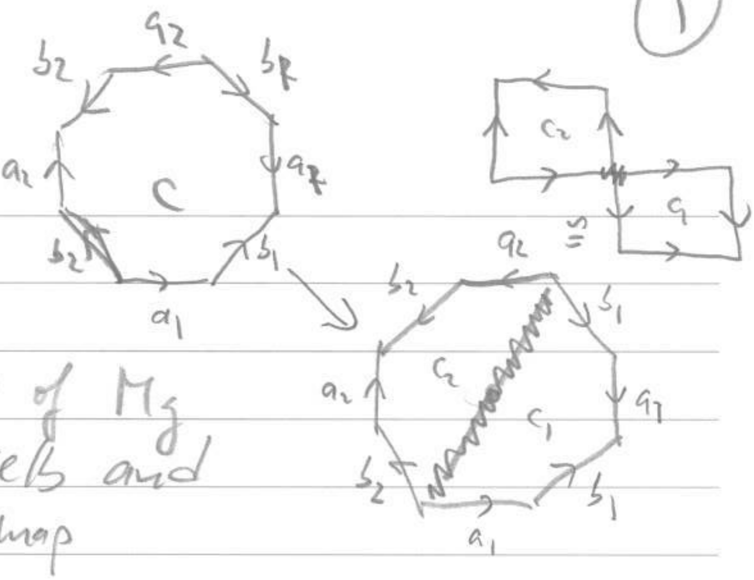
INSTITUT MITTAG-LEFFLER
THE ROYAL SWEDISH ACADEMY OF SCIENCES



Hatcher, Chapter 3.2

Ex 1: Use the cell structure of M_g with 1 0-cell, $2g$ 1-cells and 1 2-cell. The collapse map

$$M_g \longrightarrow \underbrace{T^2 \vee \cdots \vee T^2}_{g \text{ copies}}$$

takes the 2-cell of M_g to the sum of the g 2-cells in $\bigvee_{i=1}^{g} T^2$.

Dually,

$$\tilde{H}^* \bigvee_{i=1}^{g} T^2 \cong \bigoplus_{i=1}^{g} \tilde{H}^* T^2 \longrightarrow \tilde{H}^* M_g$$

is an isomorphism in degree 1,

$$\bigoplus_{i=1}^{g} \mathbb{Z}\{\alpha_i, \beta_i\} \xrightarrow{\cong} \mathbb{Z}\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$$

and maps each generator $\alpha_i \cup \beta_i \in H^2(T^2)$ to the generator $\gamma \in H^2(M_g)$ dual to c.

Hence

$$\alpha_i \cup \beta_i = \gamma = -\beta_i \cup \alpha_i \quad \text{for } i = 1, \dots, g$$

while the remaining pairs of generators multiply to 0.

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$$
\begin{array}{c|c}
 & \begin{array}{cccccccc} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & & & \alpha_g & \beta_g \end{array} \\
\begin{array}{c} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \alpha_g \\ \beta_g \end{array} &
\begin{bmatrix}
0 & -1 & 0 & 0 & & & 0 & 0 \\
1 & 0 & 0 & 0 & & \cdots & 0 & 0 \\
0 & 0 & 0 & -1 & & & 0 & 0 \\
0 & 0 & 1 & 1 & & & 0 & 0 \\
 & \vdots & & & \ddots & & \vdots & \\
0 & 0 & 0 & 0 & & \cdots & 0 & -1 \\
0 & 0 & 0 & 0 & & & 1 & 0
\end{bmatrix}
\end{array}
$$

<u>Ex. 11:</u> $f: S^{k+l} \to S^k \times S^l$ induces

$$f_*: H_*(S^{k+l}) \to H_*(S^k \times S^l)$$

dual to

$$f^*: H^*(S^k \times S^l) \to H^*(S^{k+l})$$
$$\cong \Lambda_{\mathbb{Z}}(x,y) \qquad = \Lambda_{\mathbb{Z}}(z)$$

with $|x|=k$, $|y|=l$, $|z|=k+l$. Then $f^*(x)=0$ and $f^*(y)=0$ live in trivial groups, so $f^*(x \smile y)$ $= f^*x \smile f^*y = 0$ in degree $k+l$. Hence f_* is also trivial in degree $k+l$.

