Milnor - Stasheff

2-A $n = 2m-1$ odd, $S^n = S(\mathbb{C}^m) = S(\mathbb{R}^{2m})$

Let

$$v(z_1, \dots, z_m) = (iz_1, \dots, iz_m)$$

for $|z_1|^2 + \dots + |z_m|^2 = 1$, or equivalently

$$v(x_1, y_1, \dots, x_m, y_m) = (-y_1, x_1, \dots, -y_m, x_m)$$

for $x_1^2 + y_1^2 + \dots + x_m^2 + y_m^2 = 1$.

Then $v(z) \perp z$ for each $z = (z_1, \dots, z_m) \in S^n$, so $v(z) \in T_z S^n$ and $z \mapsto v(z)$ defines a vector field on S^n. It is nowhere zero, because $|v(z)| = |iz| = |z| = 1 \neq 0$.

Let $n(z) = z$. Then $n(z) \perp T_z S^n$ so $n(z) \in N_z S^n$ and $z \mapsto n(z)$ defines a section in the normal bundle of S^n. Hence there is an isomorphism

$$\begin{tikzcd} S^n \times \mathbb{R} \arrow[rr, "\cong"] \arrow[dr] & & NS^n \arrow[dl] \\ & S^n & \end{tikzcd}$$

$$(z, \lambda) \longmapsto \lambda \cdot n(z) \quad (= \lambda z)$$

showing that $NS^n \to S^n$ is trivial.

2-B If $v: S^n \to TS^n$ is nowhere zero, let $w: S^n \to TS^n$ be given by $w(z) = \frac{v(z)}{\|v(z)\|}$ so that $\|w(z)\| = 1$ for all $z \in S^n$.

Consider

$$h(z,t) = \cos \pi t \cdot z + \sin \pi t \cdot w(z).$$

Here $\|h(z,t)\| = \cos^2 \pi t \cdot \|z\| + \sin^2 \pi t \cdot \|w(z)\| = 1$ since $z \perp w(z)$, so

$$h: S^n \times [0,1] \to S^n$$

is a homotopy from

$$z \mapsto h(z,0) = z$$

to

$$z \mapsto h(z,1) = -z,$$

i.e., from the identity map of S^n to the antipodal map of S^n.

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For $n = 2m$ even the reflection

$$r(x_1, x_2, \dots, x_{n+1}) = (-x_1, x_2, \dots, x_{n+1})$$

is homotopic to the antipodal map

$$a(x_1, x_2, \dots, x_{n+1}) = (-x_1, -x_2, \dots, -x_{n+1})$$

via the homotopy

$$h((x_1, \dots, x_{n+1}), t) = (-x_1, \cos \pi t \cdot x_2 + \sin \pi t \cdot x_3,$$
$$-\sin \pi t \cdot x_2 + \cos \pi t \cdot x_3,$$
$$\vdots$$
$$\cos \pi t \cdot x_{2m} + \sin \pi t \cdot x_{2m+1},$$
$$-\sin \pi t \cdot x_{2m} + \cos \pi t \cdot x_{2m+1})$$

$\left(\begin{array}{l}\text{for } x_1^2 + \dots + x_{n+1}^2 = 1 \\ 0 \le t \le 1\end{array}\right)$

or equivalently

$$h(x_1, z_1, \dots, z_m, t) = (-x_1, e^{i\pi t} z_1, \dots, e^{i\pi t} z_m)$$

for $x_1 \in \mathbb{R}$, $z_1, \dots, z_m \in \mathbb{C}$, $x_1^2 + |z_1|^2 + \dots + |z_m|^2 = 1$

Hence $\deg(a) = \deg(r) = -1$.

If S^n admits a nonzero vector field then $\deg(a) = \deg(id) = 1$, so n cannot be even.

In particular, for $n \geq 2$ even, S^n cannot be parallelizable.

2-C Let $\xi : E \xrightarrow{\pi} B$ be a vector bundle, with B paracompact. Choose an open cover $\{U_\alpha\}_\alpha$ of B with trivializations

$$\begin{tikzcd} U_\alpha \times \mathbb{R}^n \arrow[r, "h_\alpha", "\cong"'] \arrow[dr, "proj_1"'] & \pi^{-1}(U_\alpha) \arrow[d] \subset E \arrow[d, "\pi"] \\ & U_\alpha \subset B \end{tikzcd}$$

By paracompactness, there exists a locally finite refinement $\{V_\beta\}_\beta$ of $\{U_\alpha\}_\alpha$, such that for each β there exists an α with $V_\beta \subset U_\alpha$, and each point of B has a neighborhood that only meets finitely many of the V_β. (Of course, $B = \bigcup_\alpha U_\alpha = \bigcup_\beta V_\beta$.)
Furthermore, there exists a partition of unity $\{\varphi_\beta\}_\beta$, with $\varphi_\beta : B \to I$, $\operatorname{supp} \varphi_\beta \subset V_\beta$, $\sum_\beta \varphi_\beta = 1$, subordinate to $\{V_\beta\}_\beta$.

