Delta(S^1)$. Similarly, determine the chain complex $G \otimes \Delta_*(S^1)$ and the cochain complex $\mathrm{Hom}(\Delta_*(S^1), G)$, and compute the simplicial homology and cohomology groups $H_*^\Delta(S^1; G)$ and $H^*_\Delta(S^1; G)$.

Exercise 2.13.7. Give the torus T^2 a Δ-complex structure with one 0-simplex v, three 1-simplices a, b and c, and two 2-simplices U and L, as in Section 2.1 of Hatcher [1]. Determine the simplicial chain complex $\Delta_*(T^2)$ and compute the simplicial homology groups $H_*^\Delta(T^2)$. Similarly, determine the chain complex $G \otimes \Delta_*(T^2)$ and the cochain complex $\mathrm{Hom}(\Delta_*(T^2), G)$, and compute the simplicial homology and cohomology groups $H_*^\Delta(T^2; G)$ and $H^*_\Delta(T^2; G)$.

Exercise 2.13.8. Give T^2 a CW-complex structure with one 0-cell v, two 1-cells a and b, and one 2-cell T. Under the homeomorphism $S^1 \times S^1 \cong T^2$, the cartesian product of each i-cell in the first S^1 and each j-cell in the second S^1 is an $(i+j)$-cell in T^2. Determine the cellular complex $C_*^{CW}(T^2)$ and compute the simplicial homology groups $H_*^{CW}(T^2)$. Similarly, determine the chain complex $C_*^{CW}(T^2; G)$ and the cochain complex $C^*_{CW}(T^2; G)$, and compute the cellular homology and cohomology groups $H_*^{CW}(T^2; G)$ and $H^*_{CW}(T^2; G)$.

Exercise 2.13.9. Give the real projective plane $\mathbb{R}P^2$ a Δ-complex structure with two 0-simplices v and w, three 1-simplices a, b and c, and two 2-simplices U and L, as in Section 2.1 of Hatcher [1]. Determine the simplicial chain complex $\Delta_*(\mathbb{R}P^2)$, and compute the simplicial homology and cohomology groups $H_*^\Delta(\mathbb{R}P^2; G)$ and $H^*_\Delta(\mathbb{R}P^2; G)$ for $G = \mathbb{Z}$ and $G = \mathbb{Z}/2$.

Exercise 2.13.10. Give $\mathbb{R}P^2$ a CW-complex structure with one 0-cell v, one 1-cell a, and one 2-cell P. Determine the cellular chain complex $C_*^{CW}(\mathbb{R}P^2)$, and compute the cellular homology and cohomology groups $H_*^{CW}(\mathbb{R}P^2; G)$ and $H^*_{CW}(\mathbb{R}P^2; G)$ for $G = \mathbb{Z}$ and $G = \mathbb{Z}/2$.

Theorem 2.13.11 (Cellular (co-)homology)**.** *For all CW pairs (X, A) there are isomorphisms*

$$H_n(X, A; G) \cong H_n^{CW}(X, A; G)$$

and

$$H^n(X, A; G) \cong H^n_{CW}(X, A; G)\,,$$

that are natural with respect to cellular maps of pairs.

If X is a CW complex of *finite type*, meaning that it has only a finite number of n-cells for each n (but may be infinite-dimensional), then the cellular complex $C_*^{CW}(X)$ is finitely generated in each degree. It follows that the cellular homology groups $H_*^{CW}(X)$ are finitely generated in each degree, hence this is also the case for the (isomorphic) singular homology groups $H_*(X)$. Similarly for CW pairs of (relatively) finite type.

Proof. We prove the theorem in the special case of absolute integral homology, saying that there are natural isomorphisms

$$H_n(X) \cong H_n^{CW}(X)$$

for all CW complexes X. We first show that $H_n(X^{(m)}) = 0$ for $m < n$. Next we prove that $H_n(X^{(n)}) \cong \ker(d_n)$, where $d_n \colon C_n^{CW}(X) \to C_{n-1}^{CW}(X)$. Thereafter we prove that $H_n(X^{(n+1)}) \cong \ker(d_n)/\operatorname{im}(d_{n+1}) = H_n^{CW}(X)$. Finally we show that $H_n(X^{(m)}) \cong H_n^{CW}(X)$ for all $m > n$, and deduce that $H_n(X) \cong H_n^{CW}(X)$.

It is clear that $H_n(X^{(-1)}) = 0$, since $X^{(-1)} = \varnothing$. Suppose inductively that $H_n(X^{(m-1)}) = 0$ for some $m < n$. The long exact sequence in homology for the pair $(X^{(m)}, X^{(m-1)})$ contains the part

$$H_n(X^{(m-1)}) \xrightarrow{i_*} H_n(X^{(m)}) \to 0\,,$$

where the term at the right hand end is $H_n(X^{(m)}, X^{(m-1)}) \cong \tilde{H}_n(\bigvee_\alpha S^m) = \bigoplus_\alpha \tilde{H}_n(S^m) = 0$, and α runs over the m-cells in X. Hence i_* is surjective, and the inductive hypothesis implies that $H_n(X^{(m)}) = 0$.

Next consider the part

$$0 \to H_n(X^{(n)}) \xrightarrow{j_*} C_n^{CW}(X) \xrightarrow{\partial} H_{n-1}(X^{(n-1)})$$

of the long exact sequence for $(X^{(n)}, X^{(n-1)})$. The term at the left hand end is $H_n(X^{(n-1)}) = 0$, and we use the notation $C_n^{CW}(X) = H_n(X^{(n)}, X^{(n-1)})$. By exactness, $j_* \colon H_n(X^{(n)}) \to C_n^{CW}(X)$ is injective. Since n is arbitrary, we also have that $j_* \colon H_{n-1}(X^{(n-1)}) \to C_{n-1}^{CW}(X)$ is injective. Hence we have isomorphisms

$$H_n(X^{(n)}) \cong \operatorname{im}(j_*) = \ker(\partial) = \ker(j_*\partial) = \ker(d_n)$$

where $d_n = j_*\partial$ is the composite

$$C_n^{CW}(X) \xrightarrow{\partial} H_{n-1}(X^{(n-1)}) \xrightarrow{j_*} C_{n-1}^{CW}(X)\,.$$

Thereafter consider the part

$$C_{n+1}^{CW}(X) \xrightarrow{\partial} H_n(X^{(n)}) \xrightarrow{i_*} H_n(X^{(n+1)}) \to 0$$

of the long exact sequence for $(X^{(n+1)}, X^{(n)})$. The term at the right hand end is $H_n(X^{(n+1)}, X^{(n)}) = 0$, so i_* is surjective. Hence we have isomorphisms

$$H_n(X^{(n+1)}) \cong H_n(X^{(n)})/\ker(i_*) = H_n(X^{(n)})/\operatorname{im}(\partial) \cong \ker(d_n)/\operatorname{im}(d_{n+1})\,.$$

The last isomorphism is induced by j_* $H_n(X^{(n)}) \to C_n^{CW}(X)$, which takes $H_n(X^{(n)})$ isomorphically to $\operatorname{im}(j_*) = \ker(d_n)$, and takes $\operatorname{im}(\partial)$ isomorphically to $\operatorname{im}(j_*\partial) = \operatorname{im}(d_{n+1})$.

It may be helpful to combine these two steps in one diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
C^{CW}_{n+1}(X) \arrow[r, "\partial"] \arrow[dr, "d_{n+1}"'] & H_n(X^{(n)}) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_n(X^{(n+1)}) \arrow[r] & 0 \\
 & C^{CW}_n(X) \arrow[d, "\partial"] \arrow[dr, "d_n"] & & \\
0 \arrow[r] & H_{n-1}(X^{(n-1)}) \arrow[r, "j_*"] & C^{CW}_{n-1}(X) &
\end{tikzcd}$$

with exact rows and columns.

Finally we consider $m > n$ and suppose that $H_n(X^{(m)}) \cong H^{CW}_n(X)$. The part

$$0 \to H_n(X^{(m)}) \xrightarrow{i_*} H_n(X^{(m+1)}) \to 0$$

of the long exact sequence for $(X^{(m+1)}, X^{(m)})$, where the groups at the end are $H_{n+1}(X^{(m+1)}, X^{(m)}) = 0$ and $H_n(X^{(m+1)}, X^{(m)}) = 0$, shows that i_* is an isomorphism. Hence $H_n(X^{(m+1)}) \cong H^{CW}_n(X)$, and by induction this holds for all $m > n$.

To conclude the argument, we use the fact that

$$\operatorname*{colim}_m C_*(X^{(m)}) \cong C_*(X)$$

(each simplex $\sigma\colon \Delta^n \to X$ factors uniquely through some $X^{(m)}$ by compactness) to deduce that

$$\operatorname*{colim}_m H_*(X^{(m)}) \cong H_*(X)\,.$$

(See the definition, lemma and exercise below.) Since the bonding homomorphisms i_* in degree n of the direct limit system on the left are isomorphisms for all $m > n$, it follows that $H_n(X^{(n+1)}) \cong H_n(X)$, which concludes the proof. □

Definition 2.13.12. The *colimit* of a sequence

$$\dots \longrightarrow A_m \xrightarrow{f_m} A_{m+1} \longrightarrow \dots$$

of abelian groups, with $m \ge 0$, is the abelian group

$$\operatorname*{colim}_m A_m = (\bigoplus_m A_m)/\sim$$

where $\sim$ is the equivalence relation on $\bigoplus_m A_m$ that identifies $x \in A_m \subset \bigoplus_m A_m$ with its image $f_m(x) \in A_{m+1} \subset \bigoplus_m A_m$, for all $m \ge 0$. Let $i_m\colon A_m \to \operatorname{colim}_m A_m$ be the homomorphism that takes $x \in A_m$ to the equivalence class of its image in $\bigoplus_m A_m$. Then the diagram

$$\begin{tikzcd}
\dots \arrow[r] & A_m \arrow[r, "f_m"] \arrow[drr, "i_m"'] & A_{m+1} \arrow[r] \arrow[dr, "i_{m+1}"] & \dots \\
 & & & \operatorname{colim}_m A_m
\end{tikzcd}$$

commutes.

The first part of the following lemma says that $\mathrm{colim}_m A_m$ is the *categorical colimit* of the displayed sequence of abelian groups.

Lemma 2.13.13. *Given any abelian group B, and homomorphisms $j_m \colon A_m \to B$ for $m \ge 0$ that make the diagram*

$$\begin{tikzcd} \dots \arrow[r] & A_m \arrow[r, "f_m"] \arrow[drr, "j_m"'] & A_{m+1} \arrow[r] \arrow[dr, "j_{m+1}"] & \dots \\ & & & B \end{tikzcd}$$

commute, there exists a unique homomorphism $g \colon \mathrm{colim}_m A_m \to B$ such that the diagram

$$\begin{tikzcd} A_m \arrow[dr, "i_m"] \arrow[ddr, "j_m"'] & \\ & \mathrm{colim}_m A_m \arrow[d, "g"] \\ & B \end{tikzcd}$$

commutes for each $m \ge 0$. For each element $y \in \mathrm{colim}_m A_m$ there exists an $m \ge 0$ and an $x \in A_m$ such that $i_m(x) = y$. If $x \in A_m$ is such that $i_m(x) = 0$ in $\mathrm{colim}_m A_m$, then there exists an $n \ge m$ such that $(f_{n-1} \circ \dots \circ f_m)(x) = 0$ in A_n.

Proof. The homomorphism g is uniquely determined by the rule that the image $i_m(x)$ of $x \in A_m$ in $\mathrm{colim}_m A_m$ must be mapped to $j_m(x) \in B$.

Let us use the notation

$$f_m^n = f_{n-1} \circ \dots \circ f_m \colon A_m \to A_n$$

for the composite homomorphism, for $n \ge m$. Any element y of $\mathrm{colim}_m A_m$ is in the image from $\bigoplus_m A_m$, hence is a finite sum of terms $i_{m_1}(x_1) + \dots + i_{m_k}(x_k)$ in the images of $A_{m_1}, \dots, A_{m_k}$ for some $m_1 < \dots < m_k$. Let $m = m_k$, and let

$$x = \sum_{s=1}^{k} f_{m_s}^m(x_s)$$

be the sum of the images of these classes in A_m. Then $i_m(x) = y$ in $\mathrm{colim}_m A_m$.

Let $x \in A_m$ be such that $i_m(x) = 0$. Then the image of x in $\bigoplus_m A_m$ is in equivalence class of 0 for the equivalence relation $\sim$. Hence it is the (signed) sum of a finite number of terms $x_s - f_{m_s}(x_s)$ in $\bigoplus_m A_m$, for some $m_1 < \dots < m_k$, where x_s denotes the image in $\bigoplus_m A_m$ of an element $x_s \in A_{m_s}$, and $f_{m_s}(x_s)$ denotes the image in $\bigoplus_m A_m$ of $f_{m_s}(x_s) \in A_{m_s+1}$. Let $n = m_k + 1$. Then $f_m^n(x)$ in A_n is the (signed) sum of the terms $f_{m_s}^n(x_s) - f_{m_s+1}^n(f_{m_s})(x_s)$ in A_n. Here each term is 0, so $f_m^n(x) = 0$ in A_n, as claimed. □

Exercise 2.13.14. Consider a diagram

$$\dots \to C_*^m \xrightarrow{i_\#} C_*^{m+1} \to \dots$$

of chain complexes (C_*^m, ∂) and chain maps, for all $m \geq 0$. There is an induced diagram

$$\dots \to H_n(C_*^m) \xrightarrow{i_*} H_n(C_*^{m+1}) \to \dots$$

of homology groups, in all degrees n. Prove that there is an isomorphism

$$\operatorname*{colim}_m H_n(C_*^m) \cong H_n(\operatorname*{colim}_m C_*^m)$$

in all degrees n.

Exercise 2.13.15. Prove the cellular (co-)homology theorem for relative homology and cohomology groups with coefficients.

((Warning: For the cohomology of infinite-dimensional CW complexes X, one must replace the passage to colimits in the homological case to a comparison of $H^n(X; G)$ with $\lim_m H^n(X^{(m)}; G)$. This can be done with a Milnor lim-lim^1 sequence, but using the universal coefficient theorem is easier.))

Exercise 2.13.16. Let $k \geq 1$ and consider the CW structure on the unit sphere $S^k \subset \mathbb{R}^{k+1}$, with n-skeleton S^n for $n \leq k$, and two n-cells e_+^n and e_-^n for each $0 \leq n \leq k$. Determine the cellular complex $C_*^{CW}(S^k)$, and compute the cellular homology groups $H_*^{CW}(S^k; G)$ and the cellular cohomology groups $H^*_{CW}(S^k; G)$. Be careful with orientations and signs.

Exercise 2.13.17. Let $S^\infty = \bigcup_k S^k$ be the union of these CW complexes. Determine $C_*^{CW}(S^\infty)$ and compute $H_*^{CW}(S^\infty; G)$ and $H^*_{CW}(S^\infty; G)$. Can you prove that S^∞ is contractible?

Exercise 2.13.18. Let $k \geq 1$ and consider the CW structure on the real projective space $\mathbb{R}P^k = S^k/\sim$, where $x \sim -x$ for $x \in S^k$, with n-skeleton $\mathbb{R}P^n$ for $n \leq k$, and one n-cell e^n for each $0 \leq n \leq k$. Use naturality for the cellular map $f\colon S^k \to \mathbb{R}P^k$ to determine the cellular complex $C_*^{CW}(\mathbb{R}P^k)$, and compute the cellular homology groups $H_*^{CW}(\mathbb{R}P^k; G)$ and the cellular cohomology groups $H^*_{CW}(\mathbb{R}P^k; G)$. You may concentrate on the cases $G = \mathbb{Z}$, $G = \mathbb{Z}/2$ and $G = \mathbb{Z}/p$ with p an odd prime.

Exercise 2.13.19. Let $\mathbb{R}P^\infty = \bigcup_k \mathbb{R}P^k$ be the union of these CW complexes. Determine $C_*^{CW}(\mathbb{R}P^\infty)$ and compute $H_*^{CW}(\mathbb{R}P^\infty; G)$ and $H^*_{CW}(\mathbb{R}P^\infty; G)$. You may concentrate on the cases $G = \mathbb{Z}$, $G = \mathbb{Z}/2$ and $G = \mathbb{Z}/p$ with p an odd prime.

Topologists often write $\mathbb{Z}/m$ where algebraists might write $\mathbb{Z}/m\mathbb{Z}$ or $\mathbb{Z}/(m)$. Hatcher [1] writes $\mathbb{Z}_m$, as topologists used to do, but this is easily confused with the ring of p-adic integers, especially when $m = p$.

((Do $\mathbb{C}P^k$ and $\mathbb{C}P^\infty$ too.))

((Do surfaces, products of CW complexes.))

((Compare Hatcher's notations with other notations like $S_*(X)$, $\Delta_*(X)$ for singular chains, and $C_*(X)$ for cellular chains.))

Chapter 3

The universal coefficient theorems

There is a natural homomorphism

$$G \otimes H_n(X) \longrightarrow H_n(X; G)$$

taking a tensor product $g \otimes [x]$, where x is a singular n-cycle in X, to the homology class $[g \otimes x]$ of the n-cycle $g \otimes x$ in the complex $C_*(X; G)$. This homomorphism is injective, but is not an isomorphism in general.

Similarly, there is a natural homomorphism

$$H^n(X; G) \longrightarrow \mathrm{Hom}(H_n(X), G)$$

taking the cohomology class $[\varphi]$ of an n-cocycle $\varphi\colon C_n(X) \to G$ to the homomorphism $\varphi_*\colon H_n(X) \to G$, taking $[x]$ to $\varphi(x)$, where x is an n-cycle in X. Note that if x is changed by a boundary ∂y, then $\varphi(x)$ changes by $\varphi(\partial y) = (\delta\varphi)(y) = 0$, since φ was assumed to be a cocycle. This homomorphism is surjective, but is not an isomorphism in general.

For each pair of spaces (X, A), the tensor product of the abelian group G with the long exact sequence in homology is a chain complex

$$\cdots \to G \otimes H_n(A) \longrightarrow G \otimes H_n(X) \longrightarrow G \otimes H_n(X, A) \to \dots .$$

but in general this is not an exact complex. On the other hand, the chain complex

$$\cdots \to H_n(A; G) \longrightarrow H_n(X; G) \longrightarrow H_n(X, A; G) \to \dots .$$

is exact. In this sense, the functor $H_n(X; G)$ is better behaved than the functor $G \otimes H_n(X)$.

Similarly, applying $\mathrm{Hom}(-, G)$ to the long exact sequence in homology we get a cochain complex

$$\cdots \to \mathrm{Hom}(H_n(X, A), G) \longrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow \mathrm{Hom}(H_n(A), G) \to \dots .$$

but in general this is not an exact complex. On the other hand, the cochain complex

$$\cdots \to H^n(X, A; G) \longrightarrow H^n(X; G) \longrightarrow H^n(A, G) \to \dots .$$

is exact. In this sense, the functor $H^n(X;G)$ is better behaved than the functor $\mathrm{Hom}(H_n(X),G)$.

3.1 Half-exactness

Let R be a ring (associative, with unit), and let

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

be a short exact sequence of left R-modules. The case $R = \mathbb{Z}$ is of particular interest, since $\mathbb{Z}$-modules are the same as abelian groups. For commutative rings R we do not need to distinguish between left and right R-modules.

Lemma 3.1.1. *Let M and N be left and right R-modules, respectively. Then*

(a)

$$N \otimes_R A \xrightarrow{1\otimes i} N \otimes_R B \xrightarrow{1\otimes j} N \otimes_R C \to 0$$

is exact, but $1 \otimes i$ might not be injective;

(b)

$$0 \to \mathrm{Hom}_R(C,M) \xrightarrow{\mathrm{Hom}(j,1)} \mathrm{Hom}_R(B,M) \xrightarrow{\mathrm{Hom}(i,1)} \mathrm{Hom}_R(A,M)$$

is exact, but $\mathrm{Hom}(i,1)$ *might not be surjective;*

(c)

$$0 \to \mathrm{Hom}_R(M,A) \xrightarrow{\mathrm{Hom}(1,i)} \mathrm{Hom}_R(M,B) \xrightarrow{\mathrm{Hom}(1,j)} \mathrm{Hom}_R(M,C)$$

is exact, but $\mathrm{Hom}(1,j)$ *might not be surjective.*

We say that $N\otimes_R(-)$ is *right exact* and that $\mathrm{Hom}_R(-,M)$ and $\mathrm{Hom}_R(M,-)$ are *left exact*. A functor that preserves short exact sequences will also preserve long exact sequences, and will be called an *exact functor*.

Proof. (a) and (b) were treated in an earlier exercise. Case (c) is an easy exercise. □

Definition 3.1.2. (a) A (right) R-module N is said to be *flat* if

$$1 \otimes i \colon N \otimes_R A \to N \otimes_R B$$

is injective for each R-linear injection $i \colon A \to B$. This is equivalent to asking that the functor $N \otimes_R (-)$ is exact, in the sense that it preserves exact sequences.

(b) A (left) R-module M is said to be *injective* if $\mathrm{Hom}(i,1)\colon \mathrm{Hom}_R(B,M) \to \mathrm{Hom}_R(A,M)$ is surjective for each R-linear injection $i\colon A \to B$. This means that any R-module homomorphism $A \to M$ extends over B:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d] & B \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

and is equivalent to asking that the functor $\mathrm{Hom}_R(-,M)$ is exact.

