Hatcher Chapter 3.3

18, 19, 21, 24, 30, 31, 32, 33

Ex 18

Let $x \in \varinjlim_\alpha G_\alpha$. Then x is the image of $x_\alpha \in G_\alpha$ for some α. Suppose that $mx = 0$. Then mx_α maps to 0 in $\varinjlim_\alpha G_\alpha$, so for some $\beta \geq \alpha$ the image of mx_α in G_β is 0. Then the image x_β of x_α in G_β satisfies $mx_\beta = 0$. Since G_β is torsionfree, $x_\beta = 0$. Thus x is the image of $x_\beta = 0$ in G_β, hence $x = 0$.

Let $H \subseteq \varinjlim_\alpha G_\alpha$ be a finitely generated subgroup. H is a finitely generated abelian group, hence admits a finite set $x^1, \dots, x^g$ of generators, subject to the relations that follow from

$$m^1 x^1 = 0, \dots, m^r x^r = 0$$

for some $m^1, \dots, m^r \in \mathbb{N}$, $r \leq g$.
Each x^i admits a lift $x^i_{\alpha_i} \in G_{\alpha_i}$ for some α_i (for $1 \leq i \leq n$). Replacing α_i by a $\beta_i \geq \alpha_i$ we may assume that $m^i x^i_{\beta_i} = 0$ for $1 \leq i \leq r$.

Let $\beta_i = \alpha_i$ for $r < i \leq g$. Choose a γ with $\gamma \geq \beta_i$ for all i. Then the images $x^i_\gamma \in G_\gamma$ map to the generators x_γ in $\varinjlim_\alpha G_\alpha$ and satisfy $m^i x^i_\gamma = 0$ for $i \leq r$. Hence $H_\gamma = \langle x^1_\gamma, \dots, x^g_\gamma \rangle \subseteq G_\gamma$ maps isomorphically to $H < G$.

Ex. 19

$$\varinjlim_{\alpha} A_\alpha = \operatorname{colim}_\alpha A_\alpha \text{ is a quotient of } \bigoplus_\alpha A_\alpha .$$

A countable direct sum of countable abelian groups is countable, as can be seen by enumerating $\bigoplus_{i=0}^{\infty} \mathbb{N}$ (e.g. by order of increasing $\sum_{i=0}^{n} n_i$). $\ni (n_i)_{i=0}^{\infty}$

Hence the quotient $\varinjlim_\alpha A_\alpha$ is also countable.

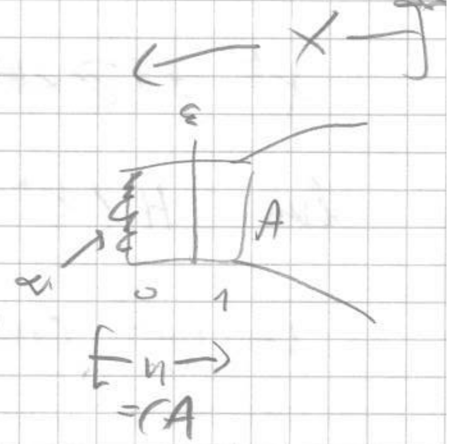
If $X \subset \mathbb{R}^n$ is open, then $X = \bigcup_{i=0}^{\infty} B(x_i; \varepsilon_i)$

where x_i ranges over the countable set $X \cap \mathbb{Q}^n$ and ε_i ranges over the rational numbers ~~with~~ ε with $B(x_i; \varepsilon) \subseteq X$.

(If $x \in X$ then $B(x, \varepsilon) \subset X$ for some $\varepsilon > 0$. Choose $x_i \in B(x, \varepsilon/2) \cap \mathbb{Q}^n$, and let $d(x, x_i) < \varepsilon_i < \varepsilon/2$.)

Let $Y_n = \bigcup_{i=0}^{n} B(x_i; \varepsilon_i)$. Then $H_* Y_n$ is isomorphic to H_* of the subcomplex of Δ^n where a set $\sigma = (i_0, \dots, i_q)$ is a face (for $0 \le i_0 < \dots < i_q \le n$) if and only if $B(x_{i_0}, \varepsilon_{i_0}) \cap \dots \cap B(x_{i_q}, \varepsilon_{i_q}) \neq \emptyset$ (in which case it is convex, hence contractible).