Ex. 18

$$H^1 M_g = \mathbb{Z}\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}, \quad H^2 M_g = \mathbb{Z}\{\gamma\}.$$

Write $\alpha = m_1\alpha_1 + n_1\beta_1 + \dots + m_g\alpha_g + n_g\beta_g$ with

$m_1, \dots, n_g \in \mathbb{Z}$. If $\alpha \neq 0$ then some m_i or $n_j \neq 0$.

If $m_i \neq 0$, then let $\beta = \beta_i$ and compute

$$\alpha \cup \beta = m_1(\alpha_1 \cup \beta_i) + \dots + m_i \alpha_i \cup \beta_i + \dots + n_g \beta_g \cup \beta_i$$

$$= 0 + \dots + m_i \cdot \gamma + \dots + 0 \neq 0$$

If $n_j \neq 0$, then let $\beta = \alpha_j$ and compute

$$\alpha \cup \beta = m_1 \alpha_1 \cup \alpha_j + \dots + n_j \beta_j \cup \alpha_j + \dots + n_g \beta_g \cup \alpha_j$$

$$= 0 + \dots + (-n_j\gamma) + \dots + 0 \neq 0.$$

If $M_g \simeq X \vee Y$ then $\tilde{H}^* M_g \cong \tilde{H}^* X \oplus \tilde{H}^* Y$

as non-unital graded rings. May assume $H^2 X \cong 0$.

and $H^2 Y \cong \mathbb{Z}$. If $H^1 X \neq 0$, choose $\alpha \in H^1 X$ nonzero.

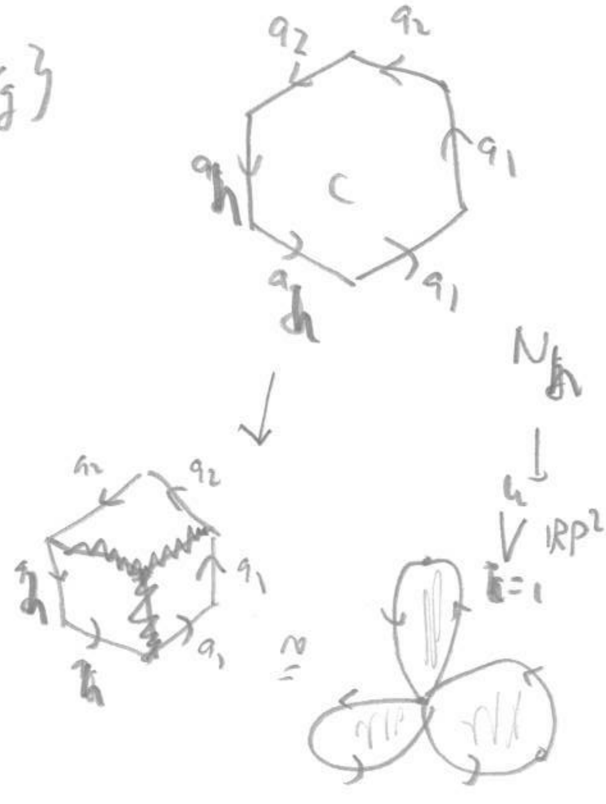
Then for any $\beta \in H^1 X$, $\alpha \cup \beta \in H^2 X = 0$, and for any $\beta \in H^1 Y$, $\alpha \cup \beta \in H^2 M$ is 0 by naturality with respect to $M \to X$ and $M \to Y$. Hence there is no $\beta \in H^1 M$ with $\alpha \cup \beta \neq 0$. $\lightning$ Contradiction.

$$H^1(N_h; \mathbb{Z}/2) = \mathbb{Z}/2\{\alpha_1, \dots, \alpha_g\}$$

$$H^2(N_h; \mathbb{Z}/2) = \mathbb{Z}/2\{\gamma\}$$

$$\alpha_1^2 = \dots = \alpha_g^2 = \gamma.$$

$$\alpha_i \cup \alpha_j = 0 \text{ for } i \neq j.$$



$$\begin{array}{c|c}
 & \begin{matrix} \alpha_1 & \cdots & & \alpha_h \end{matrix} \\
\begin{matrix} \alpha_1 \\ \vdots \\ \\ \\ \alpha_h \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}
\end{array}$$

