Milnor - Stasheff

3-F $C(B) = \mathbb{R}(B) = \{\text{maps } B \xrightarrow{f} \mathbb{R}\}$ commutative algebra

$\Gamma(\xi) = S(\xi) = \{\text{sections } B(\xi) \xrightarrow{s} E(\xi)\}$ $C(B)$-module

Fix $x \in B$ let $m_x \subset C(B)$ be the ideal $\{f \mid f(x) = 0\}$.

(a) Lemma (B Tychonoff space = completely regular and Hausdorff)

Isomorphism: $\Gamma(\xi)/m_x\Gamma(\xi) \underset{\cong}{\xrightarrow{\varphi}} F_x(\xi) \subset E(\xi)$

$$[s] \longmapsto s(x)$$

Proof For each $v \in F_x(\xi)$ there is a section s' of ξ near x with $s'(x) = v$. Using the Tychonoff property, s' can be extended to s on B with $s(x) = s'(x) = v$. So φ is surjective.

Suppose $s(x) = 0$. Choose local basis $s_1', \dots, s_n'$ near x ($n = \dim F_x(\xi)$), and write $s = \sum_{i=1}^n f_i' s_i'$ near x, for $f_1', \dots, f_n'$ defined and continuous near x. Using the Tychonoff property find $f_1, \dots, f_n \in C(B)$ with $f_i = f_i'$ near x, so that $s = \sum_{i=1}^n f_i s_i$ near x, $f_i(x) = f_i'(x) = 0$. Let $t = s - \sum_i f_i s_i$. Then $t = 0$ near x. Let $g \in C(B)$ satisfy $g(x) = 0$, and $g = 1$ whenever $t \neq 0$. Then $t = gt$, so $s = \sum f_i s_i + gt \in m_x \Gamma(\xi)$. ∎

If $\Gamma(\xi)$ is a free $C(B)$-module, choose a basis $s_1, \dots, s_n$ to get a bundle map

$$\begin{array}{ccc} B \times \mathbb{R}^n & \xrightarrow{g} & E(\xi) \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

inducing the $C(B)$-module isomorphism $C(B)\{s_1,\dots,s_n\} \xrightarrow{\cong} \Gamma(\xi)$

and $\mathbb{R}$-linear iso. $\mathbb{R}\{s_1(x),\dots,s_n(x)\} \xrightarrow{\cong} F_x(\xi)$

so that g is a bundle isomorphism.

(b) If Q is a f.g. proj $C(B)$-module, choose an idempotent $e : \underbrace{C(B)^n}_{=\Gamma(\varepsilon^n)} \longrightarrow C(B)^n$ ($e^2=e$) with $Q \cong \mathrm{im}(e)$.

(e a $C(B)$-module homomorphism)

Define $g : B\times\mathbb{R}^n \longrightarrow B\times\mathbb{R}^n$ over B by $g\circ s_i = e(s_i)$,

where $s_1,\dots,s_n$ are the n standard sections in ε^n. Hence $\Gamma(g)(s_i) = e(s_i)$ for $1\le i\le n$ and $\Gamma(g) = e$.

Then $\Gamma(g^2) = \Gamma(g)\circ\Gamma(g) = e\circ e = e = \Gamma(g)$.

This implies $g^2 = g$, because for each $(x,v)\in B\times\mathbb{R}^n$ there is a section $(b \xmapsto{s} (b,v))$ with $s(x)=v$, and

$g^2(x,v) = g^2\circ s(b) = \Gamma(g^2)(s)(b) = \Gamma(g)(s)(b) = g\circ s(b) = g(x,v)$.

Likewise $1-e$ is an idempotent, and we can find a bundle map $h : B\times\mathbb{R}^n \longrightarrow B\times\mathbb{R}^n$ over B with $\Gamma(h) = 1-e$.

Let $E = \mathrm{im}\, g \subset B\times\mathbb{R}^n$ ~~and $F = \mathrm{im}\, h \subset B\times\mathbb{R}^n$~~. Since $g+h = \mathrm{id}$ and $g\circ h = 0$ realize $e+(1-e)=1$ and $e(1-e)=0$, $\mathrm{rank}\, g_b + \mathrm{rank}\, h_b \geq n$ and $n - \mathrm{rank}\, g_b \geq \mathrm{rank}\, h_b$, so $\mathrm{rank}\, g_b + \mathrm{rank}\, h_b = n$. Since both are semicontinuous, $\mathrm{rank}\, g_b$ is locally constant, so $E \downarrow B$ is a vector bundle ξ

and $\Gamma(\xi) = \mathrm{im}\,\Gamma(g) = \mathrm{im}\, e \cong Q$.

