Mandatory assignment - Algebraic Topology II, 2014

Exercise 1. Let M be a connected n-manifold (without boundary).

a) Suppose $x \in M$, and $z \in H_n(M)$ maps trivially under

$$\rho_{x,M}\colon H_n(M) \to H_n(M, M \smallsetminus x).$$

Show that $z = 0$.

b) Show that

(i) $H_n(M) = 0$, or

(ii) $H_n(M) \cong \mathbb{Z}$, and $\rho_{x,M}$ is an isomorphism for every $x \in M$.

Exercise 2. Let R be a coefficient ring and $n \geq 2$ an integer.

a) Compute the cohomology ring $H^*(\mathbb{CP}^n/\mathbb{CP}^{n-2}; R)$.

b) Find integers $a > b$ such that $H^*(S^a \vee S^b; R) \cong H^*(\mathbb{CP}^n/\mathbb{CP}^{n-2}; R)$.

c) Let $R = \mathbb{F}_2$ and compute $Sq^2(\alpha^{n-1})$ for the degree two generator α of $H^*(\mathbb{CP}^n; \mathbb{F}_2)$.

d) With $a > b$ as in **b**) and $n \geq 4$ even, show that $S^a \vee S^b$ and $\mathbb{CP}^n/\mathbb{CP}^{n-2}$ are not homotopy equivalent.

Exercise 3. **a**) Let p be a prime number. Compute the $\lim - \lim^1$ exact sequence of the short exact sequence of inverse systems

$$0 \to (p^n\mathbb{Z})_n \to (\mathbb{Z})_n \to (\mathbb{Z}/p^n)_n \to 0.$$

In particular, conclude that $\lim^1(p^n\mathbb{Z})_n$ is uncountable.

b) Compute the degree of $S^1 \to S^1$ given by $z \mapsto z^p$. Construct a mapping telescope T with $H^2(T) \cong \mathbb{Z}_p/\mathbb{Z}$, where $\mathbb{Z}_p$ denotes the p-adic integers.