If $\alpha = m_1\alpha_1 + \dots + m_h\alpha_h \neq 0$ with each $m_i \in \mathbb{Z}/2$, then some $m_i \neq 0$. Let $\beta = \alpha_i$. Then

$$\alpha \cup \beta = m_1\alpha_1 \cup \alpha_i + \dots + m_i\alpha_i \cup \alpha_i + \dots + m_h\alpha_h \cup \alpha_i$$
$$= 0 + \dots + 1 + \dots + 0 \neq 0 \quad \text{ETC.}$$

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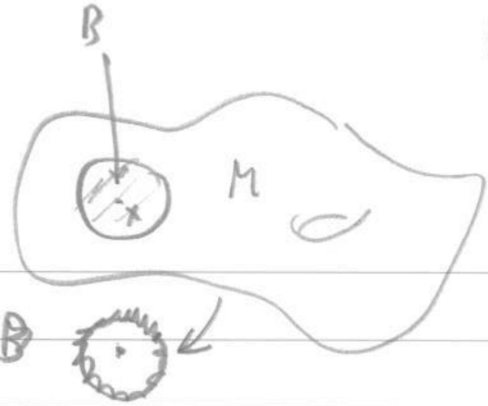
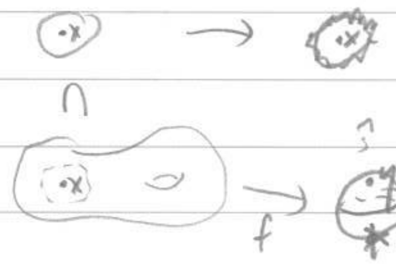


Hatcher, Chapter 3.3 $\quad S^n \cong B/\partial B$

Ex. 7: M connected closed orientable n-manifold.

Choose an $x \in B \subset M$, $(B, x) \cong (D^n, 0)$. Extend the quotient map $B \to B/\partial B \cong D^n/S^{n-1} \cong S^n$ to M, by mapping $M \setminus B$ to the base point:

$$\begin{tikzcd}
B \setminus x \arrow[dd] \arrow[rrdd, bend left] & & \\
& B \arrow[r] \arrow[d, phantom, "\cap"] & B/\partial B \arrow[d, phantom, "\cong"] \\
& M \arrow[r, "f"] & S^n \\
M \setminus x \arrow[rr] & & S^n - y
\end{tikzcd}$$



Let x map to $y \in S^n$. Then $f : M \to S^n$ maps $M - x$ to $S^n - y$.

$$\begin{tikzcd}
& H_n(B, B \setminus x) \arrow[d, "\cong"'] \arrow[dr, "\cong"] & \\
& H_n(M, M \setminus x) \arrow[r] & H_n(S^n, S^n \setminus y) \\
H_n M \arrow[ur, "\cong"] \arrow[rr, "f_*"] & & H_n S^n \arrow[u, "\cong"']
\end{tikzcd}$$

The diagram implies that f_* is an isomorphism. Orient M so that f has degree $+1$.

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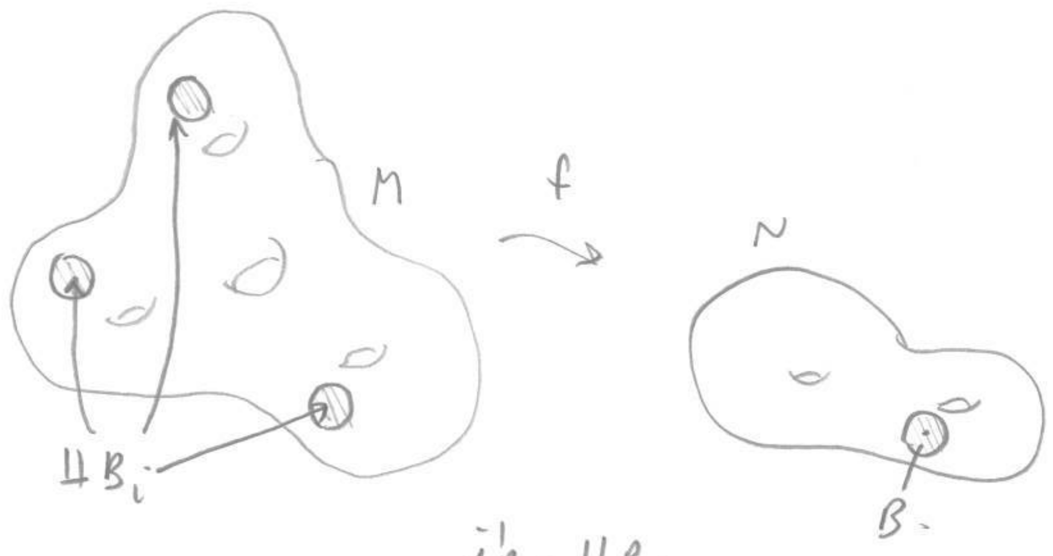
Ex 8.