(c) A (left) R-module M is said to be *projective* if

$$\mathrm{Hom}(1,j)\colon \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C)$$

is surjective for each R-linear surjection $j\colon B \to C$. This means that any R-module homomorphism $M \to C$ lifts through B:

$$\begin{tikzcd} & M \arrow[dl, dashed] \arrow[d] & \\ B \arrow[r, "j"'] & C \arrow[r] & 0 \end{tikzcd}$$

and is equivalent to asking that the functor $\mathrm{Hom}_R(M,-)$ is exact.

Exercise 3.1.3. Prove that an abelian group (viewed as a $\mathbb{Z}$-module) is flat if and only if it is torsion-free, injective if and only if it is divisible, and projective if and only if it is free.

Lemma 3.1.4. *The following are equivalent:*

(a) There is a homomorphism $r\colon B \to A$ with $ri = 1\colon A \to A$.

(b) There is a homomorphism $s\colon C \to B$ with $js = 1\colon C \to C$.

Proof. Given r we may choose s so that $ir+sj = 1\colon B \to B$, and conversely. □

In this case, we say that $0 \to A \to B \to C \to 0$ is a *split (short) exact sequence*. We call r a *retraction* and s a *section*. There are then preferred isomorphisms

$$i+s\colon A \oplus C \xrightarrow{\cong} B$$

and

$$(j,r)\colon B \xrightarrow{\cong} A \times C\,.$$

Lemma 3.1.5. *If $0 \to A \to B \to C \to 0$ is split, then $1 \otimes i\colon N \otimes_R A \to N \otimes_R B$ is split injective, while $\mathrm{Hom}(i,1)\colon \mathrm{Hom}_R(B,M) \to \mathrm{Hom}_R(A,M)$ and $\mathrm{Hom}(1,j)\colon \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C)$ are split surjective.*

Proof. If $r\colon B \to A$ is a retraction, then $(r\otimes 1)(i\otimes 1) = 1$ shows that $i\otimes 1$ is split injective, and $\mathrm{Hom}(i,1)\,\mathrm{Hom}(r,1) = 1$ shows that $\mathrm{Hom}(i,1)$ is split surjective. If $s\colon C \to B$ is the associated section, then $\mathrm{Hom}(1,j)\,\mathrm{Hom}(1,s) = 1$ shows that $\mathrm{Hom}(1,j)$ is split surjective. □

Lemma 3.1.6. *If $C = R\{T\}$ is a free R-module, then any surjection $j\colon B \to C$ admits a section. Hence any short exact sequence $0 \to A \to B \to C \to 0$ with C free is split.*

Proof. For each basis element $t \in T$, use surjectivity to choose an element $s(t) \in B$ with $js(t) = t$. Since C is free, we can extend s R-linearly to obtain the desired homomorphism $s\colon C \to B$ with $js = 1$. □

Lemma 3.1.7. *Any free module is projective. Conversely, any projective module is a direct summand of a free module.*

((Proof.))

3.2 Free resolutions

The previous lemmas show that when applied to short exact sequences $0 \to A \to B \to C \to 0$ with C a free R-module, the functors $N \otimes_R (-)$ and $\mathrm{Hom}_R(-, M)$ are exact. To control the failure of exactness for general R-modules C, we can choose to *resolve* C by free R-modules. This means that we replace C by a chain complex

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0$$

of free R-modules $F_n = R\{T_n\}$, such that $H_0(F_*, \partial) \cong C$ and $H_n(F_*, \partial) = 0$ for $n \neq 0$. An isomorphism $H_0(F_*, \partial) \cong C$ corresponds to a choice of a homomorphism $\epsilon\colon F_0 \to C$ that makes the R-module diagram

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} C \to 0$$

exact at all points. In other words, ϵ is surjective and $\mathrm{im}(\partial_1) = \ker(\epsilon)$. Such a diagram is called an *augmented chain complex.* We call the complex (F_*, ∂) a *free resolution* of the R-module C.

If we think of C is a chain complex concentrated in degree 0, then ϵ can also be viewed as a chain map

$$\epsilon\colon F_* \to C$$

that induces an isomorphism in homology. A free resolution of C is thus a chain complex F_* of free R-modules, with a quasi-isomorphism ϵ to C concentrated in degree 0.

Lemma 3.2.1 (Existence of free resolutions)**.** *Any R-module C admits a free resolution.*

Proof. We construct the free R-modules F_n by induction on $n \geq 0$. First choose any set $T_0 \subseteq C$ that generates C as an R-module, let $F_0 = R\{T_0\}$ be the free R-module generated by that set, and let $\epsilon\colon F_0 \to C$ be the R-linear homomorphism given by sending each generator $t \in T_0$ of F_0 to its image in C. Suppose inductively that we have constructed an exact diagram

$$F_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} C \to 0\,,$$

for some $n \geq 0$. Consider $\ker(\partial_n)$ (interpreted as $\ker(\epsilon)$ when $n = 0$), and choose any set $T_{n+1} \subset \ker(\partial_n)$ that generates $\ker(\partial_n)$ as an R-module. Let $F_{n+1} = R\{T_{n+1}\}$ be the free R-module generated by that set, and let the R-linear homomorphism $\partial_{n+1}\colon F_{n+1} \to F_n$ be given by sending each generator $t \in T_{n+1}$ of F_{n+1} to its image in $\ker(\partial_n) \subset F_n$. Then $\mathrm{im}(\partial_{n+1})$ is the R-submodule of F_n generated by T_{n+1}, which equals $\ker(\partial_n)$. Hence we have extended the exact diagram above one step to the left. Continuing by induction, we obtain a free chain complex (F_*, ∂) with a quasi-isomorphism to C, as desired. □

A *short free resolution* of C is a resolution by a free chain complex (F_*, ∂) concentrated in degrees 0 and 1, giving a short exact sequence

$$0 \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} C \to 0\,.$$

More generally, a *free resolution of length k* is a resolution by a free chain complex F_* with $F_n = 0$ for $n > k$:

$$0 \to F_k \xrightarrow{\partial_k} \dots \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} C \to 0$$

Lemma 3.2.2. *Any abelian group C admits a short free resolution.*

Proof. Define $\epsilon\colon F_0 = \mathbb{Z}\{T_0\} \to C$ as before, by choosing a set of abelian group generators $T_0 \subseteq C$. Then $\ker(\epsilon) \subseteq F_0$ is a subgroup of a free abelian group. It is an algebraic fact that any subgroup of a free abelian group is again a free group. This is easily seen for finitely generated free abelian groups, and the general case can be proved by an application of Zorn's lemma. Hence there is a set $T_1 \subset \ker(\epsilon)$ and an isomorphism $\mathbb{Z}\{T_1\} \cong \ker(\epsilon)$. We let $F_1 = \mathbb{Z}\{T_1\}$ as before. Then

$$0 \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} C \to 0$$

(extended by 0's to the left) is a free resolution of C of length 1, i.e., a short free resolution. □

For example, the free group $C = \mathbb{Z}$ has a (very) short free resolution

$$0 \to \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

with $F_0 = \mathbb{Z}$ and $F_1 = 0$. A finite cyclic group $C = \mathbb{Z}/m$ has a short free resolution

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z}/m \to 0$$

with $F_0 = F_1 = \mathbb{Z}$, where ∂_1 multiplies by m.

Theorem 3.2.3. *Let $\epsilon\colon F_* \to C$ and $\epsilon\colon G_* \to D$ be augmented chain complexes of R-modules, and let $\varphi\colon C \to D$ be an R-linear homomorphism. Suppose that F_n is a free R-module for each $n \ge 0$, and that $\epsilon : G_* \to D$ is a quasi-isomorphism. (For example, $\epsilon\colon F_* \to C$ and $\epsilon\colon G_* \to D$ might both be free R-module resolutions.) Then there exists an R-linear chain map $f_\#\colon F_* \to G_*$ covering φ, in the sense that the diagram*

$$\begin{tikzcd}
\dots \arrow[r] & F_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & F_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"'] & C \arrow[r] \arrow[d, "\varphi"'] & 0 \\
\dots \arrow[r] & G_1 \arrow[r, "\partial_1"] & G_0 \arrow[r, "\epsilon"] & D \arrow[r] & 0
\end{tikzcd}$$

commutes. Moreover, if $g_\#\colon F_* \to G_*$ is a second R-linear chain map covering φ, then there exists an R-linear chain homotopy $P\colon F_* \to G_{*+1}$ from $f_\#$ to $g_\#$.

Proof. We construct the components $f_n\colon F_n \to G_n$ of the chain map $f_\#$ by induction on $n \ge 0$. To start the induction, consider the homomorphism $\varphi\epsilon\colon F_0 \to D$ and the surjection $\epsilon\colon G_0 \to D$. Since F_0 is free, we can find a lift $f_0\colon F_0 \to G_0$, with $\epsilon f_0 = \varphi\epsilon$, by choosing a basis for F_0 and defining $f_0(t)$ to be a lift of $(\varphi\epsilon)(t)$ for each basis element t.

Suppose inductively that we have constructed a chain map $f_\#\colon F_* \to G_*$ in degrees $* \le n$, meaning a finite sequence of homomorphisms $f_m\colon F_m \to G_m$ for $0 \le m \le n$, such that $\epsilon f_0 = \varphi\epsilon$ and $\partial_m f_m = f_{m-1}\partial_m$ for $1 \le m \le n$. We wish to construct a homomorphism $f_{n+1}\colon F_{n+1} \to G_{n+1}$ so that $\partial_{n+1} f_{n+1} = f_n \partial_{n+1}$:

$$\begin{tikzcd}
F_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, dashed, "f_{n+1}"'] & F_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & F_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & \dots \arrow[r] & F_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"'] & C \arrow[d, "\varphi"'] \\
G_{n+1} \arrow[r, "\partial_{n+1}"] & G_n \arrow[r, "\partial_n"] & G_{n-1} \arrow[r] & \dots \arrow[r] & G_0 \arrow[r, "\epsilon"] & D
\end{tikzcd}$$

Consider the homomorphism $f_n\partial_{n+1}\colon F_{n+1} \to G_n$. The composite

$$\partial_n f_n \partial_{n+1}\colon F_{n+1} \to G_{n-1}$$

equals the composite $f_{n-1}\partial_n\partial_{n+1} = 0$ (suitably interpreted for $n = 0$), hence $f_n\partial_{n+1}$ factors through $\ker(\partial_n) = \operatorname{im}(\partial_{n+1})$ in G_n. Since F_{n+1} is free and $\partial_{n+1}\colon G_{n+1} \to \operatorname{im}(\partial_{n+1})$ is surjective, we can find a lift $f_{n+1}\colon F_{n+1} \to G_{n+1}$ with $\partial_{n+1}f_{n+1} = f_n\partial_{n+1}$, by fixing a basis T_{n+1} for F_{n+1}, defining the lift f_{n+1} on each basis element by appealing to the necessary surjectivity, and extending f_{n+1} linearly, as before. This completes the inductive step. Hence there exists a chain map $f_\#\colon F_* \to G_*$ (in all degrees) covering φ, as desired.

Suppose now that $g_\#\colon F_* \to G_*$ is a second such chain map. We construct the components $P_n\colon F_n \to G_{n+1}$ of a chain homotopy P from $f_\#$ to $g_\#$, by induction on $n \geq 0$. For $n = 0$ we know that $\epsilon f_0 = \varphi\epsilon = \epsilon g_0$, so $g_0 - f_0\colon F_0 \to G_0$ lands in $\ker(\epsilon) = \operatorname{im}(\partial_1) \subseteq G_0$. Since F_0 is free, this means that we can find a homomorphism $P_0\colon F_0 \to G_1$ with $\partial_1 P_0 = g_0 - f_0$.

Suppose inductively that we have constructed a homomorphisms $P_m\colon F_m \to G_{m+1}$ for $0 \leq m \leq n$, such that $\partial_1 P_0 = g_0 - f_0$, and $\partial_{m+1}P_m + P_{m-1}\partial_m = g_m - f_m$ for $1 \leq m \leq n$. We wish to construct a homomorphism $P_{n+1}\colon F_{n+1} \to G_{n+2}$ such that $\partial_{n+2}P_{n+1} + P_n\partial_{n+1} = g_{n+1} - f_{n+1}$:

$$\begin{tikzcd}
 & F_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dl, dashed, "P_{n+1}"'] \arrow[d, "g_{n+1}-f_{n+1}"] & F_n \arrow[r, "\partial_n"] \arrow[dl, "P_n"'] \arrow[d, "g_n-f_n"] & F_{n-1} \arrow[dl, "P_{n-1}"] \\
G_{n+2} \arrow[r, "\partial_{n+2}"'] & G_{n+1} \arrow[r, "\partial_{n+1}"'] & G_n &
\end{tikzcd}$$

We know that $\partial n + 1P_n + P_{n-1}\partial_n = g_n - f_n$, so

$$\partial_{n+1}P_n\partial_{n+1} + P_{n-1}\partial_n\partial_{n+1} = g_n\partial_{n+1} - f_n\partial_{n+1}\,,$$

which implies that

$$\partial_{n+1}P_n\partial_{n+1} = \partial_{n+1}g_{n+1} - \partial_{n+1}f_{n+1}$$

since $\partial_n\partial_{n+1} = 0$ and $f_\#$ and $g_\#$ are chain maps. Pence $g_{n+1} - f_{n+1} - P_n\partial_{n+1}\colon F_{n+1} \to G_{n+1}$ lands in $\ker(\partial_{n+1}) = \operatorname{im}(\partial_{n+2})$. Since F_{n+1} is free, this means that we can find a homomorphism $P_{n+1}\colon F_{n+1} \to G_{n+2}$ with

$$\partial_{n+2}P_{n+1} = g_{n+1} - f_{n+1} - P_n\partial_{n+1}$$

which gives the desired relation $\partial_{n+2}P_{n+1} + P_n\partial_{n+1} = g_{n+1} - f_{n+1}$. Continuing by induction, we obtain the full chain homotopy P. □

Proposition 3.2.4 (Essential uniqueness of free resolutions)**.** *Let $\epsilon\colon F_* \to C$ and $\epsilon\colon G_* \to C$ be any two free resolutions of the same R-module C.*

(a) There exist chain maps $f_\#\colon F_* \to G_*$ and $g_\#\colon G_* \to F_*$, both covering the identity map of C, and chain homotopies from $g_\# f_\#\colon F_* \to F_*$ to the identity of F_*, and from $f_\# G_\#\colon G_* \to G_*$ to the identity of G_*. In other words, the two resolutions are chain homotopy equivalent over C.

(b) Any two chain homotopy equivalences $f_\#\colon F_* \to G_*$ and $g_\#\colon F_* \to G_*$ covering the identity map of C are chain homotopic.

Proof. (a) We apply the first part of the theorem to the free resolutions F_* and G_* of C and the identity homomorphism $\varphi = 1_C : C \to C$, to get the chain map $f_\# : F_* \to G_*$ covering 1_C. Switching the roles of F_* and G_*, we get the chain map $g_\# : G_* \to F_*$ covering 1_C. The composite $g_\# f_\#$ and the identity 1_{F_*} are both chain maps $F_* \to F_*$ that cover the identity on C, so by the second part of the theorem there exists a chain homotopy from $g_\# f_\#$ to 1_{F_*}. Likewise, the composite $f_\# g_\#$ and the identity 1_{G_*} are both chain maps $G_* \to G_*$ that cover the identity on C, so by the second part once more there exists a chain homotopy from $f_\# g_\#$ to 1_{G_*}.

(b) Chain homotopy equivalences are chain maps, so this follows from the second part of the theorem. □

3.3 Tor **and** Ext

The failure of exactness of the tensor product $N \otimes_R C$ is measured by the homology of the chain complex $N \otimes_R F_*$ obtained by replacing C by a free resolution F_*. Similarly, the failure of exactness of $\mathrm{Hom}_R(C, M)$ is measured by the cohomology of the cochain complex $\mathrm{Hom}_R(F_*, M)$. These homology and cohomology groups are called Tor- and Ext-groups.

Definition 3.3.1. Let C and M be left R-modules, and let N be a right R-module. Choose any free resolution $\epsilon : F_* \to C$ of C. The Tor-*groups* of N and C are the homology groups

$$\mathrm{Tor}_n^R(N, C) = H_n(N \otimes_R F_*)$$

of the complex $(N \otimes_R F_*, 1 \otimes \partial)$, for $n \geq 0$. The Ext-*groups* of C and M are the cohomology groups

$$\mathrm{Ext}_R^n(C, M) = H^n(\mathrm{Hom}_R(F_*, M))$$

of the cochain complex $(\mathrm{Hom}_R(F_*, M), \mathrm{Hom}(\partial, 1))$, for $n \geq 0$.

We shall soon discuss the mild dependence of the Tor- and Ext-groups on the choice of free resolution, and their resulting functoriality, but first we make some sample calculations.

Example 3.3.2. Consider the case $R = \mathbb{Z}$. A $\mathbb{Z}$-module is the same as an abelian group. Each abelian group C admits a short free resolution

$$0 \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} C \to 0$$

with $F_n = 0$ for $n \geq 2$. Consider any abelian group G. By definition, $\mathrm{Tor}_n^{\mathbb{Z}}(G, C)$ is the n-th homology group of the complex

$$0 \to G \otimes F_1 \xrightarrow{1 \otimes \partial_1} G \otimes F_0 \to 0$$

and $\mathrm{Ext}_{\mathbb{Z}}^n(C, G)$ is the n-th cohomology group of the cochain complex

$$0 \to \mathrm{Hom}(F_0, G) \xrightarrow{\mathrm{Hom}(\partial_1, 1)} \mathrm{Hom}(F_1, G) \to 0\,.$$

By right exactness of $G \otimes (-)$ we deduce that $\mathrm{Tor}_0^{\mathbb{Z}}(G, C) \cong G \otimes C$. By left exactness of $\mathrm{Hom}(-, G)$ we deduce that $\mathrm{Ext}_{\mathbb{Z}}^0(C, G) = \mathrm{Hom}(C, G)$. The vanishing of F_n for $n \geq 2$ implies that $\mathrm{Tor}_n^{\mathbb{Z}}(G, C) = 0$ and $\mathrm{Ext}^n_{\mathbb{Z}}(C, G) = 0$ for these n. Hence the only new Tor-group for $\mathbb{Z}$-modules is

$$\mathrm{Tor}(G, C) = \mathrm{Tor}_1^{\mathbb{Z}}(G, C) = \ker(1 \otimes \partial_1)$$

and the only new Ext-group for $\mathbb{Z}$-modules is

$$\mathrm{Ext}(C, G) = \mathrm{Ext}^1_{\mathbb{Z}}(C, G) = \mathrm{cok}(\mathrm{Hom}(\partial_1, 1))\,.$$

We have exact sequences

$$0 \to \mathrm{Tor}(G, C) \to G \otimes F_1 \xrightarrow{1\otimes\partial_1} G \otimes F_0 \to G \otimes C \to 0$$

and

$$0 \to \mathrm{Ext}(C, G) \to \mathrm{Hom}(F_0, G) \xrightarrow{\mathrm{Hom}(\partial_1,1)} \mathrm{Hom}(F_1, G) \to \mathrm{Hom}(C, G) \to 0\,.$$

For $C = \mathbb{Z}$, we can take $F_0 = \mathbb{Z}$ and $F_1 = 0$. Then

$$\mathrm{Tor}(G, \mathbb{Z}) = \mathrm{Ext}(\mathbb{Z}, G) = 0$$

for any G. For $C = \mathbb{Z}/m$ with $m \geq 1$, we can take $F_0 = F_1 = \mathbb{Z}$ and ∂_1 multiplies by m. Then $G \otimes \mathbb{Z} \cong G$ and

$$\mathrm{Tor}(G, \mathbb{Z}/m) \cong \ker(G \xrightarrow{m} G) = G[m]$$

is the subgroup of elements of order dividing m in G. Furthermore, $\mathrm{Hom}(\mathbb{Z}, G) \cong G$ and

$$\mathrm{Ext}(\mathbb{Z}/m, G) \cong \mathrm{cok}(G \xrightarrow{m} G) = G/mG$$

is the quotient by elements divisible by m in G.

A general finitely generated abelian group $C = \bigoplus_\alpha C_\alpha$ can be written as a (finite) direct sum of cyclic abelian groups of the form $\mathbb{Z}$ or $\mathbb{Z}/m$ for $m \geq 1$. (The case $m = 1$ is trivial.) Then

$$\mathrm{Tor}(G, C) \cong \bigoplus_\alpha \mathrm{Tor}(G, C_\alpha)$$

and

$$\mathrm{Ext}(C, G) \cong \prod_\alpha \mathrm{Ext}(C_\alpha, G)$$

are determined by the special cases we already considered.

Remark 3.3.3. The Tor-group $\mathrm{Tor}(G, C)$ only depends on the torsion subgroups of G and C, and is sometimes called the *torsion product* of G and C. The Ext-group $\mathrm{Ext}(C, G)$ classifies the abelian group extensions of C by G, up to isomorphism.