The trivializations h_α restrict to trivializations

$$\begin{tikzcd} V_\beta \times \mathbb{R}^n \arrow[r, "h_\beta", "\cong"'] \arrow[dr, "proj_1"'] & \pi^{-1}(V_\beta) \arrow[d, "\pi"] \subset E \\ & V_\beta \subset B \end{tikzcd}$$

(let $h_\beta = h_\alpha | V_\beta \times \mathbb{R}^n$ for one choice of α)

Define a local Euclidean metric

$$\mu_\beta : \pi^{-1}(V_\beta) \to \mathbb{R}$$

by $\mu_\beta(v) = x_1^2 + \dots + x_n^2$ if $k_\beta^{-1}(v) = (b, x_1, \dots, x_n)$.
Then μ_β is quadratic and positive definite on each fiber $\pi^{-1}(b)$, for $b \in V_\beta$. Alternatively form

$$\langle\, , \rangle_\beta : \pi^{-1}(V_\beta) \times_{V_\beta} \pi^{-1}(V_\beta) \to \mathbb{R}$$

by $\langle v, w \rangle_\beta = x_1 y_1 + \dots + x_n y_n$ if

$k_\beta^{-1}(v) = (b, x_1, \dots, x_n)$ and $k_\beta^{-1}(b, y_1, \dots, y_n)$.
Then $\langle\, , \rangle_\beta$ is bilinear, symmetric and positive definite on each fiber $\pi^{-1}(b) \times_{\{b\}} \pi^{-1}(b)$ for $b \in V_\beta$.

Then $\varphi_\beta \pi \cdot \mu_\beta : \pi^{-1}(V_\beta) \to \mathbb{R}$ has support in $\pi^{-1}\operatorname{supp}\varphi_\beta \subset \pi^{-1}(V_\beta)$, and can be ~~extended~~ continuously extended by 0 on $E \setminus \pi^{-1}V_\beta$. Write $\varphi_\beta \mu_\beta : E \to \mathbb{R}$ for this extension,
Alternatively, form $\varphi_\beta \pi \langle\, , \rangle_\beta : E \times_B E \to \mathbb{R}$.

The sum

$$\sum_\beta \varphi_\beta \mu_\beta : \overset{E(B)}{\overset{\shortparallel}{E}} \longrightarrow \mathbb{R}$$

only has finitely many nonzero terms in a neighborhood of any point, hence is continuous. It is quadratic on each fiber $\pi^{-1}(b) = E_b$, and positive definite since each $\varphi_\beta(b)\mu_\beta$ is non-negative definite and for at least one β we have $\varphi_\beta(b) > 0$, so that $\varphi_\beta\mu_\beta$ is positive definite.

Alternatively consider the sum

$$\sum_\beta \varphi_\beta \langle\, ,\rangle_\beta : E \times_B E \longrightarrow \mathbb{R}$$
$$= E(\xi\oplus\xi)$$

which is bilinear and symmetric on each fiber $\pi^{-1}(b) \times_{\{b\}} \pi^{-1}\{b\} = E_b \times E_b$, and is positive definite because each $\varphi_\beta(b) \geq 0$ and some $\varphi_\beta(b) > 0$.

3-A

Let $f: B\times\mathbb{R}^m \to B\times\mathbb{R}^n$ be a continuous map of trivial bundles over B. For each $c\in B$ let $f_c:\mathbb{R}^m\to\mathbb{R}^n$ be the linear homomorphism so that $f(c,v)=(c,f_c(v))$.

$$\begin{tikzcd}
\ker f_c \arrow[dr, hook, "i_c"] & & & & \operatorname{cok} f_c \\
& \mathbb{R}^m \arrow[rr, "f_c"] \arrow[dr, two heads] & & \mathbb{R}^n \arrow[ur, two heads, "p_c"] & \\
& & \operatorname{im} f_c \arrow[ur, hook] & &
\end{tikzcd}$$

Let $i_c:\ker f_c\to\mathbb{R}^m$ and $p_c:\mathbb{R}^n\to\operatorname{cok} f_c$ be the canonical homomorphisms.

<u>Fix</u> $b\in B$ and suppose that $\operatorname{rank} f_b=\dim\operatorname{im} f_b=k$. Then $\dim\ker f_b=m-k$ and $\dim\operatorname{cok} f_b=n-k$.

