MANDATORY ASSIGNMENT FOR MAT9540 FALL 2017

JOHN ROGNES

Give a 30 to 45 minute presentation about either Problem 1 or Problem 3, e.g. on Wednesday November 8th 2017. (There is no Problem 2.)

Problem 1

For abelian groups A and B let $\mathrm{Ext}(A, B) = \mathrm{Ext}^1_{\mathbb{Z}}(A, B)$.

(a) If A is free, show that $\mathrm{Ext}(A, B) = 0$ for any B.

(b) If A is finitely generated, and $\mathrm{Ext}(A, \mathbb{Z}) = 0$, show that A is free.

(c) For a general abelian group A, show that if $\mathrm{Ext}(A, B) = 0$ for each B then A is free. Hint: Consider a free resolution of A, and use this to choose a suitable B.

(d) The Whitehead problem asks: "Is every abelian group A with $\mathrm{Ext}(A, \mathbb{Z}) = 0$ a free abelian group?" Find information about the status of this problem, including references to the literature.

Problem 3

Let $T^2 = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$ be the torus surface. Take as known that $H^*(T^2) = \Lambda_{\mathbb{Z}}(x, y)$.

(a) Show that it is impossible to cover T^2 with only two coordinate charts U_1 and U_2. Here we assume that the U_i are open subsets of T^2, each homeomorphic to $\mathbb{R}^2$, with $U_1 \cup U_2 = T^2$.

(b) Find three coordinate charts U_1, U_2 and U_3 that cover T^2. Hint: Let U_1 be the homeomorphic image of $(0, 1)^2 \subset \mathbb{R}^2$, and give similar descriptions of U_2 and U_3.

(c) Let M_g be a closed, connected, orientable surface of genus $g \geq 2$. What is the minimal number of coordinate charts needed to cover M_g?

(d) The Lusternik–Schnirelmann category $\mathrm{cat}(X)$ of a space X is the minimal integer k for which X can be covered by k open subsets $U_1, \ldots, U_k$ such that each inclusion $U_i \to X$ is null-homotopic. (Some authors use a different normalization.) The Ganea conjecture states that $\mathrm{cat}(X \times S^n) = \mathrm{cat}(X) + 1$ for each space X and any $n \geq 1$. Find information about the status of this problem, including references to the literature.