Lemma 3.3.4. *There are natural isomorphisms*

$$\mathrm{Tor}_0^R(N, C) \cong N \otimes_R C$$

and

$$\mathrm{Ext}^0_R(C, M) \cong \mathrm{Hom}_R(C, M)\,.$$

Proof. There are exact sequences

$$N \otimes_R F_1 \xrightarrow{1\otimes\partial_1} N \otimes_R F_0 \xrightarrow{1\otimes\epsilon} N \otimes_R C \to 0$$

and

$$0 \to \mathrm{Hom}_R(C, M) \xrightarrow{\mathrm{Hom}(\epsilon,1)} \mathrm{Hom}_R(F_0, M) \xrightarrow{\mathrm{Hom}(\partial_1,1)} \mathrm{Hom}_R(F_1, M)$$

by the right exactness of $N \otimes_R (-)$ and the left exactness of $\mathrm{Hom}_R(-, M)$. □

Lemma 3.3.5. *(a) If N is flat then $\mathrm{Tor}_n^R(N, C) = 0$ for all $n \geq 1$.*

(b) If M is injective then $\mathrm{Ext}_R^n(C, M) = 0$ for all $n \geq 1$.

(c) If C is free (or projective) then $\mathrm{Tor}_n^R(N, C) = 0$ and $\mathrm{Ext}_R^n(C, M) = 0$ for all $n \geq 1$.

Proof. If N is flat then

$$N \otimes_R F_{n+1} \xrightarrow{1\otimes\partial_{n+1}} N \otimes_R F_n \xrightarrow{1\otimes\partial_n} N \otimes_R F_{n-1}$$

is exact for all $n \geq 1$.

If M is injective then

$$\mathrm{Hom}_R(F_{n-1}, M) \xrightarrow{\mathrm{Hom}(\partial_n,1)} \mathrm{Hom}_R(F_n, M) \xrightarrow{\mathrm{Hom}(\partial_{n+1},1)} \mathrm{Hom}_R(F_{n+1}, M)$$

is exact for all $n \geq 1$.

If C is free, then we can take $F_0 = C$ and $F_n = 0$ for all $n \geq 1$. If C is projective, then it is a direct summand of a free module C', and $\mathrm{Tor}_n^R(N, C)$ and $\mathrm{Ext}_R^n(C, M)$ are direct summands of the trivial groups $\mathrm{Tor}_n^R(N, C')$ and $\mathrm{Ext}_R^n(C', M)$. □

Proposition 3.3.6. *The groups $\mathrm{Tor}_n^R(N, C)$ and $\mathrm{Ext}_n^R(C, M)$ are well-defined, up to a preferred isomorphism.*

Proof. We must discuss the dependence of $\mathrm{Tor}_n^R(N, C)$ on the choice of free resolution of C.

Let $\epsilon\colon F_* \to C$ and $\epsilon\colon G_* \to C$ be two free resolutions of C. Then there exists a chain homotopy equivalence $f_\#\colon F_* \to G_*$ covering the identity on C, with chain homotopy inverse $g_\#\colon G_* \to F_*$, by the essential uniqueness of free resolutions. Then

$$1 \otimes f_\#\colon N \otimes_R F_* \to N \otimes_R G_*$$

is a chain homotopy equivalence, with chain homotopy inverse $1 \otimes g_\#$. Hence the induced homomorphism $f_*\colon H_n(N \otimes_R F_*) \to H_n(N \otimes_R G_*)$ is an isomorphism, so $\mathrm{Tor}_n^R(N, C)$ computed using F_* is isomorphic to $\mathrm{Tor}_n^R(N, C)$ computed using G_*.

More precisely, any two chain homotopy equivalences $f_\#\colon F_* \to G_*$ and $g_\#\colon F_* \to G_*$ (not related to the chain homotopy inverse above), there exists a chain homotopy $P\colon F_* \to G_{*+1}$ from $f_\#$ to $g_\#$. Then $1 \otimes P\colon N \otimes_R F_* \to N \otimes_R G_{*+1}$ is a chain homotopy from $1 \otimes f_\#$ to $1 \otimes g_\#$, so the induced homomorphisms $f_*\colon H_n(N \otimes_R F_*) \to H_n(N \otimes_R G_*)$ and g_* are equal. Hence the groups $\mathrm{Tor}_n^R(N, C)$ computed using F_* are isomorphic to the ones computing G_*, by a preferred isomorphism.

The Ext-case is very similar. Any two resolutions F_* and G_* of C are connected by a chain homotopy equivalence $f_\#\colon F_* \to G_*$ covering the identity on C, which well-defined up to chain homotopy. Applying $\mathrm{Hom}_R(-, M)$ gives a chain homotopy equivalence

$$\mathrm{Hom}(f_\#, 1)\colon \mathrm{Hom}_R(G_*, M) \to \mathrm{Hom}_R(F_*, M)\,,$$

which is also well-defined up to chain homotopy. Passing to (co-)homology, we get an isomorphism $f^*\colon H_n(\mathrm{Hom}_R(G_*, M)) \to H_n(\mathrm{Hom}_R(F_*, M))$, well-defined with no ambiguity. Hence $\mathrm{Ext}^n_R(C, M)$ computed using F_* is isomorphic to $\mathrm{Ext}^n_R(C, M)$ computed using G_*, by a preferred isomorphism. □

Remark 3.3.7. One way to interpret this form of uniqueness is to say that for each left R-module C we choose a free resolution $F_* \to C$, once and for all, and use that for the definition of $\mathrm{Tor}^R_n(N, C)$ and $\mathrm{Ext}^n_R(C, M)$. Any other free resolution $G_* \to C$ is then chain homotopy equivalent to the chosen resolution, by a chain homotopy equivalence that is well-defined up to chain homotopy. Applying $N \otimes_R (-)$ or $\mathrm{Hom}_R(-, M)$ we then get induced chain homotopy equivalences that are well-defined up to chain homotopy. These induce isomorphisms in homology that are well-defined, with no ambiguity.

Proposition 3.3.8. *The* Tor*-groups are covariantly functorial in N and C, defining a functor*

$$\mathrm{Tor}^R_n\colon (\mathbf{Mod}{-}R) \times (R{-}\mathbf{Mod}) \longrightarrow \mathbf{Ab}\,.$$

The Ext*-groups are contravariantly functorial in C and covariantly functorial in M, defining a functor*

$$\mathrm{Ext}^n_R\colon (R{-}\mathbf{Mod})^{op} \times (R{-}\mathbf{Mod}) \longrightarrow \mathbf{Ab}\,.$$

Proof. Let $\varphi\colon C \to D$ and $\nu\colon N \to N'$ be left and right R-module homomorphisms, respectively. Choose free resolutions $\epsilon\colon F_* \to C$ and $\epsilon\colon G_* \to D$. There is then chain map $f_\#\colon F_* \to G_*$ covering φ, unique up to chain homotopy. Hence there is a chain map

$$\nu \otimes f_\#\colon N \otimes_R F_* \to N' \otimes_R G_*\,,$$

well-defined up to chain homotopy. The induced map in homology

$$(\nu, \varphi)_*\colon\ \mathrm{Tor}^R_n(N, C) \longrightarrow \mathrm{Tor}^R_n(N', D)$$

is well-defined. This specifies the functor Tor^R_n on morphisms.

To check compatibility with compositions, let $\psi\colon D \to E$ and $\nu'\colon N' \to N''$ be two more homomorphisms. Choose a free resolution $\epsilon\colon H_* \to E$, and a chain map $g_\#\colon G_* \to H_*$ covering ψ. Then the composite $g_\# f_\#\colon F_* \to H_*$ is a chain map covering $\psi\varphi\colon C \to E$, so the composite

$$(\nu' \otimes g_\#)(\nu \otimes f_\#) = \nu'\nu \otimes g_\# f_\#\colon N \otimes_R F_* \to N'' \otimes_R H_*$$

induces $(\nu'\nu, \psi\varphi)_*\colon\ \mathrm{Tor}^R_n(N, C) \to \mathrm{Tor}^R_n(N'', E)$ in homology, and equals the composite of $(\nu, \varphi)_*$ and $(\nu', \psi)_*\colon\ \mathrm{Tor}^R_n(N', D) \to \mathrm{Tor}^R_n(N'', E)$.

Compatibility with identities is equally obvious, hence Tor^R_n is covariantly functorial in both variables. The Ext-case is very similar, replacing $\otimes_R$ by Hom_R. □

Remark 3.3.9. We may say that Tor_n^R is a *bifunctor* in $\mathbf{Mod}-R$ and $R-\mathbf{Mod}$. For each fixed right R-module N, the rule $C \mapsto \mathrm{Tor}_n^R(N,C)$ defines a functor $R-\mathbf{Mod} \to \mathbf{Ab}$, taking $\varphi\colon C \to D$ to $\varphi_*\colon \mathrm{Tor}_n^R(N,C) \to \mathrm{Tor}_n^R(N,D)$. Similarly, for each fixed left R-module C, the rule $N \mapsto \mathrm{Tor}_n^R(N,C)$ defines a functor $\mathbf{Mod}-R \to \mathbf{Ab}$, taking $\nu\colon N \to N'$ to $\nu_*\colon \mathrm{Tor}_n^R(N,C) \to \mathrm{Tor}_n^R(N',C)$. Having a bifunctor is more than just having a rule that is separately functorial in each variable. The two functors must also be compatible in the sense that the diagram

$$\begin{tikzcd}
\mathrm{Tor}_n^R(N,C) \arrow[r, "\varphi_*"] \arrow[d, "\nu_*"'] & \mathrm{Tor}_n^R(N,D) \arrow[d, "\nu_*"] \\
\mathrm{Tor}_n^R(N',C) \arrow[r, "\varphi_*"] & \mathrm{Tor}_n^R(N',D)
\end{tikzcd}$$

commutes, since $(\nu,1_D)(1_N,\varphi) = (\nu,\varphi) = (1_{N'},\varphi)(\nu,1_C)$. In briefer notation, we must have $\nu_*\varphi_* = \varphi_*\nu_*$. In fact, this condition is necessary and sufficient for the rule (separately functorial in each variable) to constitute a bifunctor (simultaneously functorial in both variables).

Similarly, Ext_R^n is a bifunctor in $(R-\mathbf{Mod})^{op}$ and $R-\mathbf{Mod}$. The homomorphisms $\varphi_*\colon \mathrm{Ext}_R^n(N,C) \to \mathrm{Ext}_R^n(N,D)$ (for each N) and $\nu^*\colon \mathrm{Ext}_R^n(N',C) \to \mathrm{Ext}_R^n(N,C)$ (for each C) satisfy $\nu^*\varphi_* = \varphi_*\nu^*$.

Exercise 3.3.10. Do Exercises 2 and 3 from Section 3.1 of Hatcher [1].

The following long exact sequences are much better suited for inductive arguments, e.g. by the five-lemma, than the half-exact sequences given only by their lowest three terms.

Proposition 3.3.11. *Let C be a left R-module, and let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of right R-modules. Then there is a long exact sequence of* Tor*-groups*

$$\cdots \to \mathrm{Tor}_n^R(N',C) \to \mathrm{Tor}_n^R(N,C) \to \mathrm{Tor}_n^R(N'',C) \xrightarrow{\partial} \mathrm{Tor}_{n-1}^R(N',C) \to \dots$$

(natural in C and the short exact sequence), that ends

$$\cdots \to \mathrm{Tor}_1^R(N'',C) \xrightarrow{\partial} N' \otimes_R C \to N \otimes_R C \to N'' \otimes_R C \to 0\,.$$

In particular, for $R = \mathbb{Z}$ and $0 \to G' \to G \to G'' \to 0$ a short exact sequence of abelian groups there is a six-term exact sequence

$$\begin{aligned}0 \to \mathrm{Tor}(G',C) \to \mathrm{Tor}(G,C) \to \mathrm{Tor}(G'',C)& \\ \xrightarrow{\partial} G' \otimes C \to G \otimes C \to G'' \otimes C \to 0\,.&\end{aligned}$$

Proof. Let F_* be a free resolution of C. There is then a short exact sequence of chain complexes

$$0 \to N' \otimes_R F_* \to N \otimes_R F_* \to N'' \otimes_R F_* \to 0\,.$$

(In degree n it is a direct sum of copies of the short exact sequence $0 \to N' \to N \to N'' \to 0$.) The associated long exact sequence of homology groups is the desired long exact sequence of Tor-groups. □

Proposition 3.3.12. *Let C be a left R-module, and let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of left R-modules. Then there is a long exact sequence of* Ext*-groups*

$$\dots \to \mathrm{Ext}^n_R(C, M') \to \mathrm{Ext}^n_R(C, M) \to \mathrm{Ext}^n_R(C, M'') \xrightarrow{\delta} \mathrm{Ext}^{n+1}_R(C, M') \to \dots$$

(natural in C and the short exact sequence), that begins

$$0 \to \mathrm{Hom}_R(C, M') \to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(C, M'') \xrightarrow{\delta} \mathrm{Ext}^1_R(C, M') \to \dots$$

In particular, for $R = \mathbb{Z}$ and $0 \to G' \to G \to G'' \to 0$ a short exact sequence of abelian groups there is a six-term exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(C, G') \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(C, G'')& \\ \xrightarrow{\delta} \mathrm{Ext}(C, G') \to \mathrm{Ext}(C, G) \to \mathrm{Ext}(C, G'') \to 0\,.& \end{aligned}$$

Proof. Let F_* be a free resolution of C. There is then a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_R(F_*, M') \to \mathrm{Hom}_R(F_*, M) \to \mathrm{Hom}_R(F_*, M'') \to 0\,.$$

(In degree n it is a product of copies of the short exact sequence $0 \to M' \to M \to M'' \to 0$.) The associated long exact sequence of cohomology groups is the desired long exact sequence of Ext-groups. □

((Exercises: Can compute Ext using projective and/or resolutions and Tor using left and/or right flat resolutions. Balance Ext and Tor.))

3.4 The universal coefficient theorem in homology

We will prove the following theorem, expressing homology with arbitrary coefficients in terms of integral homology.

Theorem 3.4.1 (Universal coefficient theorem)**.** *Let (X, A) be a pair of topological spaces, and let G be an abelian group. There is a natural short exact sequence*

$$0 \to G \otimes H_n(X, A) \xrightarrow{\alpha} H_n(X, A; G) \to \mathrm{Tor}(G, H_{n-1}(X, A)) \to 0$$

for each n. The sequence is split, but not naturally split.

Example 3.4.2. I particular, α is always an isomorphism if G is flat, i.e., a torsion free abelian group. For example, we have isomorphisms $\mathbb{Q} \otimes H_*(X) \cong H_*(X; \mathbb{Q})$ and $\mathbb{Z}_{(p)} \otimes H_*(X) \cong H_*(X; \mathbb{Z}_{(p)})$ for any prime p, where $\mathbb{Z}_{(p)}$ denotes the integers localized at p.

More generally, let R be a ring, and N a right R-module. The chain complex $C_*(X, A; R) = R \otimes C_*(X, A)$ is a complex of free left R-modules, so that $H_*(X, A; R)$ is a left R-module, and there is a natural isomorphism

$$C_*(X, A; N) \cong N \otimes_R C_*(X, A; R)\,.$$

There is a natural homomorphism

$$\alpha\colon N \otimes_R H_*(X, A; R) \longrightarrow H_*(X, A; N)$$

taking $n \otimes [x]$ to $[n \otimes x]$, where $n \in N$ and x is a cycle in $C_*(X, A; R)$.

Definition 3.4.3. A commutative ring R is said to be a *principal ideal domain* if it is an integral domain (any product of nonzero elements in nonzero) and each ideal is principal (it can be generated by a single element).

The ring $R = \mathbb{Z}$ is a principal ideal domain, as is any field. When R is a principal ideal domain, every submodule of a free R-module is free. This leads to the vanishing of $\mathrm{Tor}_n^R(N, C)$ and $\mathrm{Ext}_R^n(C, M)$ for all $n \geq 2$, as in the case of the integers. The theorem above is the special case $R = \mathbb{Z}$ and $N = G$ of the following result.

Theorem 3.4.4. *Let (X, A) be a pair of topological spaces, let R be a principal ideal domain, and let N be a right R-module. There is a natural short exact sequence*

$$0 \to N \otimes_R H_n(X, A; R) \xrightarrow{\alpha} H_n(X, A; N) \to \mathrm{Tor}_1^R(N, H_{n-1}(X, A)) \to 0$$

for each n. The sequence is split, but not naturally split.

This topological theorem is in turn the special case $C_* = C_*(X, A; R)$ of the following algebraic proposition.

Proposition 3.4.5. *Let R be a principal ideal domain, let (C_*, ∂) be a chain complex of free left R-modules, and let N be a right R-module. There is a natural short exact sequence*

$$0 \to N \otimes_R H_n(C_*) \xrightarrow{\alpha} H_n(N \otimes_R C_*) \to \mathrm{Tor}_1^R(N, H_{n-1}(C_*)) \to 0$$

for each n. The sequence is split, but not naturally split.

Proof. Let

$$B_n = \mathrm{im}(\partial) \subseteq Z_n = \ker(\partial) \subseteq C_n\,,$$

as usual. Each left R-module C_n is free, hence so is each submodule B_n and Z_n, since R is a PID. For each n there is a short exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0\,.$$

Since B_{n-1} is free, this sequence is split. Tensoring with N on the left, we get a (split) short exact sequence

$$0 \to N \otimes_R Z_n \to N \otimes_R C_n \xrightarrow{\partial} N \otimes_R B_{n-1} \to 0$$

for each n, where we abbreviate $1 \otimes \partial$ to ∂. These fit together in a commutative diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & N \otimes_R Z_{n+1} \arrow[r] \arrow[d, "0"'] & N \otimes_R C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"'] & N \otimes_R B_n \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & N \otimes_R Z_n \arrow[r] \arrow[d, "0"'] & N \otimes_R C_n \arrow[r, "\partial"] \arrow[d, "\partial"'] & N \otimes_R B_{n-1} \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & N \otimes_R Z_{n-1} \arrow[r] \arrow[d] & N \otimes_R C_{n-1} \arrow[r, "\partial"] \arrow[d] & N \otimes_R B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

which we view as a short exact sequence of chain complexes

$$0 \to (N \otimes_R Z_*, 0) \longrightarrow (N \otimes_R C_*, \partial) \xrightarrow{\partial} (N \otimes_R B_{*-1}, 0) \to 0\,.$$

Here $(N \otimes_R Z_*, 0)$ denotes the chain complex with $N \otimes_R Z_n$ in degree n and zero maps as boundary homomorphisms, while $(N \otimes_R B_{*-1}, 0)$ denotes the chain complex with $N \otimes_R B_{n-1}$ in degree n and zero boundaries.

The associated long exact sequence in homology contains the terms

$$N \otimes_R B_n \xrightarrow{k_n} N \otimes_R Z_n \longrightarrow H_n(N \otimes_R C_*) \longrightarrow N \otimes_R B_{n-1} \xrightarrow{k_{n-1}} N \otimes_R Z_{n-1}\,,$$

where k_n denotes the connecting homomorphism. Hence there is a natural short exact sequence

$$0 \to \operatorname{cok}(k_n) \xrightarrow{\alpha} H_n(N \otimes_R C_*) \to \ker(k_{n-1}) \to 0\,.$$

Chasing the definition of the connecting homomorphism, we see that $k_n = 1 \otimes \iota_n$ is equal to the tensor product of N with the inclusion $\iota_n \colon B_n \subseteq Z_n$, in each degree n.

((Exercise: Make that chase.))

By the definition of homology, there is a short exact sequence

$$0 \to B_n \xrightarrow{\iota_n} Z_n \xrightarrow{\epsilon_n} H_n(C_*) \to 0\,.$$

Since B_n and Z_n are free left R-modules, this is a short free resolution (F_*, ∂) of the homology R-module $H_n(C_*)$. In the notation used above, $F_1 = B_n$, $F_0 = Z_n$ and $\partial_1 = \iota_n$. Hence the Tor-groups of N and $H_n(C_*)$ are the homology groups of the complex $N \otimes_R F_*$, so that there is an exact sequence

$$0 \to \operatorname{Tor}_1^R(N, H_n(C_*)) \to N \otimes_R B_n \xrightarrow{1 \otimes \iota_n} N \otimes_R Z_n \to N \otimes_R H_n(C_*) \to 0\,.$$

In other words, there are natural isomorphisms $\ker(k_n) \cong \operatorname{Tor}_1^R(N, H_n(C_*))$ and $\operatorname{cok}(k_n) \cong N \otimes_R H_n(C_*)$, for all n. Hence we have a natural short exact sequence

$$0 \to N \otimes_R H_n(C_*) \xrightarrow{\alpha} H_n(N \otimes_R C_*) \longrightarrow \operatorname{Tor}_1^R(N, H_{n-1}(C_*)) \to 0\,.$$

By inspection of the definitions, the left hand homomorphism α takes $n \otimes [x]$ in $N \otimes_R H_n(C_*)$ to $[n \otimes x]$ in $H_n(N \otimes_R C_*)$, for each n-cycle $x \in Z_n$.

