STARTING QUESTIONS FOR MAT4540/MAT9540 FALL 2016

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Problem 1

(a) State the Poincaré duality theorem for closed, connected, oriented n-manifolds.

Let $M = (S^1 \times S^2)\#(S^1 \times S^2)$ be the connected sum of two copies of the product $S^1 \times S^2$.

(b) Explain why M is a closed, connected 3-manifold. Is M orientable?

(c) Determine the homology groups $H_i(M)$ and the cohomology groups $H^i(M)$ for $0 \le i \le 3$. What can you say about the cup product pairings $H^1(M) \times H^1(M) \to H^2(M)$ and $H^1(M) \times H^2(M) \to H^3(M)$?

Problem 2

Consider

$$G(\mathbb{R}) = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, x \in \mathbb{R} \right\}$$

as a topological group with respect to matrix multiplication, let $G(\mathbb{Z}) \subset G(\mathbb{R})$ be the discrete subgroup where $x, y, z \in \mathbb{Z}$ are integers, and let $M = G(\mathbb{R})/G(\mathbb{Z})$ be the space of left cosets.

(a) Explain why M is a closed, connected 3-manifold. Is M orientable?

(b) Determine the fundamental group $\pi_1(M)$. Is M aspherical?

(c) Determine the homology groups $H_i(M)$ and the cohomology groups $H^i(M)$ for $0 \le i \le 3$. What can you say about the cup product pairings $H^1(M) \times H^1(M) \to H^2(M)$ and $H^1(M) \times H^2(M) \to H^3(M)$?

Problem 3

(a) State the Poincaré duality theorem for closed, connected, oriented n-manifolds.

Let M and N be closed, connected and oriented n-manifolds, with fundamental classes $[M] \in H_n(M)$ and $[N] \in H_n(N)$. Let $f \colon M \to N$ be a map of degree 1, meaning that $f_*([M]) = [N]$.

(b) Show that $[N] \cap \varphi = f_*([M] \cap f^*(\varphi))$, for each $\varphi \in H^*(N)$.

(c) Prove that $f_* \colon H_*(M) \to H_*(N)$ is surjective and $f^* \colon H^*(N) \to H^*(M)$ is injective, in each degree.