$$f^{-1}B = \coprod_i B_i$$

$$f|B_i : B_i \xrightarrow{\cong} B$$

$$\begin{tikzcd}
\text{gen. } [M] \in H_n M \arrow[r, "f_*"] \arrow[d] & H_n N \ni \deg(f)[N] \arrow[d, "\cong"] \\
H_n(M, M - f^{-1}B) \arrow[r] & H_n(N, N-B) \\
\bigoplus_i \tilde{H}_n(B_i/\partial B_i) \arrow[u, "\cong"] \arrow[r, "(\varepsilon_i)"] & \tilde{H}_n B/\partial B \arrow[u, "\cong"]
\end{tikzcd}$$

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^t \qquad \sum_i \varepsilon_i$$

$$\deg(f) = \sum_i \varepsilon_i$$

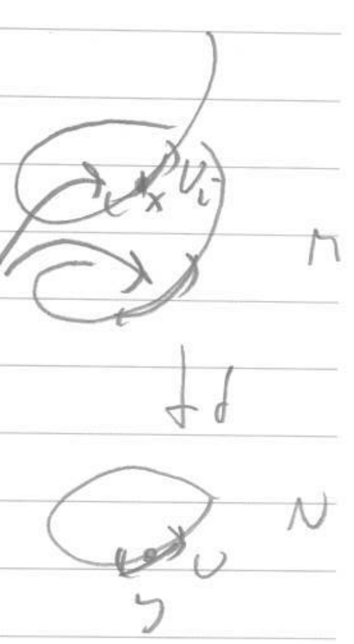
$$(f|B_i)_* : [B_i] = \varepsilon_i [B] \qquad \varepsilon_i = \pm 1$$

$\varepsilon_i = \pm 1$.

Ex. 9 $f: M \to N$ p-sheeted covering space,

for each $x \in M$, $f(x) \in N$ lies in an evenly covered nbhd $U \cong \mathbb{R}^n$, so $x \in f^{-1}U \cong \coprod_i U_i$.

$\coprod_i U_i = f^{-1}U$



$f|_{U_i} : U_i \xrightarrow{\cong} U$.

$U_i \subset M$

$\cong \downarrow \; \downarrow f$

$U \subset N$

$f(x)$

$$\begin{tikzcd}
H_n M \arrow[r] \arrow[d, "f_*"] & H_n(M, M-x) \arrow[d, "f_*"] & H_n(U_i, U_i - x) \arrow[l, "\cong"'] \arrow[d, "f|_{U_i}", "\cong"'] \\
H_n N \arrow[r] & H_n(N, N-f(x)) & H_n(U, U-f(x)) \arrow[l, "\cong"']
\end{tikzcd}$$

$[M] \mapsto o_x^M$

$[N] \mapsto o^N_{f(x)}$

f is orientation preserving / reversing at x if $f_*(o_x^M) = \pm o^N_{f(x)}$.

This is locally constant in x:

$$\begin{tikzcd}
o_x^M \in H_n(M, M-x) \arrow[ddd, bend right, "f_*^x"'] & H_n(M, M-B_i) \arrow[l, "\cong"'] \arrow[r] & H_n(M, M-y) \ni o_y^M \arrow[ddd, bend left, "f_*^y"] \\
H_n(U_i, U_i-x) \arrow[u, "\cong"] \arrow[d, "\cong"] & H_n(U_i, U_i-B_i) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[u, "\cong"] \arrow[d, "f_*^B", "\cong"'] & H_n(U_i, U_i-y) \arrow[u, "\cong"] \arrow[d, "\cong"] \\
H_n(U, U-f(x)) \arrow[d, "\cong"] & H_n(U, U-B) \arrow[l] \arrow[r] \arrow[d, "\cong"] & H_n(U, U-f(y)) \arrow[d, "\cong"] \\
H_n(N, N-f(x)) & H_n(N, N-B) \arrow[l] \arrow[r] & H_n(N, N-f(y))
\end{tikzcd}$$

$o^M_{B_i} \in H_n(M, M-B_i)$ restricts to o^M_x and o^M_y,

and maps to $\pm\, o^N_B \in H_n(N, N-B)$,

which restricts to $\pm\, o^N_{fx}$ and $\pm\, o^N_{fy}$.