To see that the universal coefficient short exact sequence admits a splitting, choose a retraction $r \colon C_n \to Z_n$ in the split short exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$. The composite

$$\epsilon_n r \colon C_n \to Z_n \to H_n(C_*)$$

defines a chain map $(C_*, \partial) \to (H_*(C_*), 0)$, since r restricts to the identity on Z_n, so that $\epsilon_n r$ is zero on B_n. Tensoring with N on the left, we get a chain map

$$(N \otimes_R C_*, \partial) \longrightarrow (N \otimes_R H_*(C_*), 0)$$

and an induced map in homology

$$H_n(C_*; N) \xrightarrow{\rho} N \otimes_R H_n(C_*)$$

for each n. This is a retraction for the universal coefficient sequence, since $\rho\alpha$ takes $n \otimes [x]$ via $[n \otimes x]$ to $n \otimes \epsilon_n r(x)$, and $\epsilon_n r(x) = [x]$ for any $x \in Z_n$. $\square$

Exercise 3.4.6. Adapt Exercise 11 from Section 3.1 of Hatcher [1] to prove that the splitting in the universal coefficient theorem for homology cannot be natural.

Exercise 3.4.7. Let R be a PID. Let $f \colon (X, A) \to (Y, B)$ be a map of pairs such that $f_* \colon H_*(X, A; R) \to H_*(Y, B; R)$ is an isomorphism in all degrees. Prove that $f_* \colon H_*(X, A; N) \to H_*(Y, B; N)$ is an isomorphism in all degrees, for every right R-module N.

3.5 The universal coefficient theorem in cohomology

We now turn to the theorem expressing cohomology with arbitrary coefficients in terms of integral homology.

Theorem 3.5.1 (Universal coefficient theorem)**.** *Let (X, A) be a pair of topological spaces, and let G be an abelian group. There is a natural short exact sequence*

$$0 \to \mathrm{Ext}(H_{n-1}(X, A), G) \to H^n(X, A; G) \xrightarrow{\beta} \mathrm{Hom}(H_n(X, A), G) \to 0$$

for each n. The sequence is split, but not naturally split.

Example 3.5.2. In particular, β is always an isomorphism if G is injective, i.e., a divisible abelian group. For example, we have isomorphisms $H^n(X; \mathbb{Q}) \cong \mathrm{Hom}(H_n(X), \mathbb{Q})$ and $H^n(X; \mathbb{Q}/\mathbb{Z}_{(p)}) \cong \mathrm{Hom}(H_n(X), \mathbb{Q}/\mathbb{Z}_{(p)})$ for any prime p, where $\mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ can be identified under the isomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$ with the group of p-th power roots of unity in $S^1 \subset \mathbb{C}^*$.

More generally, let R be a ring and M a left R-module. There is a natural isomorphism

$$C^*(X, A; M) = \mathrm{Hom}(C_*(X, A), M) \cong \mathrm{Hom}_R(C_*(X, A; R), M)\,,$$

and a natural homomorphism

$$\beta\colon H^*(X,A;M) \longrightarrow \mathrm{Hom}_R(H_*(X,A;R),M)$$

taking $[\varphi]$ to $[x] \mapsto \varphi(x)$, where the group homomorphism $\varphi\colon C_*(X,A) \to M$ extends R-linearly in a unique way to $\varphi\colon C_*(X,A;R) \to M$ and x is a cycle in $C_*(X,A;R)$. Here is a more general version of the theorem above.

Theorem 3.5.3. *Let (X,A) be a pair of topological spaces, let R be a principal ideal domain, and let M be a left R-module. There is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X,A;R),M) \to H^n(X,A;M) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(X,A;R),M) \to 0$$

for each n. The sequence is split, but not naturally split.

Again, this follows from the following result in the case $C_* = C_*(X,A;R)$.

Proposition 3.5.4. *Let R be a principal ideal domain, let (C_*,∂) be a chain complex of free left R-modules, and let M be a left R-module. There is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*),M) \to H^n(\mathrm{Hom}_R(C_*,M)) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(C_*),M) \to 0$$

for each n. The sequence is split, but not naturally split.

Proof. Let $B_n = \mathrm{im}(\partial) \subseteq Z_n = \ker(\partial) \subseteq C_n$. These are free R-modules, since R is a PID and C_n is assumed to be free. For each n there is a short exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0\,,$$

which splits since B_{n-1} is free. Applying $\mathrm{Hom}_R(-,M)$, we get a (split) short exact sequence

$$0 \to \mathrm{Hom}_R(B_{n-1},M) \xrightarrow{\delta} \mathrm{Hom}_R(C_n,M) \longrightarrow \mathrm{Hom}_R(Z_n,M) \to 0$$

for each n, where we abbreviate $\mathrm{Hom}(\partial,1)$ to δ. These fit together in a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_R(B_{n-2},M) \arrow[r, "\delta"] \arrow[d, "0"'] & \mathrm{Hom}_R(C_{n-1},M) \arrow[r] \arrow[d, "\delta"'] & \mathrm{Hom}_R(Z_{n-1},M) \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(B_{n-1},M) \arrow[r, "\delta"] \arrow[d, "0"'] & \mathrm{Hom}_R(C_n,M) \arrow[r] \arrow[d, "\delta"'] & \mathrm{Hom}_R(Z_n,M) \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(B_n,M) \arrow[r, "\delta"] \arrow[d] & \mathrm{Hom}_R(C_{n+1},M) \arrow[r] \arrow[d] & \mathrm{Hom}_R(Z_{n+1},M) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

which we view as a short exact sequence of cochain complexes

$$0 \to (\mathrm{Hom}_R(B_{*-1}, M), 0) \xrightarrow{\delta} (\mathrm{Hom}_R(C_*, M), \delta) \longrightarrow (\mathrm{Hom}_R(Z_*, M), 0) \to 0\,.$$

Here $(\mathrm{Hom}_R(B_{*-1}, M), 0)$ denotes the cochain complex with $\mathrm{Hom}_R(B_{n-1}, M)$ in degree n and zero maps as coboundary homomorphisms, and $(\mathrm{Hom}_R(Z_*, M), 0)$ denotes the cochain complex with $\mathrm{Hom}_R(Z_n, M)$ in degree n and zero coboundaries.

The associated long exact sequence in cohomology contains the terms

$$\begin{aligned}\mathrm{Hom}_R(Z_{n-1}, M) \xrightarrow{k^{n-1}} \mathrm{Hom}_R(B_{n-1}, M) \longrightarrow H^n(\mathrm{Hom}_R(C_*, M)) \\ \longrightarrow \mathrm{Hom}_R(Z_n, M) \xrightarrow{k^n} \mathrm{Hom}_R(B_n, M)\,.\end{aligned}$$

Hence there is a natural short exact sequence

$$0 \to \mathrm{cok}(k^{n-1}) \to H^n(\mathrm{Hom}_R(C_*, M)) \xrightarrow{\beta} \ker(k^n) \to 0\,.$$

Chasing the definition of the connecting homomorphism, we see that $k^n = \mathrm{Hom}(\iota_n, 1)$ is Hom-dual to the inclusion $\iota_n\colon B_n \subseteq Z_n$, in each degree n.

By the definition of homology, there is a short exact sequence

$$0 \to B_n \xrightarrow{\iota_n} Z_n \xrightarrow{\epsilon_n} H_n(C_*) \to 0\,.$$

Since B_n and Z_n are free, this is a short free resolution (F_*, ∂) of the homology R-module $H_n(C_*)$. Hence the Ext-groups of $H_n(C_*)$ and M are the cohomology groups of the cochain complex $\mathrm{Hom}_R(F_*, M)$, so that there is an exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_R(H_n(C_*), M) \longrightarrow \mathrm{Hom}_R(Z_n, M) \xrightarrow{\mathrm{Hom}(\iota_n,1)} \mathrm{Hom}_R(B_n, M) \\ \longrightarrow \mathrm{Ext}^1_R(H_n(C_*), M) \to 0\,.\end{aligned}$$

In other words, there are natural isomorphisms $\ker(k^n) \cong \mathrm{Hom}_R(H_n(C_*), M)$ and $\mathrm{cok}(k^n) \cong \mathrm{Ext}^1_R(H_n(C_*), M)$, for all n. Hence we have a natural short exact sequence

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*), M) \to H^n(\mathrm{Hom}_R(C_*, M)) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(C_*), M) \to 0\,.$$

By inspection of the definitions, the right hand homomorphism β takes $[\varphi]$ in $H^n(\mathrm{Hom}_R(C_*, M))$ to the homomorphism mapping $[x]$ in $H_n(C_*)$ to $\varphi(x)$ in M.

To see that the universal coefficient short exact sequence admits a splitting, choose a retraction $r\colon C_n \to Z_n$ in the split short exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$. The composite $\epsilon_n r\colon C_n \to Z_n \to H_n(C_*)$ defines a chain map $(C_*, \partial) \to (H_*(C_*), 0)$, since r restricts to the identity on Z_n, so that $\epsilon_n r$ is zero on B_n. Applying Hom into M we get a cochain map

$$(\mathrm{Hom}_R(H_*(C_*), M), 0) \longrightarrow (\mathrm{Hom}_R(C_*, M), \delta)$$

and an induced map in cohomology

$$\mathrm{Hom}_R(H_n(C_*), M) \xrightarrow{\sigma} H^n(\mathrm{Hom}_R(C_*, M))$$

for each n. This is a section for the universal coefficient sequence. □

Exercise 3.5.5. Do Exercise 11 from Section 3.1 in Hatcher [1], to see that the splitting in the universal coefficient theorem for cohomology cannot be natural.

Exercise 3.5.6. Let R be a PID. Let $f\colon (X,A) \to (Y,B)$ be a map of pairs such that $f_*\colon H_*(X,A;R) \to H_*(Y,B;R)$ is an isomorphism in all degrees. Prove that $f^*\colon H^*(Y,B;M) \to H^*(X,A;M)$ is an isomorphism in all degrees, for every left R-module M.

Exercise 3.5.7. Let X be a CW complex. Recall from proof of the cellular homology theorem that the inclusion $X^{(m)} \to X$ induces isomorphisms $H_n(X^{(m)}) \cong H_n(X)$ for all $n < m$. Deduce that there are isomorphisms $H^n(X;G) \cong H^n(X^{(m)};G)$ for all $n < m$, where G is any abelian group. Use this to complete the proof of the cellular cohomology theorem, in the case of infinite-dimensional X.

3.6 Some calculations

Example 3.6.1. Consider $X = \mathbb{R}P^3$ with the minimal CW structure, having one cell e^n in each dimension for $0 \le n \le 3$. The cellular complex $C_* = C_*^{CW}(\mathbb{R}P^3)$ is

$$0 \to \mathbb{Z}\{e^3\} \xrightarrow{0} \mathbb{Z}\{e^2\} \xrightarrow{2} \mathbb{Z}\{e^1\} \xrightarrow{0} \mathbb{Z}\{e^0\} \to 0\,.$$

Hence the integral homology groups are

$$H_n(\mathbb{R}P^3) = \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}/2 & \text{for } n = 1 \\ 0 & \text{for } n = 2 \\ \mathbb{Z} & \text{for } n = 3 \\ 0 & \text{for } n > 3. \end{cases}$$

It follows that

$$G \otimes H_n(\mathbb{R}P^3) = \begin{cases} G & \text{for } n = 0,\ n = 3 \\ G/2G & \text{for } n = 1 \\ 0 & \text{for } n = 2,\ n > 3 \end{cases}$$

and

$$\mathrm{Tor}(G, H_n(\mathbb{R}P^3)) = \begin{cases} G[2] & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$H_n(\mathbb{R}P^3; G) \cong \begin{cases} G & \text{for } n = 0 \\ G/2G & \text{for } n = 1 \\ G[2] & \text{for } n = 2 \\ G & \text{for } n = 3 \\ 0 & \text{for } n > 3. \end{cases}$$

Similarly,

$$\mathrm{Hom}(H_n(\mathbb{R}P^3), G) = \begin{cases} G & \text{for } n = 0,\ n = 3 \\ G[2] & \text{for } n = 1,\ n > 3 \\ 0 & \text{for } n = 2 \end{cases}$$

and

$$\operatorname{Ext}(H_n(\mathbb{R}P^3), G) = \begin{cases} G/2G & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$H^n(\mathbb{R}P^3; G) \cong \begin{cases} G & \text{for } n = 0 \\ G[2] & \text{for } n = 1 \\ G/2G & \text{for } n = 2 \\ G & \text{for } n = 3 \\ 0 & \text{for } n > 3. \end{cases}$$

For instance,

$$H^n(\mathbb{R}P^3; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n = 1 \\ \mathbb{Z}/2 & \text{for } n = 2 \\ \mathbb{Z} & \text{for } n = 3 \\ 0 & \text{for } n > 3. \end{cases}$$

Notice how the torsion in $H_n(X)$ is shifted up to $H^{n+1}(X;\mathbb{Z})$, while the free part of $H_n(X)$ reappears in $H^n(X;\mathbb{Z})$.

((State a proposition.)) Since $H_0(X)$ is always free, $\operatorname{Tor}(G, H_0(X)) = 0$ and $\operatorname{Ext}(H_0(X), G) = 0$, so for $n \le 1$ there are isomorphisms

$$\alpha\colon G \otimes H_n(X) \xrightarrow{\cong} H_n(X; G)$$

and

$$\beta\colon H^n(X; G) \xrightarrow{\cong} \operatorname{Hom}(H_n(X), G)\,.$$

Similarly for relative (co-)homology.

Exercise 3.6.2. Do Exercise 5 in Section 3.1 of Hatcher [1], proving that there is an isomorphism

$$H^1(X; G) \cong \operatorname{Hom}(\pi_1(X), G)$$

for path-connected X.

((Handle as field case.)) The group $\mathbb{Q}$ is torsion free and divisible, so $\operatorname{Tor}(\mathbb{Q}, H_{n-1}(X)) = 0$ and $\operatorname{Ext}(H_{n-1}(X), \mathbb{Q}) = 0$, and there are isomorphisms

$$\alpha\colon H_n(X) \otimes \mathbb{Q} \xrightarrow{\cong} H_n(X; \mathbb{Q})$$

and

$$\beta\colon H^n(X; \mathbb{Q}) \xrightarrow{\cong} \operatorname{Hom}(H_n(X), \mathbb{Q})$$

for all n. It follows that there is an isomorphism

$$H^n(X; \mathbb{Q}) \cong \operatorname{Hom}_{\mathbb{Q}}(H_n(X; \mathbb{Q}), \mathbb{Q})$$

identifying $H^n(X;\mathbb{Q})$ with the dual $\mathbb{Q}$-vector space of $H_n(X;\mathbb{Q})$.

((Handle for PID's.)) If $H_*(X)$ is of *finite type*, meaning that $H_n(X)$ is finitely generated for each n, then we can write

$$H_n(X) = T_n \oplus F_n$$

where T_n is a finite abelian group, and F_n is a finitely generated free abelian group. If $F_n \cong \mathbb{Z}^r$ we say that $H_n(X)$ has *rank* r. Note that $\mathbb{Q} \otimes T_n = 0$ and $\mathbb{Q} \otimes \mathbb{Z}^r = \mathbb{Q}^r$, so the rank of $H_n(X)$ equals the dimension of $H_n(X;\mathbb{Q}) \cong \mathbb{Q} \otimes H_n(X)$ as a $\mathbb{Q}$-vector space. The rank of $H_n(X)$ is also known as the *n-th Betti number* of X.

3.7 Field coefficients

((Recall reduced homology.))

Proposition 3.7.1. *Let X be any space. Then $\tilde{H}_*(X) = 0$ if and only if $\tilde{H}_*(X;\mathbb{Q}) = 0$ and $\tilde{H}_*(X;\mathbb{Z}/p) = 0$ for all primes p.*

Corollary 3.7.2. *A map $f\colon X \to Y$ induces isomorphisms*

$$f_*\colon H_*(X) \xrightarrow{\cong} H_*(Y)$$

in integral homology if and only if it induces isomorphisms

$$f_*\colon H_*(X;F) \xrightarrow{\cong} H_*(Y;F)$$

with coefficients in the fields $F = \mathbb{Q}$ and $F = \mathbb{Z}/p$, for all primes p.

This follows by passage to the mapping cone C_f, using the long exact sequence

$$\dots \to H_n(X;G) \xrightarrow{f_*} H_n(Y;G) \longrightarrow \tilde{H}_n(C_f;G) \to \dots$$

for $G = \mathbb{Z}$ and $G = F$.

Proof. The forward implication is clear from the universal coefficient theorem in homology. For the converse, assume that $\tilde{H}_*(X;\mathbb{Q}) = 0$ and $\tilde{H}_*(X;\mathbb{Z}/p)$ for all primes p. From the short exact sequence

$$0 \to \tilde{H}_n(X)/p \to \tilde{H}_n(X;\mathbb{Z}/p) \to \tilde{H}_{n-1}(X)[p] \to 0$$

we deduce that $\tilde{H}_n(X)/p = 0$ and $\tilde{H}_{n-1}(X)/[p] = 0$, so multiplication by p on $\tilde{H}_n(X)$ is an isomorphism. Hence $\tilde{H}_n(X) \to \tilde{H}_n(X) \otimes \mathbb{Q}$, which inverts every prime, is already an isomorphism. But

$$\tilde{H}_n(X) \otimes \mathbb{Q} \cong \tilde{H}_n(X;\mathbb{Q})$$

is zero by assumption, so $\tilde{H}_n(X) = 0$. □

Lemma 3.7.3. *Suppose that $R = F$ is a field. Then $\operatorname{Tor}_n^F(N,C) = 0$ and $\operatorname{Ext}_F^n(C,M) = 0$ for all $n \geq 1$.*

Proof. Any module over a field is free, so C (concentrated in degree 0) is a very short free resolution of itself. □

Proposition 3.7.4. *Let (X,A) be a pair of topological spaces, and let F be a field. There is a natural isomorphism*

$$\beta\colon H^n(X,A;F) \xrightarrow{\cong} \operatorname{Hom}_F(H_n(X,A;F),F) = H_n(X,A;F)^*$$

There is a more general version of the universal coefficient theorems, for a principal ideal domain R and an R-module M. Replacing $C_*(X)$ by $C_*(X;R)$ one is led to work with a chain complex C_* of free R-modules. The assumption that R is a PID ensures that the submodules B_* and Z_* are still free. This leads to the split short exact sequences

$$0 \to H_n(X;R) \otimes_R M \to H_n(X;M) \to \mathrm{Tor}_1^R(H_{n-1}(X;R), M) \to 0$$

and

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R), M) \to H^n(X;M) \to \mathrm{Hom}_R(H_n(X;R), M) \to 0\,,$$

and similarly for relative (co-)homology. In the case where R is a field F the derived functors Tor_1^F and Ext^1_F vanish. This leads to the stated isomorphism.

Corollary 3.7.5. *$\tilde{H}_*(X) = 0$ if and only if $\tilde{H}^*(X;\mathbb{Q}) = 0$ and $\tilde{H}^*(X;\mathbb{Z}/p) = 0$ for all primes p.*

3.8 Bockstein homomorphisms

((Discuss $\beta\colon H_n(X;C) \to H_{n-1}(A)$ and $\beta\colon H^n(X;C) \to H^{n+1}(X;A)$ associated to a SES $0 \to A \to B \to C \to 0$. How to recover $H_*(X)$ from $H_*(X;\mathbb{Q})$ and $H_*(X;\mathbb{Z}/p)$ for all p?))

Chapter 4

Cup product

We turn to a method of introducing product structures

$$\cup\colon C^k(X;R)\otimes_R C^\ell(X;R)\longrightarrow C^{k+\ell}(X;R)$$

and

$$\cup\colon H^k(X;R)\otimes_R H^\ell(X;R)\longrightarrow H^{k+\ell}(X;R)$$

on the singular cochains and singular cohomology of a space X, where R is a ring, called the cochain cup product and the cohomology cup product, respectively.

The basic idea is to use the diagonal map $\Delta\colon X\to X\times X$, taking x to $\Delta(x)=(x,x)$, and the induced cochain map

$$\Delta^{\#}\colon C^*(X\times X;R)\to C^*(X;R)\,.$$

This requires an understanding of the relationship between the tensor product of cochains $C^k(X;R)\otimes_R C^\ell(X;R)$ and the cochains of the cartesian product $C^{k+\ell}(X\times X;R)$. This will be provided by an Alexander–Whitney homomorphism

$$AW\colon C^k(X;R)\otimes_R C^\ell(Y;R)\longrightarrow C^{k+\ell}(X\times Y;R)$$

in the special case $X=Y$. To motivate the introduction of this homomorphism for singular cochains, we first take a closer look at the cellular and simplicial cases.

4.1 Cellular diagonal approximations

Suppose first that X is a CW complex. Then the product $X\times X$ also has a natural CW structure, and we can describe the cellular (co-)chain complex of $X\times X$ in terms of the one for X. It is just as easy to describe the CW structure on $X\times Y$, where Y is another CW complex. This added generality also has the advantage of keeping the two factors apart in the following discussion.

