1.14.11. Being universal covers, $\tilde{X}$ and $\tilde{Y}$ are simply connected and so $\pi_1(\tilde{X}) = \pi_1(\tilde{Y}) = 0$. Since $\tilde{f}_* : H_n(\tilde{X}) \xrightarrow{\cong} H_n(\tilde{Y})$ is an isomorphism $\forall n$ by assumption, Corollary 1.7.3 yields that $\tilde{f}$ is a homotopy equivalence.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}", "\simeq"'] \arrow[d, "p"'] & \tilde{Y} \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Hence $q_* \circ \tilde{f}_* : \pi_n(\tilde{X}) \longrightarrow \pi_n(Y)$ is an isomorphism for $n \geq 2$.

Since $q_* \circ \tilde{f}_* = (q \circ \tilde{f})_* = (f \circ p)_* = f_* \circ p_*$,

it follows that f_* is an iso on $\pi_{\geq 2}$ as well.

Since f_* is iso on π_1 by assumption, we see that f_* is iso on π_n $\forall n$. Hence Whitehead's theorem yields that f is a homotopy equivalence.

(2)

1.14.14. $S^k \to S^m \xrightarrow{p} S^n$ being a fiber bundle, we can find an open $U \subseteq S^n$ such that

$$\begin{array}{c} p^{-1}(U) \cong U \times S^k \\ \searrow \quad \swarrow \\ U \end{array}$$

Hence $m = n+k$. Thus it is enough to show that $k = n-1$.

We may assume $n > 1$. Moreover, we know that $k > 0$. For if $k = 0$ then S^n would be a 2-sheeted covering of itself, which is impossible since S^n is simply connected for $n > 1$.

Consider the L.E.S.

$$\cdots \to \pi_n S^k \to \underset{\substack{\parallel \\ 0}}{\pi_n S^m} \to \underset{\substack{\parallel \\ \mathbb{Z}}}{\pi_n S^n} \xrightarrow{\partial} \pi_{n-1} S^k \to \cdots$$

Since $n < m$ we have $\pi_n S^m = 0$, hence the map $\partial : \pi_n S^n = \mathbb{Z} \to \pi_{n-1} S^k$ is injective. In particular, $\pi_{n-1} S^k \neq 0$, so $n-1 \geq k$.

Similarly, consider

$$\cdots \to \pi_{k+1} S^n \xrightarrow{\partial} \underset{\substack{\parallel \\ \mathbb{Z}}}{\pi_k S^k} \to \underset{\substack{\parallel \\ 0}}{\pi_k S^m} \to \cdots$$

Since $k < m$, $\pi_k S^m = 0$ and thus the map $\partial : \pi_{k+1}(S^n) \to \pi_k(S^k) = \mathbb{Z}$ is surjective. Hence $k+1 \geq n$.

We conclude that $k = n-1$.

4.2.13. Let $f: X \to Y$ be a map of n-dimensional CW complexes, and let $\tilde{f}: \tilde{X} \to \tilde{Y}$ be a lift of f to the universal cover of X, Y.
Since $\pi_1(\tilde{X}) = \pi_1(Y) = 0$ and $\pi_i(\tilde{X}) \cong \pi_i(X)$, $\pi_i(\tilde{Y}) \cong \pi_i(Y)$, $\tilde{f}$ induces an iso. on π_i for $i \leq n$.

By replacing $\tilde{Y}$ w/ the mapping cylinder $M_{\tilde{f}}$, we may assume that $\tilde{X}$ is a subcomplex of $\tilde{Y}$.
Then $(\tilde{Y}, \tilde{X})$ is n-connected (since $\pi_i(\tilde{X}) \cong \pi_i(\tilde{Y})$ for $i \leq n$). Hence $H_i(\tilde{Y}, \tilde{X}) = 0$ for $i \leq n$ by Hurewicz. But then the long exact homology sequence gives $H_i(\tilde{X}) \cong H_i(\tilde{Y})$ for $i \leq n$.
Since $\tilde{X}, \tilde{Y}$ are n-connected, $H_i(\tilde{X}) = H_i(\tilde{Y}) = 0$ for $i > n$. Hence $H_i(\tilde{X}) \cong H_i(\tilde{Y})$ $\forall n$.
By Exercise 1.14.11, f is a homotopy equivalence

(4)

4.2.30.
$$\begin{tikzcd} E \subseteq \mathbb{R}^2 \arrow[d, "p"'] \arrow[dl, "pr_1"] \\ \mathbb{R} \end{tikzcd}$$

Consider the fiber $p^{-1}(0)$ over $0 \in \mathbb{R}$. Then $p^{-1}(0) \subseteq \{0\} \times \mathbb{R}$ must be homeomorphic to $\mathbb{R}$. Hence the possibilities are:

$$p^{-1}(0) = (\{0\} \times \mathbb{R}) \setminus \begin{cases} \emptyset & (i) \\ (-\infty, a] & (ii) \\ [b, \infty) & (iii) \\ (-\infty, a] \cup [b, \infty),\ a < b & (iv) \end{cases}$$

Here the case (i) corresponds to the trivial bundle, i.e., $E = \mathbb{R}^2 \xrightarrow{p = pr_1} \mathbb{R}$.

In fact, all the remaining cases (ii) – (iv) define fiber bundles as well.

We illustrate how to show this in the case where $E = \mathbb{R}^2 \setminus (\{0\} \times (-\infty, 0])$.

Indeed, we will define a homeomorphism $\varphi: \mathbb{R} \times (0, \infty) \xrightarrow{\cong} E = \mathbb{R}^2 \setminus (\{0\} \times (-\infty, 0])$ such that the diagram

$$\begin{tikzcd} \mathbb{R} \times (0,\infty) \arrow[r, "\varphi", "\cong"'] \arrow[dr, "pr_1"'] & E \arrow[d, "p"] \\ & \mathbb{R} \end{tikzcd}$$

commutes.

Let $\varphi(x,y) := \left(x, y^2 - \left(\frac{x}{2y}\right)^2\right)$, then φ is a homeomorphism with inverse $\psi(x,z) = \left(x, \sqrt{\frac{1}{2}\left(z + \sqrt{x^2+z^2}\right)}\right)$.

4.2.31. By assumption, the map $\pi_n(F) \to \pi_n(E)$ is trivial for $n > 0$. Hence we get a short exact sequence

$$0 \to \pi_n(E) \to \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to 0 \quad \forall\, n > 1.$$

We must construct a splitting $s: \pi_{n-1}(F) \to \pi_n(B)$ (i.e., s.t. $\partial \circ s = id$).

Let $H: F \times I \to E$ be a homotopy from the inclusion $F \hookrightarrow E$ to the constant map c_e.

Consider a class $[f] \in \pi_{n-1}(F)$, represented by a map $f: S^{n-1} \to F$.

We extend f to a map $\tilde{f}: D^n \to E$ by letting

$$\tilde{f}(v) = \begin{cases} H\left(f\left(\frac{v}{|v|}\right), |v|\right), & v \neq 0 \\ e, & v = 0 \end{cases} \qquad (e = \text{base pt. of } E)$$

We then define $s([f]) := [\tilde{f}] \in \pi_n(E, F) = \pi_n(B)$.

Then $s([f])$ does not depend on the homotopy class of f, and moreover $\partial s([f]) = \partial([\tilde{f}])$ is given as

$v \longmapsto H(f(v), 1) = f(v)$, $v \in \partial D^n = S^{n-1}$.

So $\partial \circ s = id$, and hence the exact sequence splits.