Hence $H_* Y_n$ is finitely generated and countable in each degree. Finally $X = \operatorname{colim}_n Y_n$ and $H_* X = \varinjlim_n H_* Y_n$ is countable in each degree.



Ex. 2) If $\infty \in U \subseteq X^+ = X \cup \{\infty\}$
with $U \cong CA = A \times [0,1] / (A \times \{0\} = \infty)$

each compact $K \subset X$ lies in the complement of $U_\varepsilon = A \times [0,\varepsilon) / A \times \{0\} \subset X^+$ for some $0 < \varepsilon < 1$, so $X^+ - K$ contains some such U_ε.

Hence

$$H^*_c(X;G) = \varinjlim_{K} H^*(X, X-K; G)$$

$$\cong \varinjlim_{\varepsilon} H^*(X, U_\varepsilon \setminus \{\infty\}; G)$$

and $X^+ = X \cup U_\varepsilon$ with $X \cap U_\varepsilon = U_\varepsilon \setminus \{\infty\}$

$$\cong \uparrow \text{ excision}$$

$$\varinjlim_{\varepsilon} H^*(X^+, U_\varepsilon; G)$$

$$\cong \downarrow \ (U_\varepsilon \simeq *)$$

$$\varinjlim_{\varepsilon} H^*(X^+, \{\infty\}; G)$$

$$= \underline{\underline{H^*(X^+, \{\infty\}; G)}}$$

If $X = \mathbb{Z} \times \mathbb{R}$,
$K = \{-n, \dots, n\} \times [-n, n]$ is a cofinal family of compact subspaces,
and

$$H^1(X, X-K) \cong \bigoplus_{i=-n}^{n} H^1(\mathbb{R}, \mathbb{R} \setminus [-n,n]) = \bigoplus_{i=-n}^{n} \mathbb{Z}$$

so $H^1_c(X) = \bigoplus_{i=-\infty}^{\infty} \mathbb{Z}$, while

$X^+ = (\mathbb{Z}\times\mathbb{R}) \cup \{\infty\} \cong \bigvee^{\infty} S^1$

has $H^1(X^+) = \prod^{\infty} H^1(S^1) = \prod^{\infty} \mathbb{Z}$

is not countable. Answer: <u>No</u>.

A continuous bijection of compact Hausdorff spaces is a homeomorphism.

<u>Ex 24</u>

If M is orientable, $H_2(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z})$

$\cong \mathrm{Hom}(H_1(M),\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}^r \oplus F, \mathbb{Z}) \cong \mathbb{Z}^r$

by Poincaré duality and UCT.

If M is not orientable, $H^2(M;\mathbb{Z}/2) \cong$

$\mathrm{Ext}^1(H_1(M),\mathbb{Z}/2) \oplus \mathrm{Hom}(H_2(M),\mathbb{Z}/2)$

$\cong \mathrm{Ext}^1(F,\mathbb{Z}/2) \oplus \mathrm{Hom}(H_2(M),\mathbb{Z}/2)$

is isomorphic to $H_1(M;\mathbb{Z}/2) \cong H_1(M)\otimes\mathbb{Z}/2$

$\cong (\mathbb{Z}/2)^r \oplus (F\otimes\mathbb{Z}/2)$.

Here $\#\mathrm{Ext}^1(F,\mathbb{Z}/2) = \# F\otimes\mathbb{Z}/2$, so

$\mathrm{Hom}(H_2M,\mathbb{Z}/2) \cong (\mathbb{Z}/2)^r$.

Take as granted that H_2M is finitely generated.

By Corollary 3.28 tors $H_2M = \mathbb{Z}/2$. Thus

$H_2M = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2$.

For orientable M, we have

$$H_1(S^1 \times S^2) = \mathbb{Z}$$

and

$$H_1(\underset{\underset{L^3(m)}{\shortparallel}}{S^3/\mathbb{Z}/m}) = \mathbb{Z}/m$$

Taking connected sums, any $H_1 M = \mathbb{Z}^r \oplus F$ can be realized (Use Ex. 6)

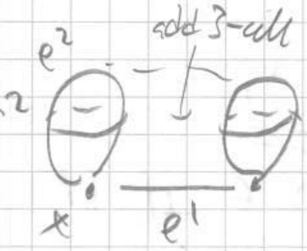
For non-orientable M, start with

$$N = S^2 \times I / (x,0) \sim (\rho x, 0)$$

where $\rho : S^2 \to S^2$ is a reflection (of degree -1).