Definition 4.1.1. Let X and Y be CW complexes. Then the product $X\times Y$ has a CW structure with n-skeleton

$$(X\times Y)^{(n)}=\bigcup_{k+\ell=n}X^{(k)}\times Y^{(\ell)}\,.$$

For each k-cell e^k_α of X and each ℓ-cell e^ℓ_β of Y, there is a $(k+\ell)$-cell

$$e^{k+\ell}_{\alpha,\beta} = e^k_\alpha \times e^\ell_\beta$$

of $X \times Y$, with characteristic map

$$\Phi_{\alpha,\beta} \colon D^{k+\ell} \cong D^k \times D^\ell \xrightarrow{\Phi_\alpha \times \Phi_\beta} X^{(k)} \times Y^{(\ell)} \subset (X \times Y)^{(k+\ell)} .$$

The attaching map of $e^k_\alpha \times e^\ell_\beta$ is

$$\begin{aligned} \varphi_{\alpha,\beta} \colon S^{k+\ell-1} &\cong S^{k-1} \times D^\ell \cup D^k \times S^\ell \\ &\xrightarrow{\varphi_\alpha \times \Phi_\beta \cup \Phi_\alpha \times \varphi_\beta} X^{(k-1)} \times Y^{(\ell)} \cup X^{(k)} \times Y^{(\ell-1)} \subset X^{(k+\ell-1)} . \end{aligned}$$

Note that we have to choose a homeomorphism $D^{k+\ell} \cong D^k \times D^\ell$. We can do this in a systematic way, by fixing homeomorphisms $D^n \cong I^n = I \times \cdots \times I$ for all $n \geq 0$, and then use the composite isomorphism

$$D^{k+\ell} \cong I^{k+\ell} \cong I^k \times I^\ell \cong D^k \times D^\ell .$$

Its restriction to the boundary then gives the homeomorphism

$$S^{k+\ell-1} \cong S^{k-1} \times D^\ell \cup D^k \times S^{\ell-1} ,$$

where

$$S^{k-1} \times D^\ell \cap D^k \times S^{\ell-1} = S^{k-1} \times S^{\ell-1} .$$

Since the set of n-cells of $X \times Y$ is in one-to-one correspondence with the pairs of k-cells of X and ℓ-cells of Y, as k and ℓ range over the pairs of non-negative integers with $k+\ell = n$, we get an isomorphism

$$\times \colon \bigoplus_{k+\ell=n} C^{CW}_k(X) \otimes C^{CW}_\ell(Y) \xrightarrow{\cong} C^{CW}_n(X \times Y)$$

that takes the generator $e^k_\alpha \otimes e^\ell_\beta$ on the left to the generator $e^{k+\ell}_{\alpha,\beta} = e^k_\alpha \times e^\ell_\beta$ on the right.

Geometrically, this corresponds to the homeomorphisms

$$\begin{aligned} \bigvee_{k+\ell=n} X^{(k)}/X^{(k-1)} \wedge Y^{(\ell)}/Y^{(\ell-1)} &\xleftarrow{\cong} \bigvee_{k+\ell=n} \bigvee_\alpha S^k \wedge \bigvee_\beta S^\ell \\ &\cong \bigvee_{\alpha,\beta,k+\ell=n} S^{k+\ell} \xrightarrow{\cong} (X \times Y)^{(n)}/(X \times Y)^{(n-1)} \end{aligned}$$

under the identifications $C^{CW}_k(X) = H_k(X^{(k)}, X^{(k-1)}) \cong \tilde{H}_k(X^{(k)}/X^{(k-1)})$, etc.

Lemma 4.1.2. *The cellular boundary operators in $C^{CW}_*(X)$, $C^{CW}_*(Y)$ and $C^{CW}_*(X \times Y)$ are related by the formula*

$$\partial_n(e^k_\alpha \times e^\ell_\beta) = \partial_k(e^k_\alpha) \times e^\ell_\beta + (-1)^k e^k_\alpha \times \partial_\ell(e^\ell_\beta) .$$

((ETC: Discuss proof!))

This leads us to make the following algebraic definition.

Definition 4.1.3. Let (C_*, ∂) and (D_*, ∂) be chain complexes, of right and left R-modules, respectively. Let the $(C_* \otimes_R D_*, \partial)$ be the chain complex of abelian groups with

$$(C_* \otimes_R D_*)_n = \bigoplus_{k+\ell=n} C_k \otimes_R D_\ell$$

and with boundary homomorphism

$$\partial\colon (C_* \otimes_R D_*)_n \longrightarrow (C_* \otimes_R D_*)_{n-1}$$

given by the formula

$$\partial(a \otimes b) = \partial(a) \otimes b + (-1)^k a \otimes \partial(b)$$

for $a \in C_k$ and $b \in D_\ell$, with $k + \ell = n$. This defines a chain complex, since

$$\begin{aligned}\partial(\partial(a \otimes b)) &= \partial(\partial(a) \otimes b) + (-1)^k \partial(a \otimes \partial(b)) \\ &= \partial^2 a \otimes b + (-1)^{k-1} \partial a \otimes \partial b + (-1)^k \partial a \otimes \partial b - a \otimes \partial^2 b = 0\,.\end{aligned}$$

Proposition 4.1.4. *The cross product $\times$ defines an isomorphism of chain complexes*

$$\times\colon C_*^{CW}(X) \otimes C_*^{CW}(Y) \xrightarrow{\cong} C_*^{CW}(X \times Y)\,.$$

Returning to the case $X = Y$, the next step would be to consider the chain map $\Delta_\#$ induced by the diagonal $\Delta\colon X \to X \times X$. However, Δ takes the interior of an n-cell e^n_γ in X to the interior of the $2n$-cell $e^n_\gamma \times e^n_\gamma$ in $X \times X$. Hence Δ does not preserve the skeleton filtration of the CW structures on X and $X \times X$, so Δ is not a cellular map, and it does not immediately induce a chain map of cellular chain complexes. (It does induce a chain map of singular chain complexes, but for the proposition above only concerns the cellular chain complexes.)

To sidestep this problem, we would like to use the cellular approximation theorem, see [1, Thm. 4.8.].

Theorem 4.1.5 (Cellular approximation)**.** *Any map $f\colon X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, then the homotopy may be taken to be constant on that subcomplex.*

Hence there exists a cellular map $D\colon X \to X \times X$ that is homotopic to Δ, called a diagonal approximation. It induces a chain map $D_\#\colon C_*^{CW}(X) \to C_*^{CW}(X \times X)$. Any other cellular map $D'\colon X \to X \times X$ that is homotopic to Δ is also homotopic to D. The homotopy $H\colon X \times I \to X \times X$ from D to D' need not be cellular, but it is cellular on the subcomplex $X \times \{0, 1\}$, so it is homotopic to another homotopy $P\colon X \times I \to X \times X$ that is cellular. Then P induces a chain homotopy from $D_\#$ to $D'_\#$, so the chain homotopy class of $D_\#$ is well-defined.

Suppose then, that $\varphi\colon C_k^{CW}(X) \to R$ and $\psi\colon C_\ell^{CW}(X) \to R$ are given cellular cochains on X with values in a ring R. Then their product $\varphi \cdot \psi$, given by the composite

$$C_k^{CW}(X) \otimes C_\ell^{CW}(X) \xrightarrow{\varphi \otimes \psi} R \otimes R \xrightarrow{\cdot} R\,,$$

can be extended by zero to the other summands of

$$C_*^{CW}(X) \otimes C_*^{CW}(X)$$

in degree $n = k + \ell$. Using the cross product isomorphism to $C_n^{CW}(X \times X)$ we can form the composite homomorphism

$$C_n^{CW}(X) \xrightarrow{D_n} C_n^{CW}(X \times X) \cong [C_*^{CW}(X) \otimes C_*^{CW}(X)]_n \xrightarrow{\varphi \cdot \psi} R$$

to get a cochain $\varphi \cup \psi$ in $C^n_{CW}(X; R)$. In this way we get the desired pairing

$$C^k_{CW}(X; R) \otimes C^\ell_{CW}(X; R) \xrightarrow{\cup} C^{k+\ell}_{CW}(X; R) .$$

((Discuss when this induces a well-defined pairing in cohomology.)) The precise cochain will depend on the choice of cellular diagonal approximation D, but any two choices differ by a chain homotopy.

Exercise 4.1.6. Consider the CW structure on $\mathbb{R}P^2 = \mathbb{R}P^1 \cup e^2$, with 1-skeleton $\mathbb{R}P^1$ and a single 2-cell attached by the double covering map $\varphi \colon S^1 \to \mathbb{R}P^1$. Construct a cellular approximation $D \colon \mathbb{R}P^2 \to \mathbb{R}P^2 \times \mathbb{R}P^2$ to the diagonal map $\Delta \colon \mathbb{R}P^2 \to \mathbb{R}P^2 \times \mathbb{R}P^2$. Hint: D takes $\mathbb{R}P^1$ to

$$\mathbb{R}P^1 \vee \mathbb{R}P^1 \subset \mathbb{R}P^1 \times \mathbb{R}P^1 \subset \mathbb{R}P^2 \times \mathbb{R}P^2$$

by first going around one of the 1-cells, and then the other one. The challenge is how to extend D over the 2-cell of $\mathbb{R}P^2$, i.e., to show that $D\varphi$ is null-homotopic.

One difficulty with this approach, is that the choice of diagonal approximation D is not explicit. If we restrict further, to the case of Δ-complexes X, then Alexander and Whitney (independently?) gave an explicit formula for a natural choice of such a diagonal approximation.

Suppose, then, that X is a Δ-complex. Then the simplicial chain complex $\Delta_*(X)$ is isomorphic to the cellular chain complex $C_*^{CW}(X)$. Note, however, that the product CW structure on $X \times X$ is not a Δ-complex, since the characteristic map of a product of simplices $\sigma^k_\alpha \colon \Delta^k \to X$ and $\sigma^\ell_\beta \colon \Delta^\ell \to X$ is a map $\Delta^k \times \Delta^\ell \to X \times X$ from a product of simplices, rather than from a $(k + \ell)$-simplex, and the attaching map of its boundary is not compatible with any identification of $\Delta^k \times \Delta^\ell$ with $\Delta^{k+\ell}$.

We shall describe a chain map

$$\begin{aligned}\Psi_\# \colon \Delta_*(X) &= C_*^{CW}(X) \\ &\xrightarrow{D_\#} C_*^{CW}(X \times X) \cong C_*^{CW}(X) \otimes C_*^{CW}(X) = \Delta_*(X) \otimes \Delta_*(X)\end{aligned}$$

induced by a cellular, but not simplicial, map $D \colon X \to X \times X$.

We use the notation $\Delta^n = [v_0, \dots, v_n]$, meaning that Δ^n is the convex hull of the points $v_0, \dots, v_n$.

In degree 0, the cellular map takes a 0-simplex $\rho \colon \Delta^0 \to X$ to the 0-cell $(\rho, \rho) \colon \Delta^0 \to X \times X$ in $X \times X$, which corresponds to $\rho \otimes \rho$ in $C_0^{CW}(X) \otimes C_0^{CW}(X)$. Hence

$$\Psi_0(\rho) = \rho \otimes \rho$$

on cellular chains.

In degree 1, the cellular map takes a 1-simplex $\sigma \colon \Delta^1 \to X$ to the composite map

$$\Delta^1 \cong [v_0] \times [v_0, v_1] \cup [v_0, v_1] \times [v_1] \subset \Delta^1 \times \Delta^1 \xrightarrow{\sigma \times \sigma} X \times X .$$

Hence

$$\Psi_1(\sigma) = \sigma|[v_0] \otimes \sigma + \sigma \otimes \sigma|[v_1]$$

on cellular chains. For instance,

$$a \longrightarrow b$$

maps to

$$\begin{tikzcd}
a \times b \arrow[r] & b \times b \\
a \times a \arrow[u] &
\end{tikzcd} .$$

In degree 2, the cellular map takes a 2-simplex $\tau \colon \Delta^2 \to X$ to the composite map

$$\Delta^2 \cong [v_0]\times[v_0, v_1, v_2]\cup[v_0, v_1]\times[v_1, v_2]\cup[v_0, v_1, v_2]\times[v_2] \subset \Delta^2\times\Delta^2 \xrightarrow{\tau\times\tau} X\times X\,.$$

Hence

$$\Psi_2(\tau) = \tau|[v_0] \otimes \tau + \tau|[v_0, v_1] \otimes \tau|[v_1, v_2] + \tau \otimes \tau|[v_2]$$

on cellular chains. For instance,

$$\begin{tikzcd}
c & \\
& b \arrow[ul] \\
a \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

maps to

$$\begin{tikzcd}
& & & c \times c \\
& a \times c \arrow[r] \arrow[urr] & b \times c \arrow[ur] & \\
& a \times b \arrow[r] \arrow[u] & b \times b \arrow[u] & \\
a \times a \arrow[uur] \arrow[ur] & & &
\end{tikzcd} .$$

((Compatible with boundary. ETC.))

In hindsight, we may note that these formulas for Ψ_n make sense in a natural way for arbitrary singular simplices in X, independent of its Δ-complex structure. This leads to the construction of the diagonal approximation in singular (co-)homology, in the following section.

4.2 The Alexander–Whitney diagonal approximation

Let $k, \ell \geq 0$. Inside the standard $(k+\ell)$-simplex

$$\Delta^{k+\ell} = [v_0, \dots, v_k, \dots, v_{k+\ell}]$$

there is a *front* k-simplex

$$\Delta^k \cong [v_0, \dots, v_k] \subseteq \Delta^{k+\ell}$$

where $t_{k+1} = \dots = t_{k+\ell} = 0$, and a *back* ℓ-simplex

$$\Delta^\ell \cong [v_k, \dots, v_{k+\ell}] \subseteq \Delta^{k+\ell}$$

where $t_0 = \dots = t_{k-1} = 0$. These meet in the single vertex v_k, where $t_k = 1$. Let $\lambda^k_{k+\ell} \colon \Delta^k \to \Delta^{k+\ell}$ and $\rho^\ell_{k+\ell} \colon \Delta^\ell \to \Delta^{k+\ell}$ be the two affine linear embeddings.

To each singular $(k+\ell)$-simplex

$$\sigma \colon \Delta^{k+\ell} \to X$$

in a topological space X, we can associate the *front k-face*

$$\sigma\lambda^k_{k+\ell} = \sigma|[v_0, \dots, v_k] \colon \Delta^k \to X$$

and the *back ℓ-face*

$$\sigma\rho^\ell_{k+\ell} = \sigma|[v_k, \dots, v_{k+\ell}] . \colon \Delta^\ell \to X$$

Their tensor product defines a homomorphism

$$\Psi_{k,\ell} \colon C_{k+\ell}(X) \longrightarrow C_k(X) \otimes C_\ell(X)$$

that takes σ to

$$\sigma\lambda^k_{k+\ell} \otimes \sigma\rho^\ell_{k+\ell} = \sigma|[v_0, \dots, v_k] \otimes \sigma|[v_k, \dots, v_{k+\ell}] .$$

For $k = \ell = 0$, the homomorphism $\Psi_{0,0} \colon C_0(X) \to C_0(X) \otimes C_0(X)$ corresponds to the diagonal map $\Delta \colon X \to X \times X$ taking a point $p \in X$ to $(p,p) \in X \times X$, under the correspondences $C_0(X) \cong \mathbb{Z}\{X\}$ and $C_0(X) \otimes C_0(X) \cong \mathbb{Z}\{X\} \otimes \mathbb{Z}\{X\} \cong \mathbb{Z}\{X \times X\}$.

For two chain complexes (C_*, ∂) and (D_*, ∂), we define the *tensor product* chain complex $(C_* \otimes D_*, \partial)$ to be given in degree n by

$$(C \times D)_n = \bigoplus_{k+\ell=n} C_k \otimes D_\ell$$

with boundary homomorphism given by

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^k x \otimes \partial y$$

for $x \in C_k$ and $y \in D_\ell$. Note that

$$\begin{aligned}
\partial^2(x \otimes y) &= \partial(\partial x \otimes y + (-1)^k x \otimes \partial y) \\
&= \partial^2 x \otimes y + (-1)^{k-1} \partial x \otimes \partial y + (-1)^k \partial x \otimes \partial y + (-1)^{2k} x \otimes \partial^2 y \\
&= 0
\end{aligned}$$

so that $(C_*\otimes D_*, \partial)$ is a chain complex. The sign $(-1)^k$ can be justified geometrically, since we are commuting the passage to a boundary past the k-dimensional object k, or algebraically, to make sure that the two middle terms in the above sum cancel.

For each $n \geq 0$, we can form the sum over all (k, ℓ) with $k + \ell = n$ of the homomorphisms $\Psi_{k,\ell}$, to get the homomorphism

$$\Psi_n : C_n(X) \longrightarrow \bigoplus_{k+\ell=n} C_k(X) \otimes C_\ell(X)$$

taking $\sigma \colon \Delta^n \to X$ to

$$\bigoplus_{k+\ell=n} = \sigma|[v_0, \dots, v_k] \otimes \sigma|[v_k, \dots, v_{k+\ell}] \,.$$

Lemma 4.2.1. *The identity*

$$\Psi_{k,\ell} \circ \partial = (\partial \otimes 1)\Psi_{k+1,\ell} + (-1)^k (1 \otimes \partial)\Psi_{k,\ell+1}$$

holds, so the homomorphisms $(\Psi_n)_n$ *define a chain map*

$$\Psi_\# : C_*(X) \longrightarrow C_*(X) \otimes C_*(X) \,.$$

Since Ψ_0 is compatible with the diagonal map, we call $\Psi_\#$ a *diagonal approximation.*

Proof. We must prove that the diagram

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "\Psi_{n+1}"'] & C_n(X) \arrow[d, "\Psi_n"'] \\
(C_*(X) \otimes C_*(X))_{n+1} \arrow[r, "\partial"] & (C_*(X) \otimes C_*(X))_n
\end{tikzcd}$$

commutes, for each n. We check that for each $\sigma \colon \Delta^{n+1} \to X$ and each pair (k, ℓ) with $k + \ell = n$, the images of σ under $\Psi_n \partial$ and $\partial \Psi_{n+1}$ have the same components in $C_k(X) \otimes C_\ell(X)$.

The (k, ℓ)-th component of $\partial\Psi_{n+1}(\sigma)$ is the sum of two contributions. One comes from the composite

$$(\partial \otimes 1)\Psi_{k+1,\ell} \colon C_{n+1}(X) \to C_{k+1}(X) \otimes C_\ell(X) \to C_k(X) \otimes C_\ell(X)$$

and the other comes from the composite

$$(-1)^k (1 \otimes \partial)\Psi_{k,\ell+1} \colon C_{n+1}(X) \to C_k(X) \otimes C_{\ell+1}(X) \to C_k(X) \otimes C_\ell(X) \,.$$

The first takes $\sigma \colon \Delta^{n+1} \to X$ to

$$\begin{aligned}
&(\partial \otimes 1)\sigma|[v_0, \dots, v_{k+1}] \otimes \sigma|[v_{k+1}, \dots, v_{n+1}] \\
&\qquad = \sum_{i=0}^{k+1} (-1)^i \sigma|[v_0, \dots, \hat{v}_i, \dots, v_{k+1}] \otimes \sigma|[v_{k+1}, \dots, v_{n+1}]
\end{aligned}$$

and the second takes σ to

$$
\begin{aligned}
(-1)^k(1\otimes\partial)\sigma|[v_0,\dots,v_k]\otimes\sigma|[v_k,\dots,v_{n+1}]
&= (-1)^k\sum_{j=0}^{\ell+1}(-1)^j\sigma|[v_0,\dots,v_k]\otimes\sigma|[v_k,\dots,\hat v_{k+j},\dots,v_{n+1}]\\
&= \sum_{i=k}^{n+1}(-1)^i\sigma|[v_0,\dots,v_k]\otimes\sigma|[v_k,\dots,\hat v_i,\dots,v_{n+1}]
\end{aligned}
$$

Notice that the term $i=k+1$ in the first sum is equal to the term $i=k$ in the second sum, up to a sign. Hence these two terms cancel when we add the expressions together, so that the (k,ℓ)-th component of $\partial\Psi_n(\sigma)$ is

$$
\sum_{i=0}^{k}(-1)^i\sigma|[v_0,\dots,\hat v_i,\dots,v_{k+1}]\otimes\sigma|[v_{k+1},\dots,v_{n+1}]
+\sum_{i=k+1}^{n+1}(-1)^i\sigma|[v_0,\dots,v_k]\otimes\sigma|[v_k,\dots,\hat v_i,\dots,v_{n+1}]\,.
$$

On the other hand, the (k,ℓ)-th component of $\Psi_n\partial(\sigma)$ is

$$
\begin{aligned}
&\Psi_{k,\ell}(\sum_{i=0}^{n+1}(-1)^i\sigma|[v_0,\dots,\hat v_i,\dots,v_{n+1}])\\
&\quad=\sum_{i=0}^{k}(-1)^i\sigma|[v_0,\dots,\hat v_i,\dots,v_{k+1}]\otimes\sigma|[v_{k+1},\dots,v_{n+1}]\\
&\qquad+\sum_{i=k+1}^{n+1}(-1)^i\sigma|[v_0,\dots,v_k]\otimes\sigma|[v_k,\dots,\hat v_i,\dots,v_{n+1}]\,.
\end{aligned}
$$

These expressions are the same, proving the claim. □

4.3 The cochain cup product

Let R be a ring, and consider cochains and cohomology with coefficients in R.

