$$(U \cap U') \times T(R^{n_1}, \ldots, R^{n_k}) \xrightarrow{h^{-1} \circ h'} (U \cap U') \times T(R^{n_1}, \ldots, R^{n_k})$$

is continuous. This follows from the continuity of T.

It is now clear that $\pi : E \to B$ is continuous, and that the resulting vector bundle $T(\xi_1, \ldots, \xi_k)$ satisfies the local triviality condition. ■

REMARK 1. This construction can be translated into Steenrod's terminology as follows. Let $GL_n = GL_n(R)$ denote the group of automorphisms of the vector space R^n. Then T determines a continuous homomorphism from the product group $GL_{n_1} \times \ldots \times GL_{n_k}$ to the group GL' of automorphisms of the vector space $T(R^{n_1}, \ldots, R^{n_k})$. Hence given bundles $\xi_1, \ldots, \xi_k$ over B with structural groups $GL_{n_1}, \ldots, GL_{n_k}$ respectively, there corresponds a bundle $T(\xi_1, \ldots, \xi_k)$ with structural group GL' and with fiber $T(R^{n_1}, \ldots, R^{n_k})$. For further discussion, see [Hirzebruch, 1966, §3.6].

REMARK 2. Given bundles $\xi_1, \ldots, \xi_k$ over distinct base spaces, a similar construction gives rise to a vector bundle $\hat{T}(\xi_1, \ldots, \xi_k)$ over $B(\xi_1) \times \ldots \times B(\xi_k)$, with typical fiber $T(F_{b_1}(\xi_1), \ldots, F_{b_k}(\xi_k))$. This yields a functor $\hat{T}$ from the category of vector bundles and bundle maps into itself. As an example, starting from the direct sum functor $\oplus$ on the category $\mathcal{V}$ one obtains the Cartesian product functor

$$\xi, \eta \mapsto \xi \mathbin{\hat{\oplus}} \eta = \xi \times \eta$$

for vector bundles.

REMARK 3. If $\xi_1, \ldots, \xi_k$ are smooth vector bundles, then $T(\xi_1, \ldots, \xi_k)$ can also be given the structure of a smooth vector bundle. The proof is similar to that of 3.6. It is necessary to make use of the fact that the isomorphism $T(f_1, \ldots, f_k)$ is a smooth function of the isomorphisms $f_1, \ldots, f_k$. This follows from [Chevalley, p. 128].

As an illustration, let $f : M \to N$ be a smooth map. Then $\mathrm{Hom}(\tau_M, f^*\tau_N)$ is a smooth vector bundle over M. Note that Df gives rise to a smooth cross-section of this vector bundle.

As a second illustration, if $M \subset N$ with normal bundle ν, where N is a smooth Riemannian manifold, then the "second fundamental form" can be defined as a smooth symmetric cross-section of the bundle $\mathrm{Hom}(\tau_M \otimes \tau_M, \nu)$. (Compare [Bishop and Crittenden], as well as Problem 5-B.)

Here are six problems for the reader.

Problem 3-A. A smooth map $f : M \to N$ between smooth manifolds is called a *submersion* if each Jacobian

$$Df_x : DM_x \to DN_{f(x)}$$

is surjective (i.e., is onto). Construct a vector bundle κ_f built up out of the kernels of the Df_x. If M is Riemannian, show that

$$\tau_M \cong \kappa_f \oplus f^*\tau_N .$$

Problem 3-B. Given vector bundles $\xi \subset \eta$ define the *quotient bundle* η/ξ and prove that it is locally trivial. If η has a Euclidean metric, show that

$$\xi^{\perp} \cong \eta/\xi .$$

Problem 3-C. More generally let ξ, η be arbitrary vector bundles over B and let f be a cross-section of the bundle $\mathrm{Hom}(\xi, \eta)$. If the rank of the linear function

$$f(b) : F_b(\xi) \to F_b(\eta)$$

is locally constant as a function of b, define the kernel $\kappa_f \subset \xi$ and the cokernel ν_f, and prove that they are locally trivial.

Problem 3-D. If a vector bundle ξ possesses a Euclidean metric, show that ξ is isomorphic to its dual bundle $\mathrm{Hom}(\xi, \varepsilon^1)$.

Problem 3-E. Show that the set of isomorphism classes of 1-dimensional vector bundles over B forms an abelian group with respect