In many expositions of the subject, the tangent vector (x, v) is identified with this linear operator X.

One defect of the above presentation is that the "smoothness" of a manifold M is made to depend on some particular embedding of M in a coordinate space. It is possible however to canonically embed any smooth manifold M in one preferred coordinate space.

Given a smooth manifold $M \subset R^A$ let $F = C^\infty(M, R)$ denote the set of all smooth functions from M to the real numbers R. Define the embedding

$$i : M \to R^F$$

by $i_f(x) = f(x)$. Let M_1 denote the image $i(M) \subset R^F$.

LEMMA 1.5. *This image* M_1 *is a smooth manifold in* R^F, *and the canonical map* $i : M \to M_1$ *is a diffeomorphism.*

The proof is straightforward.

Thus any smooth manifold has a canonical embedding in an associated coordinate space. This suggests the following definition.

Let M be a set and let F be a collection of real valued functions on M which separates points. (That is, given $x \neq y$ in M there exists $f \in F$ with $f(x) \neq f(y)$.) Then M can be identified with its image under the canonical imbedding $i : M \to R^F$.

DEFINITION. The collection F is a *smoothness structure* on M if the subset $i(M) \subset R^F$ is a smooth manifold, and if F is precisely the set of all smooth real valued functions on this smooth manifold.*

Note: This definition of "smoothness" is similar to that given by [Nomizu]. In the classical point of view the "smoothness structure" of a manifold is prescribed by the collection of local parametrizations. (See

* If only the first condition is satisfied, then F might be called a "basis" for a smoothness structure on M.

for example [Steenrod, 1951, p. 21].) In still another point of view, one uses collections of smooth functions on open subsets. (Compare [de Rham].) All of these definitions are equivalent.

In conclusion here are three problems for the reader. The first two of these will play an important role in later sections.

Problem 1-A. Let $M_1 \subset R^A$ and $M_2 \subset R^B$ be smooth manifolds. Show that $M_1 \times M_2 \subset R^A \times R^B$ is a smooth manifold, and that the tangent manifold $D(M_1 \times M_2)$ is canonically diffeomorphic to the product $DM_1 \times DM_2$. Note that a function $x \mapsto (f_1(x), f_2(x))$ from M to $M_1 \times M_2$ is smooth if and only if both $f_1 : M \to M_1$ and $f_2 : M \to M_2$ are smooth.

Problem 1-B. Let P^n denote the set of all lines through the origin in the coordinate space R^{n+1}. Define a function

$$q : R^{n+1} - \{0\} \to P^n$$

by $q(x) = Rx =$ line through x. Let F denote the set of all functions $f : P^n \to R$ such that $f \circ q$ is smooth.

a) Show that F is a smoothness structure on P^n. The resulting smooth manifold is called the *real projective space* of dimension n.

b) Show that the functions $f_{ij}(Rx) = x_i x_j / \Sigma x_k^2$ define a diffeomorphism between P^n and the submanifold of $R^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.

c) Show that P^n is compact, and that a subset $V \subset P^n$ is open if and only if $q^{-1}(V)$ is open.

Problem 1-C. For any smooth manifold M show that the collection $F = C^\infty(M, R)$ of smooth real valued functions on M can be made into a ring, and that every point $x \in M$ determines a ring homomorphism $F \to R$ and hence a maximal ideal in F. If M is compact, show that every maximal ideal in F arises in this way from a point of M. More generally, if there is a countable basis for the topology of M, show that every ring