Choose a basis $u_1,\dots,u_{m-k}$ for $\ker f_b$, a basis $f_b(v_1),\dots,f_b(v_k)$ for $\operatorname{im} f_b$, and a basis $p_b(w_1),\dots,p_b(w_{n-k})$ for $\operatorname{cok} f_b$.

$$\begin{tikzcd}
u_1,\dots,u_{m-k} \arrow[dd, mapsto] & \ker f_b \arrow[dr, hook, "i_b"] & & & \operatorname{cok} f_b & p_b(w_1),\dots,p_b(w_{n-k}) \\
& & \mathbb{R}^m \arrow[rr, "f_b"] \arrow[dr, two heads] & & \mathbb{R}^n \arrow[u, two heads, "p_b"] & w_1,\dots,w_{n-k} \arrow[u, mapsto] \\
i_b(u_1),\dots,i_b(u_{m-k}) \overset{+}{} v_1,\dots,v_k \arrow[dr, mapsto] & & & \operatorname{im} f_b \arrow[ur, hook] & & \overset{+}{f_b(v_1),\dots,f_b(v_k)} \\
& f_b(v_1),\dots,f_b(v_k) \arrow[urrrr, mapsto] & & & &
\end{tikzcd}$$

Then $i_b(u_1), \ldots, i_b(u_{m-k}), v_1, \ldots, v_k$ is a basis for $\mathbb{R}^m$

and $f_b(v_1), \ldots, f_b(v_k), w_1, \ldots, w_{n-k}$ is a basis for $\mathbb{R}^n$.
[Check!]

Consider

$$c \mapsto \det(f_c(v_1), \ldots, f_c(v_k), w_1, \ldots, w_{n-k})$$
$$B \to \mathbb{R}$$

It is continuous, and nonzero at b, hence nonzero in a neighborhood around b. So for c near b,

$$f_c(v_1), \ldots, f_c(v_k), w_1, \ldots, w_{n-k}$$

remains a basis for $\mathbb{R}^n$. In particular, the k vectors

$$f_c(v_1), \ldots, f_c(v_k)$$

remain linearly independent in $\operatorname{im} f_c \subset \mathbb{R}^n$.

Suppose that <u>rank f is constant</u>, i.e., that $\operatorname{rank} f_c = \dim \operatorname{im} f_c = k$ for all $c \in B$. Then

$$f_c(v_1), \ldots, f_c(v_k)$$

is a basis for $\operatorname{im} f_c$ for all c near b.

On the other hand, $u_1, \ldots, u_{m-k}$ may cease to be a basis for $\ker f_c$ for c near b, since $f_c(u_i)$ needs not be 0 $(1 \le i \le m-k)$.

Write
$$f_c(u_i) = \sum_{j=1}^{k} a_c^{ij} \cdot f_c(v_j)$$
for scalars a_c^{ij}. (This uses that $f_c(v_1), \ldots, f_c(v_k)$ is a basis for $\operatorname{im} f_c \ni f_c(u_i)$.) The a_c^{ij} depend continuously on c, [Check!] and $a_b^{ij} = 0$.

Let
$$s_i(c) = u_i - \sum_{j=1}^{k} a_c^{ij} \cdot v_j$$
define sections $s_1, \ldots, s_{m-k}$ in $\ker f = \bigcup_{c \in B} \ker f_c \subset B \times \mathbb{R}^m$ for c near b. Note that
$$f_c(s_i(c)) = f_c(u_i) - \sum_{j=1}^{k} a_c^{ij} f_c(v_j) = 0$$
so $s_i(c) \in \ker f_c$, and $s_i(b) = u_i$.

Then $i_c s_1(c), \ldots, i_c s_{m-k}(c), v_1, \ldots, v_k$ is a basis for $\mathbb{R}^m$ for $c = b$, hence by continuity also for all c near b.

Thus $s_1(c), \ldots, s_{m-k}(c)$ are linearly independent in

$\ker f_c$ of dimension $\dim f_c = m-k$ for all c near b,

hence $s_1, \dots, s_{m-k}$ define a local trivialization

$$\begin{tikzcd}
B \times \mathbb{R}^{m-k} \arrow[rr, "s_1,\dots,s_{m-k}"] \arrow[dr] & & \ker f = \bigcup_{c\in B} \ker f_c \subset B \times \mathbb{R}^m \arrow[dl] \\
& B &
\end{tikzcd}$$

over a neighborhood of b in B.

Thus $\ker f = \bigcup_{c\in B} \ker f_c \to B$ is a (locally trivial) vector bundle of rank $m-k$.