The *cochain cup product* is a pairing

$$
C^k(X;R)\otimes_R C^\ell(X;R)\xrightarrow{\cup}C^{k+\ell}(X;R)\,.
$$

For cochains $\varphi\colon C_k(X)\to R$ and $\psi\colon C_\ell(X)\to R$ the cup product is defined to be the $(k+\ell)$-cochain

$$
\varphi\cup\psi\colon C_{k+\ell}(X)\to R
$$

given as the composite

$$
C_{k+\ell}(X)\xrightarrow{\Psi_{k,\ell}}C_k(X)\otimes C_\ell(X)\xrightarrow{\varphi\otimes\psi}R\otimes R\xrightarrow{\cdot}R
$$

where $\Psi_{k,\ell}$ is as in the previous subsection and $\cdot\colon R\otimes R\to R$ is the ring multiplication.

More explicitly, the cup product $\varphi \cup \psi$ takes the value

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|[v_0, \dots, v_k]) \cdot \psi(\sigma|[v_k, \dots, v_{k+\ell}])$$

on a $(k+\ell)$-simplex $\sigma \colon \Delta^{k+\ell} \to X$.

Lemma 4.3.1. *The cochain cup product is unital and associative, with unit element $1 \in C^0(X;R)$ the cochain $\epsilon \colon C_0(X) \to R$ that sends each 0-simplex to the ring unit $1 \in R$.*

A *graded ring* is a graded abelian group $A_* = (A_n)_n$ with a unital and associative pairing

$$A_k \otimes A_\ell \to A_{k+\ell}$$

for all k, ℓ, which we can also write as a homomorphism

$$A_* \otimes A_* \to A_* \,.$$

By the lemma above the cochains $C^*(X;R)$ constitute a graded ring.

4.4 The cohomology cup product

The cochain cup product satisfies a Leibniz formula.

Lemma 4.4.1. *The identity*

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$$

holds in $C^{k+\ell+1}(X;R)$, for $\varphi \in C^k(X;R)$ and $C^\ell(X;R)$, so the cup product defines a cochain map

$$\cup \colon C^*(X;R) \otimes_R C^*(X;R) \longrightarrow C^*(X;R) \,.$$

Proof. Let $n = k+\ell$. For each $(n+1)$-simplex σ, we have

$$\delta(\varphi \cup \psi)(\sigma) = (\varphi \cup \psi)(\partial\sigma) = (\varphi \otimes \psi)(\Psi_{k,\ell} \circ \partial)(\sigma)$$

which by the lemma of the previous subsection is the sum of

$$(\varphi \otimes \psi)(\partial \otimes 1)\Psi_{k+1,\ell}(\sigma) = (\delta\varphi \otimes \psi)\Psi_{k+1,\ell}(\sigma) = (\delta\varphi \cup \psi)(\sigma)$$

and

$$(-1)^k(\varphi \otimes \psi)(1 \otimes \partial)\Psi_{k,\ell+1}(\sigma) = (-1)^k(\varphi \otimes \delta\psi)\Psi_{k,\ell+1}(\sigma) = (-1)^k(\varphi \cup \delta\psi)(\sigma) \,.$$

□

Corollary 4.4.2. *If $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$ are cocycles, then $\varphi \cup \psi \in C^{k+\ell}(X;R)$ is a cocycle. If furthermore φ is a coboundary, or ψ is a coboundary, then $\varphi \cup \psi$ is a coboundary.*

Proof. If $\delta\varphi = 0$ and $\delta\psi = 0$ then $\delta(\varphi \cup \psi) = 0 \cup \psi + (-1)^k \varphi \cup 0 = 0$. If also $\varphi = \delta\xi$ then $\delta(\xi \cup \psi) = \varphi \cup \psi + \xi \cup 0 = \varphi \cup \psi$. If instead $\psi = \delta\eta$ then $\delta(\varphi \cup \eta) = 0 \cup \eta + (-1)^k \varphi \cup \psi = (-1)^k \varphi \cup \psi$. □

The *cohomology cup product* is the induced pairing

$$H^k(X;R) \otimes_R H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X;R)$$

given by the formula

$$[\varphi] \cup [\psi] = [\varphi \cup \psi]$$

for each k-cocycle φ and each ℓ-cocycle ψ. It is well-defined by the corollary above. The cup product makes $H^*(X;R)$ a graded ring.

Lemma 4.4.3. *The cohomology cup product is unital and associative, with unit element $1 \in H^0(X;R)$ the cohomology class of the cocycle $\epsilon\colon C_0(X) \to R$ that sends each 0-simplex to the ring unit $1 \in R$.*

A cup product for simplicial cohomology can be defined by the same formula as for singular cohomology. Hence the isomorphism between singular cohomology and simplicial cohomology is compatible with the cup products, so that for simplicial complexes, or more generally, for Δ-complexes, the cup products in singular cohomology can be computed using simplicial cochains.

Example 4.4.4. The closed orientable surface M_g of genus $g \geq 1$ has a triangulation as a Δ-complex obtained by triangulating a regular $4g$-gon by starring with an interior point, and identifying the boundary edges pairwise according to the pattern

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$$

The integral homology groups are $H_0(M_g) = \mathbb{Z}$,

$$H_1(M_g) = \mathbb{Z}\{a_1, b_1, \dots, a_g, b_g\}$$

and $H_2(M_g) \cong \mathbb{Z}$. A generator of $H_2(M_g)$ is represented by the 2-cycle given by the signed sum of all of the 2-simplices in the triangulation, with sign $+1$ for the 2-simplices spanned by the center and one of the positively oriented edges a_i or b_j, and sign -1 for the 2-simplices spanned by the center and one of the negatively oriented edges a_i^{-1} or b_j^{-1}.

Dually, the integral cohomology groups are $H^0(M_g) = H^0(M_g;\mathbb{Z}) = \mathbb{Z}\{1\}$,

$$H^1(M_g) = H^1(M_g;\mathbb{Z}) = \mathbb{Z}\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$$

and $H^2(M_g) = H^2(M_g;\mathbb{Z}) \cong \mathbb{Z}\{\gamma\}$, with α_i dual to a_i and β_i dual to b_i. ((Recall what duality means for a basis.)) A generator $\gamma \in H^2(M_g)$ is represented by a 2-cochain/cocycle that takes the value $+1$ on the 2-cycle representing a generator of $H_2(M_g)$. The interesting cup product is the pairing

$$\cup\colon H^1(M_g) \otimes H^1(M_g) \longrightarrow H^2(M_g)\,.$$

To compute cup products, we must represent the cohomology classes α_i and β_j by 1-cocycles, say φ_i and ψ_j.

The condition $\delta\varphi_i = 0$ asserts that the alternating sum of values of φ_i on the three edges of each 2-simplex in M_g must be 0. To represent α_i, φ_i must evaluate to 1 on the edge a_i. By inspection, we can let φ_i evaluate to 1 on the two edges leading from the center to the end-points of a_i and a_i^{-1}, and to 0 on all other edges.

Similarly, ψ_j evaluates to 1 on the edge b_j, as well as on the two edges leading from the center to the end-points of b_j and b_j^{-1}, and to 0 on all other edges.

The cup product of two 1-cocycles φ and ψ is the 2-cocycle whose value on a 2-simplex σ is the product

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|[v_0, v_1]) \cdot \psi(\sigma|[v_1, v_2]) .$$

The 2-simplices of M_g fall into g groups of four triangles each. The cocycles φ_i and ψ_i are zero outside of the i-th group, so if $i \neq j$ the cup product of φ_i or ψ_i with φ_j or ψ_j is zero on all 2-simplices. Hence these cup products are zero on the simplicial cochain level, and

$$a_i \cup a_j = a_i \cup b_j = b_i \cup a_j = b_i \cup b_j = 0$$

for $i \neq j$.

Fortunately, the case $i = j$ is more interesting. The cup product $\varphi_i \cup \psi_i$ takes the value

$$\varphi_i(a_i) \cdot \psi_i(b_i) = 1 \cdot 1 = 1$$

on the 2-simplex spanned by the center and the edge b_i, and is zero on the other 2-simplices. Hence this cup product evaluates to $+1$ on the 2-cycle representing the generator of $H_2(M_g)$, so the cohomology cup product

$$\alpha_i \cup \beta_i = \gamma$$

equals the dual generator of $H^2(M_g)$.

The cup product $\psi_i \cup \varphi_i$ takes the value

$$\psi_i(b_i) \cdot \varphi_i(a_i) = 1 \cdot 1 = 1$$

on the 2-simplex spanned by the center and the edge a_i^{-1}, and is zero on the other 2-simplices. Hence this cup product evaluates to -1 on the 2-cycle representing the generator of $H_2(M_g)$, so the cohomology cup product

$$\beta_i \cup \alpha_i = -\gamma$$

equals the negative of the dual generator of $H^2(M_g)$.

The cup products $\varphi_i \cup \varphi_i$ and $\psi_i \cup \psi_i$ are zero on all 2-simplices, so the cohomology cup products $\alpha_i \cup \alpha_i$ and $\beta_i \cup \beta_i$ are both zero.

The bilinear pairing $H^1(M_g) \times H^1(M_g) \to H^2(M_g)$ is thus identified with the bilinear pairing $\mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}$ represented by the skew-symmetric $2g \times 2g$ matrix

$$\begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{bmatrix}$$

with g copies of the *hyperbolic form* $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ along the diagonal, and zeroes elsewhere.

With a different choice of basis, the cup product pairing corresponds to a different matrix. The natural choice made corresponds to a matrix that has as many vanishing entries as is possible. We assumed that $g \geq 1$. The conclusion holds as stated in the case $g = 0$ with $M_0 = S^2$, in a somewhat trivial way.

4.5 Relative cup products, naturality

Let $A, B \subseteq X$ be subspaces. If $\varphi \in C^k(X, A; R)$ vanishes on chains in A and $\psi \in C^\ell(X, B; R)$ vanishes on chains in B, then

$$\varphi \cup \psi \in C^{k+\ell}(X, A + B; R)$$

vanishes on chains in A or in B. Hence there is a relative cup product

$$H^k(X, A; R) \otimes_R H^\ell(X, B; R) \xrightarrow{\cup} H^{k+\ell}(X, A + B; R)\,.$$

If $\{A, B\}$ is *excisive*, so that $H_*(A + B) \to H_*(A \cup B)$ is an isomorphism, then $H^*(X, A \cup B; R) \to H^*(X, A + B; R)$ is an isomorphism by the long exact sequence and universal coefficient theorem. (This applies, for instance, when A and B are open subsets, or X is a CW complex and A and B are subcomplexes.) Then the cup product lifts through the isomorphism to

$$H^k(X, A; R) \otimes_R H^\ell(X, B; R) \xrightarrow{\cup} H^{k+\ell}(X, A \cup B; R)\,.$$

Some important special cases are the relative cup products

$$\begin{aligned} H^k(X, A; R) \otimes_R H^\ell(X; R) &\xrightarrow{\cup} H^{k+\ell}(X, A; R) \\ H^k(X; R) \otimes_R H^\ell(X, A; R) &\xrightarrow{\cup} H^{k+\ell}(X, A; R) \\ H^k(X, A; R) \otimes_R H^\ell(X, A; R) &\xrightarrow{\cup} H^{k+\ell}(X, A; R) \end{aligned}$$

for all pairs (X, A). The first two make $H^*(X, A; R)$ a graded bimodule over the graded ring $H^*(X; R)$. The last makes $H^*(X, A; R)$ a non-unital graded ring. (The unit $1 \in H^0(X; R)$ does not lift to $H^0(X, A; R)$ when A is non-empty).

For each map $f \colon X \to Y$ the cup product satisfies

$$f^\#(\varphi \cup \psi) = f^\#(\varphi) \cup f^\#(\psi)$$

in $C^{k+\ell}(X; R)$, for $\varphi \in C^k(Y; R)$ and $\psi \in C^\ell(Y; R)$, so the cochain cup product is natural in the sense that the diagram

$$\begin{tikzcd} C^k(Y; R) \otimes_R C^\ell(Y; R) \arrow[r, "\cup"] \arrow[d, "f^\# \otimes f^\#"] & C^{k+\ell}(Y; R) \arrow[d, "f^\#"] \\ C^k(X; R) \otimes_R C^\ell(X; R) \arrow[r, "\cup"] & C^{k+\ell}(X; R) \end{tikzcd}$$

commutes. It follows that the cohomology cup product satisfies

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

in $H^{k+\ell}(X; R)$, for $\alpha \in H^k(X; R)$ and $\beta \in H^\ell(X; R)$, so the cohomology cup product is natural in the same way. Hence the graded ring $H^*(X; R)$ is functorial for all spaces X. Similarly, the non-unital graded ring $H^*(X, A; R)$ is functorial for all pairs (X, A).

4.6 Cross product

For a pair of spaces X and Y, the natural projection maps

$$p_1 \colon X \times Y \longrightarrow X$$
$$p_2 \colon X \times Y \longrightarrow Y$$

induce cochain homomorphisms

$$p_1^{\#} \colon C^*(X;R) \longrightarrow C^*(X \times Y;R)$$
$$p_2^{\#} \colon C^*(Y;R) \longrightarrow C^*(X \times Y;R)\,.$$

When combined with the cochain cup product for $X \times Y$ we get a pairing

$$\times \colon C^k(X;R) \otimes_R C^\ell(Y;R) \overset{p_1^{\#}\otimes p_2^{\#}}{\longrightarrow} C^k(X \times Y;R) \otimes_R C^\ell(X \times Y;R) \overset{\cup}{\longrightarrow} C^{k+\ell}(X \times Y;R)$$

called the *cochain cross product*, denoted $\times$. It takes a k-cocycle $\varphi \colon C_k(X) \to R$ on X and an ℓ-cocycle $\psi \colon C_\ell(Y) \to R$ on Y to the cup product of their respective pullbacks to $X \times Y$:

$$\varphi \times \psi = p_1^{\#}(\varphi) \cup p_2^{\#}(\psi)\,.$$

Its value on a $(k+\ell)$-simplex $(\sigma, \tau) \colon \Delta^{k+\ell} \to X \times Y$ is, by definition,

$$(\varphi \times \psi)(\sigma, \tau) = \varphi(\sigma|[v_0, \dots, v_k]) \cdot \psi(\tau|[v_k, \dots, v_{k+\ell}])\,.$$

Lemma 4.6.1. *The identity*

$$\delta(\varphi \times \psi) = \delta\varphi \times \psi + (-1)^k \varphi \times \delta\psi$$

holds in $C^{k+\ell+1}(X \times Y;R)$, for $\varphi \in C^k(X;R)$ and $C^\ell(Y;R)$, so the cross product defines a cochain map

$$\times \colon C^*(X;R) \otimes_R C^*(Y;R) \longrightarrow C^*(X \times Y;R)\,.$$

Proof. This follows from the cup product Leibniz formula by naturality:

$$\begin{aligned}
\delta(\varphi \times \psi) &= \delta(p_1^{\#}(\varphi) \cup p_2^{\#}(\psi)) \\
&= \delta p_1^{\#}(\varphi) \cup p_2^{\#}(\psi) + (-1)^k p_1^{\#}(\varphi) \cup \delta p_2^{\#}(\psi) \\
&= p_1^{\#}(\delta\varphi) \cup p_2^{\#}(\psi) + (-1)^k p_1^{\#}(\varphi) \cup p_2^{\#}(\delta\psi) \\
&= \delta\varphi \times \psi + (-1)^k \varphi \cup \delta\psi
\end{aligned}$$

□

The *cohomology cross product* is the induced pairing

$$H^k(X;R) \otimes_R H^\ell(Y;R) \overset{\times}{\longrightarrow} H^{k+\ell}(X \times Y;R)$$

given by the formula

$$[\varphi] \times [\psi] = [\varphi \times \psi]$$

for each k-cocycle φ in X and each ℓ-cocycle ψ in Y. It is well-defined by the cross product Leibniz formula.

The cross product can be computed in terms of the cup product by the formula

$$\alpha \times \beta = p_1^*(\alpha) \cup p_2^*(\beta) .$$

Conversely, the cup product can be recovered from the cross product, by pullback along the diagonal map $\Delta \colon X \to X \times X$. The composite

$$H^k(X;R) \otimes_R H^\ell(X;R) \xrightarrow{\times} H^{k+\ell}(X \times X;R) \xrightarrow{\Delta^*} H^{k+\ell}(X;R)$$

is equal to the cup product, since $p_1\Delta = 1 = p_2\Delta$, so $\Delta^* p_1^* = 1 = \Delta^* p_2^*$ and

$$\Delta^*(\alpha \times \beta) = \Delta^*(p_1^*(\alpha) \cup p_2^*(\beta)) = \Delta^* p_1^*(\alpha) \cup \Delta^* p_2^*(\beta) = \alpha \cup \beta .$$

The same result holds at the cochain level.

4.7 Relative cross products, naturality

To compute the cross product in some interesting examples, we must first discuss some of its formal properties.

Let (X, A) and (Y, B) be pairs. If $\varphi \in C^k(X, A; R)$ vanishes on chains in A and $\psi \in C^\ell(Y, B; R)$ vanishes on chains in B, then $\varphi \times \psi \in C^{k+\ell}(X \times Y, A \times Y + X \times B; R)$ vanishes on chains in $A \times Y$ or in $X \times B$. Hence there is a relative cross product

$$H^k(X, A; R) \otimes_R H^\ell(Y, B; R) \xrightarrow{\times} H^{k+\ell}(X \times Y, A \times Y + X \times B; R) .$$

If A and B are open, or if X and Y are CW complexes and A and B are subcomplexes, then $H_*(A \times Y + X \times B) \to H_*(A \times Y \cup X \times B)$ is an isomorphism, so $H^*(X \times Y, A \times Y \cup X \times B; R) \cong H^*(X \times Y, A \times Y + X \times B; R)$. Then the cross product lifts to

$$H^k(X, A; R) \otimes_R H^\ell(Y, B; R) \xrightarrow{\times} H^{k+\ell}(X \times Y, A \times Y \cup X \times B; R) .$$

The target group is often written as $H^{k+\ell}((X, A) \times (Y, B); R)$, using the notation

$$(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B) .$$

As regards naturality, for each pair of maps $f \colon X \to X'$ and $g \colon Y \to Y'$ there is a map $f \times g \colon X \times Y \to X' \times Y'$, and the cochain cross product satisfies

$$(f \times g)^\#(\varphi \times \psi) = f^\#(\varphi) \times g^\#(\psi)$$

in $C^{k+\ell}(X \times Y; R)$, for $\varphi \in C^k(X'; R)$ and $\psi \in C^\ell(Y'; R)$. Hence the cohomology cross product satisfies

$$(f \times g)^*(\alpha \times \beta) = f^*(\alpha) \times g^*(\beta)$$

in $H^{k+\ell}(X \times Y; R)$, for $\alpha \in H^k(X'; R)$ and $\beta \in H^\ell(Y'; R)$, and the diagram

$$\begin{tikzcd}
H^k(X';R) \otimes_R H^\ell(Y';R) \arrow[r, "\times"] \arrow[d, "f^* \otimes g^*"] & H^{k+\ell}(X' \times Y';R) \arrow[d, "(f \times g)^*"] \\
H^k(X;R) \otimes_R H^\ell(Y;R) \arrow[r, "\times"] & H^{k+\ell}(X \times Y;R)
\end{tikzcd}$$

commutes. Similarly, the relative cross product is natural for all pairs (X, A) and (Y, B).

Naturality with respect to the connecting homomorphisms is a bit more subtle.

Lemma 4.7.1. *For all pairs (X, A) and spaces Y the natural square*

$$\begin{tikzcd} H^k(A;R) \otimes_R H^\ell(Y;R) \arrow[r, "\times"] \arrow[d, "\delta\otimes 1"] & H^{k+\ell}(A \times Y;R) \arrow[d, "\delta"] \\ H^{k+1}(X,A;R) \otimes_R H^\ell(Y;R) \arrow[r, "\times"] & H^{k+\ell+1}(X \times Y, A \times Y;R) \end{tikzcd}$$

commutes, so

$$\delta(\alpha \times \eta) = \delta\alpha \times \eta$$

for $\alpha \in H^k(A;R)$ and $\eta \in H^\ell(Y;R)$.

See Hatcher [1, p. 210] for the proof. We instead give the details in the following case.

Lemma 4.7.2. *For all spaces X and pairs (Y, B) the natural square*

$$\begin{tikzcd} H^k(X;R) \otimes_R H^\ell(B;R) \arrow[r, "\times"] \arrow[d, "1\otimes\delta"] & H^{k+\ell}(X \times B;R) \arrow[d, "\delta"] \\ H^k(X;R) \otimes_R H^{\ell+1}(Y,B;R) \arrow[r, "\times"] & H^{k+\ell+1}(X \times Y, X \times B;R) \end{tikzcd}$$

commutes up to the sign $(-1)^k$, so

$$\delta(\xi \times \beta) = (-1)^k \xi \times \delta\beta$$

for $\xi \in H^k(X;R)$ and $\beta \in H^\ell(B;R)$.