(c)

If $\Gamma(\xi) \overset{\varphi}{\cong} \Gamma(\eta)$ as $C(B)$-modules, we get a fiberwise linear bijection $\xi \xrightarrow{g} \eta$ by

$$
\begin{tikzcd}
 & \Gamma(\xi) \arrow[r, "\varphi", "\cong"'] \arrow[d] & \Gamma(\eta) \arrow[d] & \\
\Gamma(\xi) \otimes_{C(B) \ni ev_x} \mathbb{R} \arrow[r, phantom, "="] & \Gamma(\xi)/m_x\Gamma(\xi) \arrow[r, "\cong", "\varphi\otimes 1"'] \arrow[d, "\cong"] & \Gamma(\eta)/m_x\Gamma(\eta) \arrow[r, phantom, "="] \arrow[d, "\cong"] & \Gamma(\eta) \otimes_{C(B) \ni ev_x} \mathbb{R} \\
 & F_x(\xi) \arrow[r, "g_x"'] & F_x(\eta) &
\end{tikzcd}
$$

To check that g is continuous, we can work locally on B.

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4-D

If $M^n \hookrightarrow \mathbb{R}^{n+1}$ then $\tau_M \subset \varepsilon^{n+1}$

so $\varepsilon^{n+1} \cong \tau_M^n \oplus \nu_M^1$ and $w(\tau_M)w(\nu_M) = 1$

For $M = \mathbb{R}P^n$, $w(\tau_M) = (1+x)^{n+1} = 1 + (n+1)x + \ldots$

so $w_1(\nu^1) = -w_1(\tau_M) = -(n+1)x = \begin{cases} x & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$

Hence $w(\tau_M) = 1/w(\nu_M) = \begin{cases} 1/(1+x) & n \text{ even} \\ 1/1 = 1 & n \text{ odd.} \end{cases}$

For n even we must have

$$(1+x)^{n+1} \equiv \frac{1}{1+x} \mod x^{n+1}$$

and for n odd we must have

$$(1+x)^{n+1} \equiv 1 \mod x^{n+1}.$$

In the even case, $(1+x)^{n+2} \equiv 1 \mod x^{n+1}$, so

$$\binom{n+2}{i} \equiv 0 \quad \text{for } 1 \leq i \leq n$$

which only happens if $\underline{n+2 = 2^r}$ is a power of 2.

In the odd case,

$$\binom{n+1}{i} \equiv 0 \quad \text{for } 1 \leq i \leq n$$

which only happens if $\underline{n+1 = 2^r}$ is a power of 2.

4-C If $\xi_1^k \subset \tau_M^n$ then $\tau_M^n \cong \xi^k \oplus \eta^{n-k}$

so $w(\tau_M) = w(\xi^k) \cup w(\eta)$

For $M = \mathbb{R}P^n$, $w(\tau_M) = (1+x)^{n+1}$ in $P(x)/(x^{n+1})$

If n is even, $w(\tau_M) = 1 + x + \dots + x^n$. If $w(\xi) = 1$ then $w(\eta) = 1 + x + \dots + x^n$, which is impossible for η of dimension $n-1 < n$. If $w(\xi) = 1+x$ then $w(\eta) = (1+x)^{n+1}/(1+x) = (1+x)^n = 1 + \dots + x^n$, which is also impossible for η of dimension $n-1 < n$.

If n is odd, S^n admits a vector field v with $v(-x) = -v(x)$, so the line $\mathbb{R}\{v(x)\} \subset T_x S^n \xrightarrow{\cong} T_{[\pm x]} \mathbb{R}P^n$ defines a 1-dimensional subbundle of $\tau_{\mathbb{R}P^n}$

If $\mathbb{R}P^4$ admitted a field of tangent 2-planes, $\tau_{\mathbb{R}P^4} \cong \xi^2 \oplus \eta^2$, so $w(\xi) \cup w(\eta) = w(\tau_{\mathbb{R}P^4}) =$

$(1+x)^5 = 1 + x + x^4$ in $P(x)/(x^5)$

Then $w_2(\xi)\, w_2(\eta) = w_4(\tau_{\mathbb{R}P^4}) = x^2$, so

$w(\xi) = 1 + ax + x^2$ and $w(\eta) = 1 + bx + x^2$.

Hence $(1+ax+x^2)(1+bx+x^2) = 1 + x + x^4$

$\Updownarrow$ $a+b = 1$, $ab = 0$, $a+b = 0$

which has no solutions.

If $\xi^2 \subset \tau_{\mathbb{R}P^6}$ then $\tau_{\mathbb{R}P^6} \cong \xi^2 \oplus \eta^4$ so

$$w(\xi) \cup w(\eta) = w(\tau_{\mathbb{R}P^6}) = (1+x)^7 = 1+x+x^2+x^3+x^4+x^5+x^6$$

in $H^*(\mathbb{R}P^6; \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^7)$.

From $w_2(\xi) w_4(\eta) = w_6(\tau_{\mathbb{R}P^6}) = x^6$ we get

$$\begin{cases} w(\xi) = 1 + ax + x^2 \\ w(\eta) = 1 + bx + cx^2 + dx^3 + x^4 \end{cases}$$

so

$$\begin{cases} a+b = 1 \\ 1+ab+c = 1 \\ b+ac+d = 1 \\ c+ad+1 = 1 \\ d+a = 1 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=1 \\ c=0 \\ d=0 \\ (\text{fail}) \end{cases} \quad \underline{\text{or}} \quad \begin{cases} a=1 \\ b=0 \\ c=0 \\ d=1 \\ (\text{fail}) \end{cases}$$

which has $\underline{\underline{\text{no}}}$ solutions.