Similarly, $c \mapsto f_c(v_1), \dots, f_c(v_k)$ define a local trivialization

$$\begin{tikzcd}
B \times \mathbb{R}^{k} \arrow[rr, dashed] \arrow[dr] & & \operatorname{im} f = \bigcup_{c\in B} \operatorname{im} f_c \subset B \times \mathbb{R}^n \arrow[dl] \\
& B &
\end{tikzcd}$$

over a neighborhood of b, so $\operatorname{im} f = \bigcup_{c\in B} \operatorname{im} f_c \to B$ is a (locally trivial) vector bundle of rank k.

Lastly, $f_c(v_1), \ldots, f_c(v_k), w_1, \ldots, w_{n-k}$ remains a basis for $\mathbb{R}^n$ for c near b, so

$$p_c(w_1), \ldots, p_c(w_{n-k})$$

span coh f_c for all c near b. Since dim coh f_c $= n-k$, these give a basis, so the sections

$$c \mapsto p_c(w_1), \ldots, p_c(w_{n-k})$$

define a local trivialization

$$\begin{tikzcd} B \times \mathbb{R}^{n-k} \arrow[rr, dashed] \arrow[dr] & & \operatorname{coh} f = \bigcup_{c \in B} \operatorname{coh} f_c \arrow[dl] & B \times \mathbb{R}^n \arrow[l] \\ & B & \end{tikzcd}$$

near b. Hence coh f is a vector bundle of rank $n-k$.

If $f : M \to N$ is a submersion, then

$$Df : TM \longrightarrow f^*TN$$

is surjective

$$Df_x : T_xM \longrightarrow (f^*TN)_x = T_{f(x)}N$$

on each fiber. Choose local trivializations $B \times \mathbb{R}^m \cong TM|B$ and $B \times \mathbb{R}^n \cong (f^*TN)|B$ for B a neighborhood of x and consider

$$\begin{tikzcd} B \times \mathbb{R}^m \arrow[rr, "q"] \arrow[dr] & & B \times \mathbb{R}^n \arrow[dl] \\ & B & \end{tikzcd}$$

as above, with $k = n$.

Hence $\ker g$ is a vector bundle over B of rank $m-k$, so ~~ker~~ $\kappa_f = \ker D_f$ is also a vector bundle

$$0 \to \kappa_f \rightarrowtail \tau_M \twoheadrightarrow f^*\tau_N \to 0$$

If M is Riemannian, τ_M has an Euclidean inner product, and κ_f has an orthogonal complement $\kappa_f^\perp$ so that

$$\begin{tikzcd} & \kappa_f \oplus \kappa_f^\perp \arrow[r] \arrow[d, "\cong"] & \kappa_f^\perp \arrow[d] \\ \kappa_f \arrow[ur, hook] \arrow[r, hook] & \tau_M \arrow[r] & f^*\tau_N \end{tikzcd}$$

$\kappa_f \oplus \kappa_f^\perp \xrightarrow{\cong} \tau_M$, hence $\kappa_f^\perp \xrightarrow{\cong} f^*\tau_N$, and

$$\kappa_f \oplus f^*\tau_N \xrightarrow{\cong} \tau_M.$$

$\underline{\underline{3-B}}$. If $\xi \hookrightarrow \eta$ then locally $\xi|B \hookrightarrow \eta|B$ has the form

$$\begin{tikzcd} B \times \mathbb{R}^m \arrow[rr, "f"] \arrow[dr] & & B \times \mathbb{R}^n \arrow[dl] \\ & B & \end{tikzcd}$$

where f has constant rank $k = m$,

so $\operatorname{coker} f = \bigcup_{c \in B} \operatorname{cok} f_c \cong \bigcup_{c \in B} \eta_c/\xi_c$ is a vector bundle of rank ~~n~~ $n-m$ and η/ξ is such an $(n-m)$-bundle

If η has a Euclidean metric, ξ has an orthogonal complement, $\xi^\perp$, with

$$\begin{tikzcd} & \xi \oplus \xi^\perp \arrow[r] \arrow[d, "\cong"] & \xi^\perp \arrow[d] \\ \xi \arrow[ur] \arrow[r] & \eta \arrow[r] & \eta/\xi \end{tikzcd}$$

so $\xi^\perp \xrightarrow{\cong} \eta/\xi$.

<u>3-C</u> If $f : \xi \to \eta$ has locally constant rank, then it is locally of the form

$$\begin{tikzcd} B \times \mathbb{R}^m \arrow[rr, "f"] \arrow[dr] & & B \times \mathbb{R}^n \arrow[dl] \\ & B & \end{tikzcd}$$

with rank f_c constant, so ker f, im f and cok f are vector bundles, so κ_f, im $f = f(\beta)$ and ν_f are vector bundles