$C_* N$:

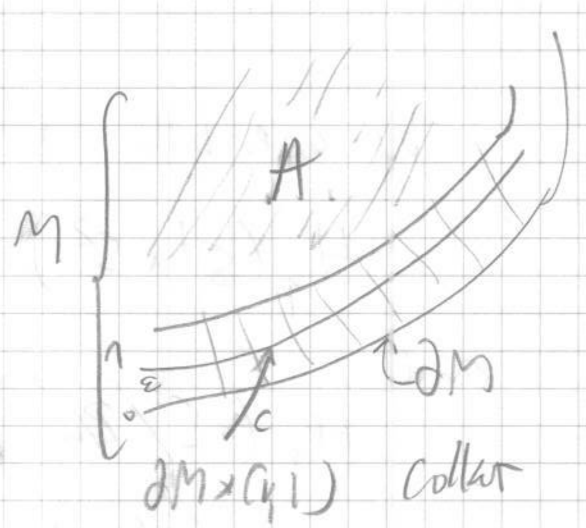
$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$



$H_* N = (0, \mathbb{Z}/2, \mathbb{Z}, \mathbb{Z})$.

This realizes $H_1 N = \mathbb{Z}$. Add $\mathbb{Z}^{r-1}$ and F using the orientable examples above and connected sums. (Use Ex 6)

Ex 30 $\quad x \in \partial M \subset M^n$



Choose a collar neighborhood

$$\begin{tikzcd}
U = \partial M \times [0,1) \arrow[r, hook, "c"] & M & \supset A \\
\partial M \times \{0\} \arrow[u, phantom, "\cup" description] \arrow[r, "\cong"] & \partial M \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

The R-orientation of $M \setminus \partial M$ restricts to an R-orientation of $U \cong \partial M \times (0,1)$. This makes ∂M R-orientable.

Can arrange that $O_x^{\partial M} \times O_t^{(0,1)} = O^U_{(x,t)} \leftarrow O^M_{c(x,t)}$