Proof. Let $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(B;R)$ be cocycles representing ξ and η, respectively. Choose an extension $\tilde{\psi} \in C^\ell(Y;R)$ of ψ. Then $\varphi \times \tilde{\psi} \in C^{k+\ell}(X \times Y;R)$ is an extension of $\varphi \times \psi \in C^{k+\ell}(X \times B;R)$, and $\delta(\xi \times \beta)$ is represented by $\delta(\varphi \times \tilde{\psi}) \in C^{k+\ell+1}(X \times Y, X \times B;R)$. Since φ is a cocycle of degree k, this equals $(-1)^k \varphi \times \delta\tilde{\psi}$ by the Leibniz formula, which represents $(-1)^k \xi \otimes \delta\beta$. $\square$

Recall that $H^1(I, \partial I;R) = R\{\alpha\}$, where α is dual to the generator of $H_1(I, \partial I;R)$ represented by the 1-cycle $\Delta^1 \cong I$. The other cohomology groups $H^m(I, \partial I;R)$ vanish.

Lemma 4.7.3. *Let Y be any space. The cross product*

$$H^1(I, \partial I;R) \otimes_R H^{n-1}(Y;R) \xrightarrow{\times} H^n(I \times Y, \partial I \times Y;R)$$

is an isomorphism. Hence each element of $H^n(I \times Y, \partial I \times Y;R)$ can be written uniquely as $\alpha \times \beta$, where $\beta \in H^{n-1}(Y;R)$. Similarly for pairs (Y, B).

Proof. The long exact sequence in cohomology for the pair $(I\times Y, \partial I\times Y)$ breaks up into short exact sequences, since the inclusion $\partial I \to I$ admits a *section up to homotopy*. Similarly for the pair $(I, \partial I)$. By naturality of the cross product, and flatness of $H^1(I, \partial I; R) = R\{\alpha\}$, we have a map of vertical short exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^0(I;R) \otimes_R H^{n-1}(Y;R) \arrow[r, "\times"] \arrow[d] & H^{n-1}(I\times Y;R) \arrow[d] \\
H^0(\partial I;R) \otimes_R H^{n-1}(Y;R) \arrow[r, "\times"] \arrow[d, "\delta\otimes 1"] & H^{n-1}(\partial I\times Y;R) \arrow[d, "\delta"] \\
H^1(I,\partial I;R) \otimes_R H^{n-1}(Y;R) \arrow[r, "\times"] \arrow[d] & H^n(I\times Y, \partial I\times Y;R) \arrow[d] \\
0 & 0
\end{tikzcd}$$

It is clear from unitality and a decomposition $\partial I \times Y \cong Y \sqcup Y$ that the upper and middle cross product maps are isomorphisms, hence so is the lower cross product. □

Let $k, \ell \geq 0$ and $n = k+\ell$. Note that $(I^k, \partial I^k) \times (I^\ell, \partial I^l) = (I^n, \partial I^n)$, since $I^k \times I^\ell = I^n$ and $\partial I^k \times I^\ell \cup I^k \times \partial I^\ell = \partial I^n$.

Corollary 4.7.4. *For $k, \ell \geq 0$ and $n = k + \ell$, the cross product*

$$H^k(I^k, \partial I^k; R) \otimes_R H^\ell(I^\ell, \partial I^l; R) \xrightarrow{\times} H^n(I^n, \partial I^n; R)$$

is an isomorphism. Hence $H^n(I^n, \partial I^n; R)$ is the free R-module generated by the n-fold cross product

$$\alpha \times \cdots \times \alpha$$

where $\alpha \in H^1(I, \partial I; R)$ is the standard generator. The remaining cohomology groups $H^m(I^n, \partial I^n; R)$ are zero.

Recall that $H^0(S^1; R) = R\{1\}$ and $H^1(S^1; R) = R\{\alpha\}$, where α is dual to the generator of $H_1(S^1; R)$ represented by the 1-cycle $\Delta^1 \to \Delta^1/\partial\Delta^1 \cong S^1$. The other cohomology groups $H^m(S^1; R)$ vanish.

Proposition 4.7.5. *Let Y be any space. The cross product*

$$H^*(S^1; R) \otimes_R H^*(Y; R) \xrightarrow{\times} H^*(S^1 \times Y; R)$$

is an isomorphism, and similarly for pairs (Y, B). Hence each element of $H^n(S^1 \times Y; R)$ can be written uniquely as a sum $\alpha \times \beta + 1 \times \gamma$, with $\beta \in H^{n-1}(Y; R)$ and $\gamma \in H^n(Y; R)$.

Proof. We use the pushout square

$$\begin{tikzcd}
\partial I \arrow[r] \arrow[d] & I \arrow[d] \\
\star \arrow[r] & S^1
\end{tikzcd}$$

where $\star = \{s_0\}$ is the base-point of S^1. The map $(I, \partial I) \to (S^1, \star)$ induces a cohomology isomorphism, and similarly when multiplied by Y. In view of the commutative square

$$\begin{tikzcd}
H^1(S^1, \star; R) \otimes_R H^{n-1}(Y; R) \arrow[r, "\times"] \arrow[d] & H^n(S^1 \times Y, \star \times Y; R) \arrow[d] \\
H^1(I, \partial I; R) \otimes_R H^{n-1}(Y; R) \arrow[r, "\times"] & H^n(I \times Y, \partial I \times Y; R)
\end{tikzcd}$$

and the previous lemma, it follows that the upper cross product is an isomorphism.

The long exact sequence for the pair $(S^1 \times Y, \star \times Y)$ also breaks up, since the inclusion $\star \to S^1$ admits a retraction, and we have another map of vertical short exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^1(S^1, \star; R) \otimes_R H^{n-1}(Y; R) \arrow[r, "\times"] \arrow[d] & H^n(S^1 \times Y, \star \times Y; R) \arrow[d] \\
{[H^*(S^1; R) \otimes_R H^*(Y; R)]^n} \arrow[r, "\times"] \arrow[d] & H^n(S^1 \times Y; R) \arrow[d] \\
H^0(\star; R) \otimes_R H^n(Y; R) \arrow[r, "\times"] \arrow[d] & H^n(\star \times Y; R) \arrow[d] \\
0 & 0
\end{tikzcd}$$

We have seen that the upper cross product is an isomorphism. Since the lower one is obviously an isomorphism, it follows that the middle map is also an isomorphism. □

Example 4.7.6. Let $T^n = S^1 \times \cdots \times S^1$ be the n-dimensional torus. The n-fold cross product

$$H^*(S^1; R) \otimes_R \cdots \otimes_R H^*(S^1; R) \xrightarrow{\times} H^*(T^n; R)$$

is an isomorphism. Hence $H^k(T^n; R)$ is a free R-module with basis the set of k-fold cup products

$$\alpha_{i_1} \cup \cdots \cup \alpha_{i_k}$$

for $1 \le i_1 < \cdots < i_k \le n$, where $\alpha_i = p_i^{\#}(\alpha) \in H^1(T^n; R)$ is the pullback of the generator $\alpha \in H^1(S^1; R)$ along the i-th projection map $p_i \colon T^n \to S^1$.

This is clear by induction on n, using the proposition above, which tells us that a basis is given by the set of n-fold cross products

$$\beta_1 \times \cdots \times \beta_n \in H^k(T^n; R)$$

where k of the classes β_i are equal to α, and the remaining $(n-k)$ of the classes β_i are equal to 1. Numbering the β_i that are equal to α as $\beta_{i_1}, \dots, \beta_{i_k}$, we get the asserted formula.

4.8 Projective spaces

Let $\mathbb{R}P^n$ be the n-dimensional real projective space, and let $\mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n$. Recall that the cellular complex $C_*^{CW}(\mathbb{R}P^n)$ has one generator e^k in each degree $0 \le k \le n$, with boundary homomorphism $\partial(e^k) = (1 + (-1)^k)e^{k-1}$. Hence $C^*_{CW}(\mathbb{R}P^n; \mathbb{Z}/2)$ has trivial coboundary, so $H^k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for each $0 \le k \le n$, where the generator in degree k evaluates to $1 \in \mathbb{Z}/2$ on e^k.

Proposition 4.8.1.

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$$

and

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]\,,$$

where $|x| = 1$.

Proof. We simplify notation by writing P^n for $\mathbb{R}P^n$ and omitting the coefficient ring $\mathbb{Z}/2$. By induction on n and naturality with respect to the inclusions $P^{n-1} \to P^n \to P^\infty$, it suffices to prove that the cup product of a generator of $H^{n-1}(P^n)$ and a generator of $H^1(P^n)$ is a generator of $H^n(P^n)$, for $n \ge 2$. It is no more difficult to prove that the cup product

$$H^i(P^n) \otimes H^j(P^n) \xrightarrow{\cup} H^n(P^n)$$

is an isomorphism, for $i + j = n$.

Consider the subspaces $\mathbb{R}^{i+1+0} \subset \mathbb{R}^{i+1+j} \supset \mathbb{R}^{0+1+j}$, which meet in $\mathbb{R}^{0+1+0}$. Passing to the spaces of lines through the origin we have the subspaces $P^i \subset P^n \supset P^j$ meeting in a single point $P^i \cap P^j = \{q\}$. Inside P^n we have an affine n-space $\mathbb{R}^n \cong U \subset P^n$ where the i-th coordinate is nonzero (counting from 0 to $i + j = n$), whose complement is a copy of P^{n-1}. The intersection $U \cap P^i \cong \mathbb{R}^i$ is an affine i-space, with complement P^{i-1} in P^i. Similarly, the intersection $U \cap P^j \cong \mathbb{R}^j$ is an affine j-space, with complement P^{j-1} in P^j.

We have a commutative diagram

$$\begin{tikzcd}
H^i(P^n) \otimes H^j(P^n) \arrow[r, "\cup"] & H^n(P^n) \\
H^i(P^n, P^n - P^j) \otimes H^j(P^n, P^n - P^i) \arrow[u] \arrow[r, "\cup"] \arrow[d, "\cong"'] & H^n(P^n, P^n - \{q\}) \arrow[u] \arrow[d, "\cong"'] \\
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \otimes H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) \arrow[r, "\cup"] & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \\
H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\}) \otimes H^j(\mathbb{R}^j, \mathbb{R}^j - \{0\}) \arrow[u, "p_1^* \otimes p_2^*"] \arrow[ur, "\times"'] &
\end{tikzcd}$$

The downward arrows are isomorphisms by excision. The cross product is an isomorphism by earlier calculations (replacing $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ by $(I^n, \partial I^n)$, etc.). The projection $p_1 \colon (\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \to (\mathbb{R}^i, \mathbb{R}^i - \{0\})$ away from a copy of $\mathbb{R}^j$ is a homotopy equivalence, and similarly for p_2, so p_1^* and p_2^* are isomorphisms. This proves that the middle horizontal cup product is an isomorphism.

The rest is maneuvering from the relative to the absolute case. The complement $P^n - P^j$ deformation retracts to P^{i-1}, since it consists of points $[x_0 : \dots : x_n]$ where at least one of the homogeneous coordinates $x_0, \dots, x_{i-1}$ is nonzero, and a deformation retraction to the subspace P^{i-1}, where all of the homogeneous coordinates $x_i, \dots, x_n$ are zero, is given by the formula

$$(t, [x_0 : \dots : x_n]) \mapsto [x_0 : \dots : x_{i-1} : tx_i : \dots : tx_n] \,.$$

Hence the homomorphism $H^i(P^n, P^n - P^j) \to H^i(P^n)$ factors as the composite

$$H^i(P^n, P^n - P^j) \longrightarrow H^i(P^n, P^{i-1}) \longrightarrow H^i(P^n)$$

where the first arrow is an isomorphism because of the deformation retraction, and the second arrow is an isomorphism by consideration of the cellular complexes. The same conclusion holds for i replaced by j or n. Hence the upper vertical arrows in the big diagram are isomorphisms, so that the upper horizontal cup product is an isomorphism.

Let $x \in H^1(P^n)$ be the generator. Let

$$x^n = x \cup \dots \cup x \in H^n(P^n)$$

denote the n-th cup power. By induction on n, we know that $x^{n-1} \in H^{n-1}(P^n)$ restricts to the generator of $H^{n-1}(P^{n-1})$, hence is the generator of $H^{n-1}(P^n)$. By what we have just shown,

$$x^n = x \cup x^{n-1}$$

is the generator of $H^n(P^n)$. □

We return to integer coefficients. Let $\mathbb{C}P^n$ be the n-dimensional complex projective space, of real dimension $2n$, and let $\mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n$. The cellular complex $C_*^{CW}(\mathbb{C}P^n)$ has one generator e^{2k} in each even degree $0 \le 2k \le 2n$, with trivial boundary homomorphisms. Hence $C^*_{CW}(\mathbb{C}P^n)$ has trivial coboundary, so $H^{2k}(\mathbb{C}P^n) \cong \mathbb{Z}$ for each $0 \le k \le n$, and the other cohomology groups are 0.

Proposition 4.8.2.

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[y]/(y^{n+1})$$

and

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y] \,,$$

where $|y| = 2$.

Let $\mathbb{H}P^n$ be the n-dimensional quaternionic projective space, of real dimension $4n$, and let $\mathbb{H}P^\infty = \bigcup_n \mathbb{H}P^n$. The cellular complex $C_*^{CW}(\mathbb{H}P^n)$ has one generator e^{4k} in degree $4k$, for $0 \le k \le n$, and trivial boundary homomorphisms. Hence $C^*_{CW}(\mathbb{H}P^n)$ has trivial coboundary, so $H^{4k}(\mathbb{H}P^n) \cong \mathbb{Z}$ for each $0 \le k \le n$, and the other cohomology groups are 0.

Proposition 4.8.3.

$$H^*(\mathbb{H}P^n) \cong \mathbb{Z}[z]/(z^{n+1})$$

and

$$H^*(\mathbb{H}P^\infty) \cong \mathbb{Z}[z]\,,$$

where $|z| = 2$.

4.9 Hopf maps

One way to detect whether a map $f\colon X \to Y$ is null-homotopic or not is to consider the cup product structure in the cohomology of the mapping cone $C_f = Y \cup CX$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "j"] & C_f \arrow[l, bend left, "r"] \end{tikzcd}$$

If f is null-homotopic, then there is a retraction $r\colon C_f \to Y$, so that the ring homomorphisms

$$H^*(Y) \xrightarrow{r^*} H^*(C_f) \xrightarrow{j^*} H^*(Y)$$

split off $H^*(Y)$ as a graded subring of $H^*(C_f)$. Therefore, if $H^*(Y)$ does not split off from $H^*(C_f)$, then f cannot be null-homotopic, i.e., it must be an *essential* map.

For example, in the CW structure on $\mathbb{C}P^2$, the 4-cell is attached to the 2-skeleton $\mathbb{C}P^1 = S^2$ by the *complex Hopf map*

$$\eta\colon S^3 \to S^2$$

taking a point in $S^3 \subset \mathbb{C}^2$ to the complex line that goes though it. The mapping cone is $C_\eta = \mathbb{C}P^2$. Here

$$H^*(\mathbb{C}P^2) = \mathbb{Z}[y]/(y^3) = \mathbb{Z}\{1, y, y^2\}$$

restricts by j^* to

$$H^*(S^2) = \mathbb{Z}[y]/(y^2) = \mathbb{Z}\{1, y\}\,,$$

but since $y^2 = 0$ in $H^*(S^2)$ and $y^2 \neq 0$ in $H^*(\mathbb{C}P^2)$ there is no ring homomorphism $r^*\colon H^*(S^2) \to H^*(\mathbb{C}P^2)$ that would be a section to j^*. Hence η cannot be null-homotopic.

As a similar example, in the CW structure on $\mathbb{H}P^2$, the 8-cell is attached to the 4-skeleton $\mathbb{H}P^1 = S^4$ by the *quaternionic Hopf map*

$$\nu\colon S^7 \to S^4$$

taking a point in $S^7 \subset \mathbb{H}^2$ to the quaternionic line that goes though it. The mapping cone is $C_\nu = \mathbb{H}P^2$. Here

$$H^*(\mathbb{H}P^2) = \mathbb{Z}[z]/(z^3) = \mathbb{Z}\{1, z, z^2\}$$

restricts by j^* to

$$H^*(S^4) = \mathbb{Z}[z]/(z^2) = \mathbb{Z}\{1, z\}\,,$$

but since $z^2 = 0$ in $H^*(S^4)$ and $z^2 \neq 0$ in $H^*(\mathbb{H}P^2)$ there is no ring homomorphism $r^*\colon H^*(S^4) \to H^*(\mathbb{H}P^2)$ that would be a section to j^*. Hence ν cannot be null-homotopic.

There is also an *octonionic plane*, denoted $\mathbb{O}P^2$, with cohomology ring $H^*(\mathbb{O}P^2) \cong \mathbb{Z}[w]/(w^3)$ with $|w| = 8$, and the attaching map

$$\sigma\colon S^{15} \to S^8$$

is an essential map known as the *octonionic Hopf map*.

A more careful argument shows that η has infinite order in $\pi_3(S^2)$, ν has infinite order in $\pi_7(S^4)$ and σ has infinite order in $\pi_{15}(S^8)$.

4.10 Graded commutativity

Recall that

$$H^*(T^2) = \mathbb{Z}\{1, \alpha, \beta, \gamma\}$$

with $|\alpha| = |\beta| = 1$ and $|\gamma| = 2$, with $\alpha \cup \beta = \gamma = -\beta \cup \alpha$. This commutativity up to a sign is typical.

Theorem 4.10.1. *Let (X, A) be a pair of space and let R be a commutative ring. Then*

$$\beta \cup \alpha = (-1)^{k\ell} \alpha \cup \beta$$

in $H^{k+\ell}(X, A; R)$, for all $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$.

We say that $H^*(X, A; R)$ is *graded commutative*. In the absolute case we say that $H^*(X; R)$ is a *commutative graded ring*. Note that if $H^*(X, A; R)$ is concentrated in even degrees, then the sign $(-1)^{k\ell}$ is always $+1$.

Proof. Let $\rho\colon C_n(X) \to C_n(X)$ be the (natural) chain map that takes $\sigma\colon \Delta^n \to X$ to

$$\rho(\sigma) = \epsilon_n \sigma|[v_n, \dots, v_0]\,,$$

where $\sigma|[v_n, \dots, v_0]$ is the composite of the affine linear map $\rho_n\colon \Delta^n \to \Delta^n$ that reverses the ordering of the vertices, and $\epsilon_n = (-1)^{n(n+1)/2}$ is the sign of the associated permutation of the vertices of Δ^n.

This is indeed a chain map, since

$$\partial\rho(\sigma) = \epsilon_n \sum_{i=0}^{n} (-1)^n \sigma|[v_n, \dots, \hat{v}_{n-i}, \dots, v_0]$$

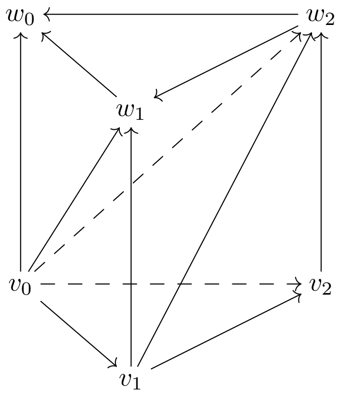
and

$$\begin{aligned} \rho\partial(\sigma) &= \epsilon_{n-1} \sum_{j=0}^{n} (-1)^j \sigma|[v_n, \dots, \hat{v}_j, \dots, v_0] \\ &= \epsilon_{n-1} \sum_{i=0}^{n} (-1)^{n-i} \sigma|[v_n, \dots, \hat{v}_{n-i}, \dots, v_0]\,, \end{aligned}$$

are equal because $\epsilon_n = (-1)^n \epsilon_{n-1}$.

There is a (natural) chain homotopy $P\colon C_n(X) \to C_{n+1}(X)$ from the identity 1 to ρ. It arises from a subdivision of $\Delta^n \times I$, with vertices $v_0, \dots, v_n$ in

$\Delta^n \times \{0\}$ and vertices $w_0, \dots, w_n$ in $\Delta^n \times \{1\}$, as the union the $(n+1)$-simplices $[v_0, \dots, v_i, w_n, \dots, w_i]$, for $0 \le i \le n$.



As subsets of $\Delta^n \times I$, these are the same simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$ as those used to show homotopy invariance of singular homology, but the ordering of the last $(n-i)$ vertices is reversed, which leads us to introduce the additional sign ϵ_{n-i}. Let

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \epsilon_{n-i} \sigma\pi|[v_0, \dots, v_i, w_n, \dots, w_i]$$

for $\sigma\colon \Delta^n \to X$ any singular n-simplex, and $\pi\colon \Delta^n \times I \to \Delta^n$ the projection. Then $P\colon C_n(X) \to C_{n+1}(X)$ satisfies

$$\partial P + P\partial = \rho - 1\,.$$

((See Hatcher [1, p. 217] for this calculation.))

We get an induced chain map $\rho^*\colon C^n(X;R) \to C^n(X;R)$ and chain homotopy P^* from 1 to ρ^*.

