MANDATORY ASSIGNMENT FOR MAT4540 FALL 2017

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Return to John Rognes by Thursday November 9th 2017. Each of the nine problem parts carry equal weight. A score over 50% is sufficient to pass. A near-pass may qualify for a second attempt. You may cooperate with other students, but your answers should reflect your own understanding.

Problem 1

For abelian groups A and B let $\mathrm{Ext}(A,B) = \mathrm{Ext}^1_{\mathbb{Z}}(A,B)$.

(a) If A is free, show that $\mathrm{Ext}(A,B) = 0$ for any B.

(b) If A is finitely generated, and $\mathrm{Ext}(A,\mathbb{Z}) = 0$, show that A is free.

(c) Let A be a general abelian group. Show that if $\mathrm{Ext}(A,B) = 0$ for each B, then A is free. Hint: Consider a free resolution of A, and use this to choose a suitable B.

Problem 2

Consider an analog clock with an hour hand and a minute hand, pointing at points h and m on the perimeter, which we identify with the circle S^1. The pair of hands thus specifies a point $(h,m) \in S^1 \times S^1 = T^2$. Let $[a] \in H_1(T^2)$ be the homology class of the cycle representing a simple closed loop by the hour hand, in the clockwise direction, keeping the minute hand fixed. Similarly, let $[b] \in H_1(T^2)$ be the class representing a simple closed loop by the minute hand, keeping the hour hand fixed. Let x and $y \in H^1(T^2)$ be dual to $[a]$ and $[b]$. Take as known that $H^*(T^2) = \Lambda_{\mathbb{Z}}(x,y)$, with $x \cup y = z$ generating $H^2(T^2)$.

(a) Let $\Delta \subset T^2$ be the simple closed loop described by letting the hour and minute hands move once around the clock face, always overlapping. Let $E \subset T^2$ be the simple closed loop described by regular motion of the hour and minute hands, showing time from 6 a.m. to 6 p.m. Express the homology classes $[\Delta]$ and $[E]$ of these cycles as linear combinations of $[a]$ and $[b]$.

(b) Poincaré duality for T^2 gives an isomorphism $D\colon H^1(T^2) \to H_1(T^2)$, mapping x and y to $D(x) = [b]$ and $D(y) = -[a]$, respectively. Find the cohomology classes δ and $\epsilon \in H^1(T^2)$ that are Poincaré dual to $[\Delta]$ and $[E]$, respectively, and calculate the cup product $\delta \cup \epsilon$.

(c) Poincaré duality also gives an isomorphism $D\colon H^2(T^2) \to H_0(T^2)$, mapping z to the homology class of a point. Calculate the Poincaré dual of $\delta \cup \epsilon$. Take as known that this class in $H_0(T^2)$ is the class of the intersection $\Delta \cap E$, interpreted as a 0-chain in T^2:

$$[\Delta \cap E] = D(D^{-1}([\Delta]) \cup D^{-1}([E]))$$

What does your answer for $D(\delta \cup \epsilon)$ say about the motion of the clock hands in the time from 6 a.m. to 6 p.m.?

Problem 3

Let $T^2 = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$ be the torus surface. Take as known that $H^*(T^2) = \Lambda_{\mathbb{Z}}(x,y)$, as in Problem 2.

(a) Show that it is impossible to cover T^2 with only two coordinate charts U_1 and U_2. Here we assume that the U_i are open subsets of T^2, each homeomorphic to $\mathbb{R}^2$, with $U_1 \cup U_2 = T^2$.

(b) Find three coordinate charts U_1, U_2 and U_3 that cover T^2. Hint: Let U_1 be the homeomorphic image of $(0,1)^2 \subset \mathbb{R}^2$, and give similar descriptions of U_2 and U_3.

(c) Let M_g be a closed, connected, orientable surface of genus $g \geq 2$. What is the minimal number of coordinate charts needed to cover M_g?