Same signs, so f is or. pres/or. rev. at x

(⟺) — " — at y.

Since M is connected, f is or. pres. or

or. rev. everywhere.

$\Rightarrow$ each $\varepsilon_i = +1$ or each $\varepsilon_i = -1$

$\Rightarrow \deg f = \sum_i \varepsilon_i = +p$ or $-p$.

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M, N connected closed oriented n-manifolds

Ex. 10 $f : M \to N$ degree 1

Pick $x_0 \in M$, consider $\pi_1(f) = f_* : \pi_1(M, x_0) \to \pi_1(N, f(x_0))$
$\cup \text{im} f_*$

Covering space theory: f lifts to
$\tilde{f} : M \to \tilde{N}$ where $\text{im}\, \pi_* = \text{im} f_*$

$$\begin{array}{ccc} & & \tilde{N} \\ & \overset{\hat{f}}{\nearrow} & \downarrow \pi \\ M & \xrightarrow{f} & N \end{array}$$

If π is finite-sheeted,
$\tilde{N}$ is compact, so

$$\begin{array}{ccc} & & H_n \tilde{N} \cong \mathbb{Z} \\ & \overset{\tilde{f}_*}{\nearrow} & \downarrow \pi_* \\ \mathbb{Z} \cong H_n M & \xrightarrow{f_*} & H_n N \cong \mathbb{Z} \end{array}$$

implies that π has degree ± 1, i.e., $\tilde{N} \cong N$ and f_* is onto.

If π is infinite-sheeted, then $\pi^{-1}(U)$ for a nonempty evenly covered $U \subset N$ contains an infinite sequence without accumulation points, so $\tilde{N}$ is not compact. Then $H_n \tilde{N} = 0$, contradicting $f_*[M] = [N]$ ($f_* \neq 0$) on H_n

[Hatcher, Prop. 3.29]

$$\begin{array}{ccc} \pi_1 M & \xrightarrow{f_*} & \pi_1 N \\ \downarrow & & \downarrow \\ H_1 M & \longrightarrow & H_1 N \end{array}$$

$\pi_1(f)$ surjective $\Rightarrow H_1(f)$ surjective

Ex. 11:

By Ex. 10, if a degree 1 map $f: M_g \to M_h$ exists, then $f_*: \underset{\cong \mathbb{Z}^{2g}}{H_1 M_g} \to H_1 M_h \cong \mathbb{Z}^{2h}$ is surjective, hence $2g \geq 2h$ and $g \geq h$.

Conversely, connected sum with a degree 1 map $T^2 \to S^2$ gives a degree 1 map $M_{h+1} \to M_h$. By induction we get such a map for $g \geq h$.

Ex. 12:

If $[x_1,x_2]\cdots[x_{2k-1},x_{2k}] = [v_1,w_1]\cdots[v_j,w_j]$ in $F = \mathrm{free}\langle x_1,\dots,x_{2k}\rangle$, then send $v_1,\dots,w_j$ til disse ordene i $x_1,\dots,x_{2k}$

$$\begin{tikzcd}
S^1 \arrow[d, "{[x_1,x_2]\cdots[x_{2k-1},x_{2k}]}"'] & S^1 \arrow[l, dashed, "="'] \arrow[d, "{[v_1,w_1]\cdots[v_j,w_j]}"] \\
\bigvee^{2k} S^1 \arrow[d] & \bigvee^{2j} S^1 \arrow[l] \arrow[d] \\
M_k \arrow[d] & M_j \arrow[l, dashed] \arrow[d] \\
S^2 & S^2 \arrow[l, "="']
\end{tikzcd}$$

of [illegible] f
$M_k \xrightarrow{f} M_j$ of degree 1 ~~of degree 1~~.

$$\begin{tikzcd}
H_2 M_k \arrow[d, "\partial"'] & H_2 M_j \arrow[l, "f_*"'] \arrow[d, "\partial"] \\
H_1 S^1 \arrow[r, equal] & H_1 S^1
\end{tikzcd}$$

$\Downarrow$

$j \geq k$.