Recall the definition of the cochain level cup product of $\varphi\colon C_k(X) \to R$ and $\psi\colon C_\ell(X) \to R$:

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|[v_0, \dots, v_k]) \cdot \psi(\sigma|[v_k, \dots, v_n])$$

for $\sigma\colon \Delta^n \to X$ as above, $n = k + \ell$. Then

$$\begin{aligned} \rho^*(\psi \cup \varphi)(\sigma) &= \epsilon_n(\psi \cup \varphi)(\sigma|[v_n, \dots, v_0]) \\ &= \epsilon_n \psi(\sigma|[v_n, \dots, v_k]) \cdot \varphi(\sigma|[v_k, \dots, v_0]) \end{aligned}$$

while

$$\begin{aligned} (\rho^*\varphi \cup \rho^*\psi)(\sigma) &= \varphi(\epsilon_k \sigma|[v_0, \dots, v_k]) \cdot \psi(\epsilon_\ell \sigma|[v_k, \dots, v_n]) \\ &= \epsilon_k \epsilon_\ell \varphi(\sigma|[v_0, \dots, v_k]) \cdot \psi(\sigma|[v_k, \dots, v_n])\,. \end{aligned}$$

Using the relation $\epsilon_n = (-1)^{k\ell} \epsilon_k \epsilon_\ell$ and commutativity of R, we get that

$$\rho^*(\psi \cup \varphi) = (-1)^{k\ell} \rho^*\varphi \cup \rho^*\psi\,.$$

Hence, at the level of cohomology groups,

$$\begin{aligned}\beta \cup \alpha &= [\psi \cup \varphi] = [\rho^*(\psi \cup \varphi)] = (-1)^{k\ell}[\rho^*\varphi \cup \rho^*\psi] \\ &= (-1)^{k\ell}[\rho^*\varphi] \cup [\rho^*\psi] = (-1)^{k\ell}[\varphi] \cup [\psi] = (-1)^{k\ell}\alpha \cup \beta\end{aligned}$$

when φ and ψ are cocycles representing α and β.

□

4.11 Tensor products of graded rings

If A_* and B_* are graded rings, we define their tensor product $A_* \otimes B_*$ to be the tensor product of graded abelian groups, with

$$[A_* \otimes B_*]_n = \bigoplus_{k+\ell=n} A_k \otimes B_\ell$$

in degree n, with the graded multiplication

$$[A_* \otimes B_*]_n \otimes [A_* \otimes B_*]_{n'} \longrightarrow [A_* \otimes B_*]_{n+n'}$$

given by

$$(\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') = (-1)^{k\ell'} \alpha\alpha' \otimes \beta\beta'$$

where $|\beta| = k$ and $|\alpha'| = \ell'$. In terms of diagrams, the multiplication on $A_* \otimes B_*$ is the composite

$$A_* \otimes B_* \otimes A_* \otimes B_* \overset{1\otimes\tau\otimes 1}{\longrightarrow} A_* \otimes A_* \otimes B_* \otimes B_* \overset{\mu\otimes\mu}{\longrightarrow} A_* \otimes B_* \,,$$

where $\tau\colon B_* \otimes A_* \to A_* \otimes B_*$ is the graded twist isomorphism that takes $\beta \otimes \alpha'$ to $(-1)^{k\ell'}\alpha' \otimes \beta$, with notation as above, and $\mu\colon A_* \otimes A_* \to A_*$ and $\mu\colon B_* \otimes B_* \to B_*$ are the multiplications in A_* and B_*.

((Example with products of spheres?))

Chapter 5

Künneth formulas

5.1 A Künneth formula in cohomology

Let (X, A) and (Y, B) be pairs of spaces, and let R be a commutative ring. Recall the notation $(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$.

Theorem 5.1.1 (Künneth formula)**.** *The cross product*

$$H^*(X, A; R) \otimes_R H^*(Y, B; R) \xrightarrow{\times} H^*((X, A) \times (Y, B); R)$$

is an isomorphisms of graded rings, if (X, A) and (Y, B) are pairs of CW complexes and $H^\ell(Y, B; R)$ is a finitely generated projective R-module, for each ℓ.

5.2 The Künneth formula in homology

Let R be a PID, throughout this section.

Theorem 5.2.1 (Künneth formula)**.** *There is a natural short exact sequence*

$$0 \to \bigoplus_{k+\ell=n} H_k(X; R) \otimes_R H_\ell(Y; R) \xrightarrow{\times} H_{k+\ell}(X \times Y; R) \longrightarrow$$
$$\longrightarrow \bigoplus_{k+\ell=n-1} \operatorname{Tor}_1^R(H_k(X; R), H_\ell(Y; R)) \to 0$$

for each n, and these sequences split.

The hypothesis of the following consequence is automatic if R is a field.

Corollary 5.2.2. *Suppose that $H_\ell(Y; R)$ is flat over R, for each ℓ. There is a natural isomorphism*

$$\times \colon H_*(X; R) \otimes_R H_*(Y; R) \xrightarrow{\cong} H_*(X \times Y; R)\,.$$

One proof of the theorem goes in two parts. One is the Eilenberg–Zilber theorem, relating the chains on $X \times Y$ to the algebraic tensor product of the chains on X and the chains on Y. The other is the algebraic Künneth theorem, computing the homology of a tensor product of chain complexes.

((Reference to Spanier's "Algebraic topology" or Mac Lane's "Homology".))

Here is the external version of the Alexander–Whitney diagonal approximation.

Definition 5.2.3. The *Alexander–Whitney homomorphism*

$$AW_n \colon C_n(X \times Y; R) \longrightarrow \bigoplus_{k+\ell=n} C_k(X; R) \otimes_R C_\ell(Y; R)$$

takes $(\sigma, \tau) \colon \Delta^n \to X \times Y$ to

$$\sigma|[v_0, \dots, v_k] \otimes \tau|[v_k, \dots, v_n] \,.$$

Theorem 5.2.4 (Eilenberg–Zilber theorem)**.** *The Alexander–Whitney homomorphism is a chain homotopy equivalence*

$$AW_\# \colon C_*(X \times Y; R) \xrightarrow{\simeq} C_*(X; R) \otimes_R C_*(Y; R) \,.$$

To prove this, one can construct a chain homotopy inverse

$$EZ_\# \colon C_*(X; R) \otimes_R C_*(Y; R) \xrightarrow{\simeq} C_*(X \times Y; R) \,.$$

This can either be done by the method of *acyclic models*, or by an explicit formula, known as the *Eilenberg–Zilber shuffle homomorphism.* ((ETC))

Theorem 5.2.5 (Algebraic Künneth formula)**.** *Let (C_*, ∂) and (D_*, ∂) be chain complexes of free R-modules. Then there is a natural short exact sequence*

$$0 \to \bigoplus_{k+\ell=n} H_k(C_*) \otimes_R H_\ell(D_*) \longrightarrow H_n(C_* \otimes D_*; R) \longrightarrow$$

$$\longrightarrow \bigoplus_{k+\ell=n-1} \mathrm{Tor}_1^R(H_k(C_*), H_\ell(D_*)) \to 0$$

for each n, and these sequences split.

The proof is similar to that of the universal coefficient theorem.

Under the assumption that $H_k(X; R)$ and $H_\ell(Y; R)$ are finitely generated projective R-modules, for each k and ℓ, we can dualize the homological Künneth isomorphism

$$H_*(X; R) \otimes_R H_*(Y; R) \xrightarrow{\cong} H_*(X \times Y; R)$$

and use the universal coefficient theorem to get a cohomological Künneth isomorphism

$$\begin{aligned}
H^*(X \times Y; R) &\cong \mathrm{Hom}_R(H_*(X \times Y; R), R) \\
&\cong \mathrm{Hom}_R(H_*(X; R) \otimes_R H_*(Y; R), R) \\
&\cong \mathrm{Hom}_R(H_*(X, R), R) \otimes_R \mathrm{Hom}_R(H_*(Y; R), R) \\
&\cong H^*(X; R) \otimes_R H^*(Y; R) \,.
\end{aligned}$$

Notice how finite generation is needed in the middle, using that the homomorphism

$$\mathrm{Hom}_R(M, R) \otimes_R \mathrm{Hom}_R(N, R) \longrightarrow \mathrm{Hom}_R(M \otimes_R N, R) \,,$$

taking the tensor product of $\varphi \colon M \to R$ and $\psi \colon N \to R$ to the composite

$$M \otimes_R N \xrightarrow{\varphi \otimes \psi} R \otimes_R R \cong R \,,$$

is an isomorphism when M and N are finitely generated projective R-modules.

5.3 Proof of the cohomology Künneth formula

We will instead give a different proof of the cohomological Künneth isomorphism, based on the study of generalized cohomology theories, which leads more directly to a result with weaker hypotheses.

The main part of the proof deals with the absolute case when $B = \varnothing$, saying that the cross product

$$H^*(X, A; R) \otimes_R H^*(Y; R) \xrightarrow{\times} H^*(X \times Y, A \times Y; R)$$

is an isomorphism of graded rings, if (X, A) is a pair of CW complexes and $H^\ell(Y; R)$ is a finitely generated projective R-module, for each ℓ. The relative case, when (Y, B) is a pair of CW complexes and $H^\ell(Y, B; R)$ is a finitely generated projective R-module, for each ℓ, follows by naturality with respect to the map $(Y, B) \to (Y/B, B/B)$ inducing isomorphisms

$$H^\ell(Y/B, B/B; R) \xrightarrow{\cong} H^\ell(Y, B; R)$$

for all ℓ, and the splittings

$$H^*(Y/B; R) \cong H^*(Y/B, B/B; R) \oplus H^*(B/B; R)$$

and

$$H^*((X, A) \times Y/B; R) \cong H^*((X, A) \times (Y/B, B/B); R) \oplus H^*((X, A) \times B/B; R) .$$

Consider the functors of CW pairs (X, A) given by

$$h^n(X, A) = \bigoplus_{k+\ell=n} H^k(X, A; R) \otimes_R H^\ell(Y; R)$$

and

$$k^n(X, A) = H^n(X \times Y, A \times Y; R) .$$

The cross product defines a natural transformation

$$\mu\colon h^n(X, A) \longrightarrow k^n(X, A)$$

for all n.

We prove that h^* and k^* are cohomology theories on the category of CW pairs, and that μ is a map of cohomology theories, i.e., a natural transformation that commutes with the connecting homomorphisms.

Proposition 5.3.1. *If a map $\mu\colon h^* \to k^*$ of cohomology theories on the category of CW pairs is an isomorphism on the pair $(\star, \varnothing)$, then it is an isomorphism for all CW pairs.*

Proof. By the map of long exact sequences

$$\begin{tikzcd}
h^{n-1}(X) \arrow[r] \arrow[d, "\mu"'] & h^{n-1}(A) \arrow[r, "\delta"] \arrow[d, "\mu"'] & h^n(X, A) \arrow[r] \arrow[d, "\mu"'] & h^n(X) \arrow[r] \arrow[d, "\mu"'] & h^n(A) \arrow[d, "\mu"'] \\
k^{n-1}(X) \arrow[r] & k^{n-1}(A) \arrow[r, "\delta"] & k^n(X, A) \arrow[r] & k^n(X) \arrow[r] & k^n(A)
\end{tikzcd}$$

and the five-lemma, it suffices to prove the proposition in the case when $A = \varnothing$.

In this absolute case first we proceed by induction on the dimension m of X. When X is 0-dimensional, it is the disjoint union $X = \coprod_\alpha \star$ of a set of points, so by the commutative diagram

$$\begin{tikzcd} h^*(\coprod_\alpha \star) \arrow[r, "\cong"] \arrow[d, "\mu"'] & \prod_\alpha h^*(\star) \arrow[d, "\prod_\alpha \mu"] \\ k^*(\coprod_\alpha \star) \arrow[r, "\cong"] & \prod_\alpha k^*(\star) \end{tikzcd}$$

and the hypothesis for $X = \star$, it follows that μ is an isomorphism for $X = \coprod_\alpha \star$.

Let $m \ge 1$, assume that μ is an isomorphism for all X of dimension less than m, and suppose that $X = X^{(m)}$ has dimension m. By the map of long exact sequences above in the case $(X, A) = (X^{(m)}, X^{(m-1)})$, the inductive hypothesis and the five-lemma, it suffices to prove that μ is an isomorphism for this CW pair. Let

$$\Phi \colon \coprod_\alpha (D^m, \partial D^m) \to (X^{(m)}, X^{(m-1)})$$

be the characteristic maps of the m-cells of X. In the commutative diagram

$$\begin{tikzcd} h^*(X^{(m)}, X^{(m-1)}) \arrow[r, "\Phi^*"] \arrow[d, "\mu"'] & h^*(\coprod_\alpha (D^m, \partial D^m)) \arrow[r, "\cong"] \arrow[d, "\mu"'] & \prod_\alpha h^*(D^m, \partial D^m) \arrow[d, "\mu"'] \\ k^*(X^{(m)}, X^{(m-1)}) \arrow[r, "\Phi^*"] & k^*(\coprod_\alpha (D^m, \partial D^m)) \arrow[r, "\cong"] & \prod_\alpha k^*(D^m, \partial D^m) \end{tikzcd}$$

the homomorphisms labeled Φ^* are isomorphisms by excision, and the right-hand horizontal arrows are isomorphisms by the product axiom. Hence it suffices to prove that μ is an isomorphism for the CW pair $(D^m, \partial D^m)$.

By the map of long exact sequences above, in the case $(X, A) = (D^m, \partial D^m)$, it suffices to know that μ is an isomorphism for $X = D^m$ and for $X = \partial D^m$. The first follows from the case $X = \star$ naturality with respect to the map $D^m \to \star$ and homotopy invariance. The second follows by induction, since the dimension of ∂D^m is less than m.

The case of infinite-dimensional X remains. For this we use that X is the colimit of its skeleta, in the sense that there is a sequence of cellular inclusions of CW complexes

$$X^{(m-1)} \subset X^{(m)} \subset \dots \subset X = \bigcup_m X^{(m)} .$$

There is a *mapping telescope*

$$T = \bigcup_m [m, m+1] \times X^{(m)} \subset \mathbb{R} \times X$$

and the composite projection $T \subset \mathbb{R} \times X \to X$ is a homotopy equivalence. See Hatcher [1, Lemma 2.34].

We can write this mapping telescope as the homotopy coequalizer of two maps

$$1, i \colon \coprod_m X^{(m)} \longrightarrow \coprod_m X^{(m)}$$

where 1 is the coproduct of the identity maps $X^{(m)} \to X^{(m)}$, while i is the coproduct of the inclusion maps $X^{(m-1)} \to X^{(m)}$. Hence there is a natural long exact sequence

$$h^{n-1}(\coprod_m X^{(m)}) \xrightarrow{1-i^*} h^{n-1}(\coprod_m X^{(m)}) \longrightarrow h^n(T) \longrightarrow h^n(\coprod_m X^{(m)}) \xrightarrow{1-i^*} h^n(\coprod_m X^{(m)})$$

which we can rewrite, using the product axiom and the homotopy equivalence $T \simeq X$, as

$$\prod_m h^{n-1}(X^{(m)}) \xrightarrow{1-i^*} \prod_m h^{n-1}(X^{(m)}) \longrightarrow h^n(X) \longrightarrow \prod_m h^n(X^{(m)}) \xrightarrow{1-i^*} \prod_m h^n(X^{(m)})$$

The kernel of the right hand $1 - i^*$ consists of the compatible sequences $(x_m)_m$ with $x_m \in h^n(X^{(m)})$ and $i^*(x_m) = x_{m-1}$ for all m, i.e., it equals the limit group

$$\ker(1 - i^*) = \lim_m h^n(X^{(m)})\,.$$

By definition, the cokernel of the left hand $1 - i^*$ is the derived limit group

$$\mathrm{cok}(1 - i^*) = \operatorname*{Rlim}_m h^{n-1}(X^{(m)})\,.$$

It vanishes if the homomorphisms $i^* \colon h^n(X^{(m)} \to h^n(X^{(m-1)}$ are surjective for sufficiently large m.

These considerations are natural in the cohomology theory h, so there is a map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Rlim}_m h^{n-1}(X^{(m)}) \arrow[r] \arrow[d, "\mu"'] & h^n(X) \arrow[r] \arrow[d, "\mu"'] & \lim_m h^n(X^{(m)}) \arrow[r] \arrow[d, "\mu"'] & 0 \\
0 \arrow[r] & \mathrm{Rlim}_m k^{n-1}(X^{(m)}) \arrow[r] & k^n(X) \arrow[r] & \lim_m k^n(X^{(m)}) \arrow[r] & 0
\end{tikzcd}$$

We have already shown that μ is an isomorphism for each finite-dimensional $X^{(m)}$, hence is induces an isomorphism of limits and derived limits. Thus μ is also an isomorphism for the general CW complex X. □

Proof of the cohomology Künneth formula. We must exhibit h^* and k^* as cohomology theories, check that μ is a map of such, and that μ is an isomorphism for the one-point space $\star$.

The connecting homomorphism $\delta \colon h^{n-1}(A) \to h^n(X, A)$ is defined as the direct sum of the tensor products

$$\delta \otimes 1 \colon H^{k-1}(A; R) \otimes_R H^\ell(Y; R) \longrightarrow H^k(X, A; R) \otimes_R H^\ell(Y; R)$$

as k ranges over the integers and $\ell = n - k$.

The connecting homomorphism $\delta \colon k^{n-1}(A) \to k^n(X, A)$ is the usual connecting homomorphism

$$\delta \colon H^{n-1}(A \times Y; R) \longrightarrow H^n(X \times Y, A \times Y; R)$$

of the pair $(X \times Y, A \times Y)$.

The tensor product of the long exact sequence

$$H^{k-1}(X;R) \xrightarrow{i^*} H^{k-1}(A;R) \xrightarrow{\delta} H^k(X,A;R) \xrightarrow{j^*} H^k(X;R) \xrightarrow{i^*} H^k(A;R)$$

with $H^\ell(Y;R)$ over R is still exact, because $H^\ell(Y;R)$ is projective, hence flat. Summing over all $k+\ell=n$ we get the long exact sequence for h^*.

The long exact sequence for k^* at (X,A) is just the usual long exact sequence for $H^n(-;R)$ at $(X\times Y, A\times Y)$.

Homotopy invariance for h^* and k^* follows immediately from homotopy invariance for ordinary cohomology.

Excision, either in the form for general topological pairs, or in the form for subcomplexes of a CW complex, is also obvious for h^*. The case of k^* is about as easy, since if $Z \subseteq A \subseteq X$ with the closure of Z contained in the interior of A, the $Z\times Y \subset A\times Y \subset X \times Y$ with the closure of $Z\times Y$ contained in the interior of $A\times Y$, and similarly for products of subcomplexes of X with Y.

The product axiom is clear for k^*, since if $X = \coprod_\alpha X_\alpha$ then $X\times Y = \coprod_\alpha (X_\alpha \times Y)$, and similarly in the relative case. The product axiom for h^* is more subtle. It amounts to the assertion that

$$(\prod_\alpha H^k(X_\alpha, A_\alpha;R)) \otimes_R H^\ell(Y;R) \longrightarrow \prod_\alpha (H^k(X_\alpha, A_\alpha;R)\otimes_R H^\ell(Y;R))$$

is an isomorphism, for all k and ℓ. This is clear if $H^\ell(Y;R)=R$, hence also if $H^\ell(Y;R)$ is finitely generated and free, since finite sums of R-modules are also finite products. By naturality in $H^\ell(Y;R)$, it also follows when $H^\ell(Y;R)$ is finitely generated and projective.

The assertion that μ is a map of generalized cohomology theories is clear from the naturality of the cross product, together with the previously proved formula $\delta(\alpha\times\eta) = \delta\alpha\times\eta$, relating the connecting homomorphism to the cross product.

The assertion that μ is an isomorphism for $(X,A) = (\star,\varnothing)$ is the assertion that

$$\times\colon H^0(\star;R)\otimes_R H^\ell(Y;R) \longrightarrow H^\ell(\star\times Y;R)$$

is an isomorphism for all ℓ, which is clear.

□

Theorem 5.3.2 (Hopf)**.** *If there is a real division algebra structure on $\mathbb{R}^n$ then n is a power of 2.*

Proof. A division algebra structure on $\mathbb{R}^n$ is a bilinear pairing $\cdot\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ such that $x\cdot y = 0$ only if $x=0$ or $y=0$. Given such a pairing, we have a map $g\colon S^{n-1}\times S^{n-1}\to S^{n-1}$ given by $g(x,y) = x\cdot y/|x\cdot y|$, such that

$$g(-x,y) = -g(x,y) = g(x,-y)\,.$$

Passing to quotients we get a map $h\colon \mathbb{R}P^{n-1}\times\mathbb{R}P^{n-1}\to\mathbb{R}P^{n-1}$. We may assume that $n>2$, in which case $\pi_1(\mathbb{R}P^{n-1})\cong\mathbb{Z}/2$. The displayed formula, and a consideration of covering spaces, implies that h induces the sum homomorphism on $\pi_1(\mathbb{R}P^{n-1})$.

Passing to cohomology, we have a graded ring isomorphism

$$H^*(\mathbb{R}P^{n-1};\mathbb{Z}/2)\cong\mathbb{Z}/2[\gamma]/(\gamma^n=0)\,,$$

with $\deg(\gamma) = 1$, and by the Künneth formula,

$$H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha, \beta]/(\alpha^n = 0, \beta^n = 0)$$

The formula for h on π_1 implies that $h^*(\gamma) = \alpha + \beta$ in H^1, so

$$0 = h^*(\gamma^n) = (\alpha + \beta)^n = \sum_{i=1}^{n-1} \binom{n}{i} \alpha^i \beta^{n-i}$$

in H^n. It is a number-theoretic fact that $\binom{n}{i} \equiv 0 \mod 2$ for all $0 < i < n$ (if and) only if n is a power of 2. $\square$

Bibliography

[1] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.