MAT4540/9540: EXERCISES 3

1. Cohomology, Ext and the universal coefficient theorem

Exercise 1. Show that $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m, \mathbf{Z}) \cong \mathbf{Z}/m$, and, more generally, $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m, B) \cong B/mB$.

Exercise 2. $\mathrm{Ext}^n_{\mathbf{Z}}(\bigoplus_\alpha A_\alpha, B) \cong \prod_\alpha \mathrm{Ext}^n_{\mathbf{Z}}(A_\alpha, B)$ and $\mathrm{Ext}^n_{\mathbf{Z}}(A, \prod_\beta B_\beta) \cong \prod_\beta \mathrm{Ext}^n_{\mathbf{Z}}(A, B_\beta)$.

Exercise 3. If F is a free abelian group, prove that $\mathrm{Ext}^1_{\mathbf{Z}}(F, A) = 0$ for any A.

Exercise 4. Let A and B be abelian groups. Recall that an extension ξ of A by B is an exact sequence $0 \to B \to E \to A \to 0$. Two extensions ξ, ξ' are equivalent if there is an isomorphism $E \cong E'$ which fits in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \big\downarrow{=} & & \big\downarrow{\cong} & & \big\downarrow{=} & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

(a) Show that if $\mathrm{Ext}^1(A, B) = 0$, then every extension $0 \to B \to E \to A \to 0$ of A by B is split.
(b) Show that there is a bijection between $\mathrm{Ext}^1(A, B)$ and equivalence classes of extensions of A by B.

From Hatcher:

- Chapter 3.1: 3.1.2, 3.1.3, 3.1.6, 3.1.11 (a).
- Chapter 3.2: 3.2.1, 3.2.4.

2. Spectral sequences

Exercise 5. Compute the cohomology ring $H^*(\Omega S^3)$.

From the notes:

- 2.13.2, 2.13.7.