for $x \in \partial M$, $t \in (0,1)$

$$A = M \setminus c(\partial M \times [0,1))$$

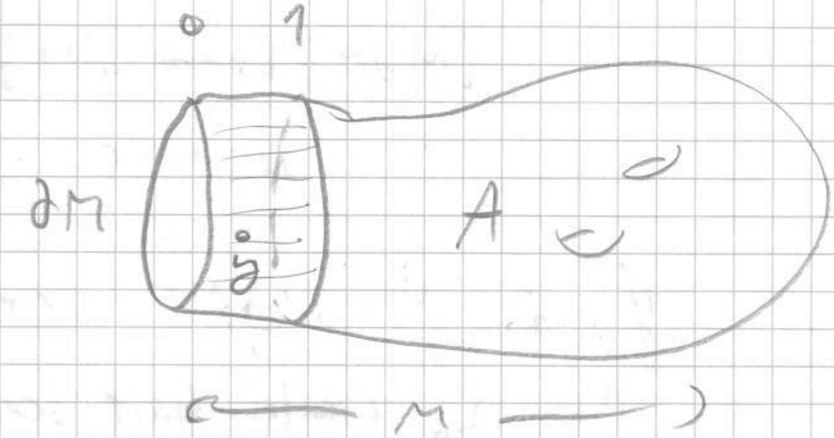
Ex 31

$$\begin{tikzcd}
 & {[M,\partial M]} \in H_n(M,\partial M) \arrow[r, "\partial"] \arrow[d] \arrow[dl, "\text{gen.}"'] & H_{n-1}(\partial M) \arrow[d] \\
H_n(M, M\setminus y) & H_n(M, \partial M \sqcup A) \arrow[l, "\rho_{M,y}"'] \arrow[r, "\partial"] & H_{n-1}(\partial M \sqcup A) \\
H_n(\partial M\times[0,1], \partial M\times[0,1]\setminus(x,t)) & H_n(\partial M\times[0,1], \partial M\times\{0,1\}) \arrow[u, "\text{exc.}", "\cong"'] \arrow[l, "\rho_{\partial M\times I,(x,t)}"'] \arrow[r, "\partial"] & H_{n-1}(\partial M\times\{0,1\}) \arrow[u] \\
 & H_{n-1}(\partial M)\times H_1([0,1],\{0,1\}) \arrow[u, "\times", "\cong"'] & {[\partial M]\times([1]-[0])} \\
 & {[\partial M]\times g} &
\end{tikzcd}$$

$$\begin{array}{ccc}
g \in H_1([0,1],\{0,1\}) & \xrightarrow{\partial} & H_0(\{0,1\}) \\
\text{generator} \quad 1 & \longmapsto & [1]-[0]
\end{array}$$

Ex. 31

$$\begin{array}{ccccc}
\partial M & \subset & M & \supset & A = M \setminus \partial M \times [0,1) \\
\cong\uparrow & & \cup & \ni^{\partial} (x,t) & \cup \\
\partial M \times \{0\} & \subset & \partial M \times [0,1] & \supset & \partial M \times \{1\}
\end{array}$$



$$\begin{tikzcd}
 & {[M,\partial M]} \arrow[ddl, bend right, mapsto] & \\
 & \cap & \\
 & H_n(M,\partial M) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(\partial M) \arrow[d, tail] \\
\text{gen.} \in H_n(M, M \setminus y) & H_n(M, \partial M \sqcup A) \arrow[l] \arrow[r, "\partial"] & H_{n-1}(\partial M \sqcup A) \\
H_n(\partial M\times[0,1], \partial M\times[0,1]-(x,t)) \ni \text{gen.} \arrow[u, "\cong"] & H_n(\partial M\times[0,1], \partial M \times\{0,1\}) \arrow[l] \arrow[u, "\cong\ \text{exc.}"'] \arrow[r, "\partial"] & H_{n-1}(\partial M\times\{0,1\}) \arrow[u] \\
 & H_{n-1}\partial M \times H_1([0,1],\{0,1\}) \arrow[u, "\cong\ \times"'] \arrow[r, "\partial"] & H_{n-1} \times H_0(\{0,1\}) \arrow[u, "\cong\ \times"'] \\
 & {[\partial M] \otimes g} \arrow[uul, bend left, mapsto] \arrow[r, mapsto] & {[\partial M] \otimes ([1]-[0])}
\end{tikzcd}$$

The image of $\partial[M,\partial M] \in H_{n-1}(\partial M)$ in $H_{n-1}(\partial M \sqcup A)$ equals the image of $\partial([\partial M]\times g) = [\partial M]\times([1]-[0])$ in the same group, so $\partial[M,\partial M] = [\partial M]$ in $H_{n-1}(\partial M)$

(and the image of this class in $H_{n-1}(A)$ is zero).

Ex 32

$$\to H_n(M,\partial M) \xrightarrow{\partial} H_{n-1}(\partial M;\mathbb{Z}/2) \xrightarrow{i_*} H_{n-1}(M;\mathbb{Z}/2)$$

$;\mathbb{Z}/2$

$H_n(M,\partial M) = \mathbb{Z}/2$, $H_{n-1}(\partial M;\mathbb{Z}/2) = \mathbb{Z}/2$

$$[M,\partial M] \longmapsto [\partial M]$$

surjective.

i_* zero, hence ~~not~~ split injective.

If $r: M \to \partial M$ were a retraction, then i_* would have been split injective.

Ex 33

M^n compact, contractible hence orientable

$$\tilde{H}_i(\partial M;\mathbb{Z}) \underset{\cong}{\xleftarrow{\partial}} H_{i+1}(M,\partial M;\mathbb{Z}) \quad \text{LES}$$

$$\cong H^{n-i-1}(M;\mathbb{Z}) \quad \text{Poincaré-Lefschetz duality}$$

$$\cong \begin{cases} \mathbb{Z} & n-i-1=0 \\ 0 & \text{else} \end{cases}$$

$$\cong \tilde{H}_i(S^{n-1}).$$