MAT4540
ALGEBRAIC TOPOLOGY II
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JOHN ROGNES

Contents

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Part 1. **Introduction**

1. August 24th lecture

- Geometry (manifolds)
- Topology (spaces)
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    - Stable homotopy theory (spectra)
- Algebra (groups, rings, modules)
  - Homological algebra (chain complexes)

1.1. **Functors from topology to algebra.**

The category Top consists of spaces X and maps $f\colon X \to Y$, etc. There is also a category Top_* of based spaces (X, x_0) and based (= basepoint-preserving) maps $f\colon (X, x_0) \to (Y, y_0)$, etc.

To these we can associate the fundamental groups $\pi_1(X, x_0)$ and induced homomorphisms $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ in the category Gp of groups. We can also associated the homology groups $H_n(X)$ and induced homomorphisms $f_*\colon H_n(X) \to H_n(Y)$ in the category Ab of abelian groups. Letting the degree n vary, we can collect these into one object $H_*(X) = (H_n(X))_n$ and one homomorphism $f_*\colon H_*(X) \to H_*(Y)$ in the category grAb of graded abelian groups.

1.2. **Kinds of spaces.** There are useful variations of the notion of a space, other than that of a set with a topology.

$$\begin{tikzcd}
\text{simplicial complexes} \arrow[d] & & \\
\Delta\text{-complexes} \arrow[d] & & \\
\text{CW complexes} \arrow[r] & \mathrm{Top} \arrow[r, "\mathrm{sing}"] & \text{simplicial sets} \arrow[ll, bend right, "|\ |"']
\end{tikzcd}$$

Simplicial complexes (= triangulations of polyhedra) are classical. CW complexes (closure finite, weak topology) were introduced by J.H.C. (Henry) Whitehead in the 1930s (published in 1949). Δ-complexes and simplicial sets were introduced by Samuel Eilenberg and Joseph Zilber in (1950), under the names 'semi-simplicial complexes' and 'complete semi-simplicial complexes', respectively.

1.3. **Singular homology.** The definition of homology goes back to Henri Poincaré in the 1890s. Their interpretation as groups is credited to Emmy Noether, around 1925. The modern definition of singular homology groups is due to Eilenberg (1944), and can be factored in three steps:

$$X \longmapsto \mathrm{sing}_\bullet(X) \longmapsto (C_*(X), \partial) \longmapsto H_*(X)\,.$$

Here $\mathrm{sing}_\bullet(X)$ is the simplicial set of singular simplices in X, given in degree n by the set

$$\mathrm{sing}_n(X) = \{\sigma \colon \Delta^n \to X\}$$

of continuous maps from the standard n-simplex. Composition with the maps $\Delta^m \to \Delta^n$ indexed by order-preserving functions $[m] \to [n] = \{0 < 1 < \cdots < n\}$ induce operators $\mathrm{sing}_n(X) \to \mathrm{sing}_m(X)$, making $\mathrm{sing}_\bullet(X)$ a simplicial set.

The passage to the singular chain complex $(C_*(X), \partial)$ involves linearization, letting

$$C_n(X) = \mathbb{Z}\{\mathrm{sing}_n(X)\}$$

be the free abelian group generated by the set $\mathrm{sing}_n(X)$. This leads to convenient algebra, but sometimes loses information. The boundary homomorphism $\partial_n \colon C_n(X) \to C_{n-1}(X)$ is defined as the alternating sum of the face operators d_i corresponding to the various order-preserving inclusions $[n-1] \to [n]$, for $0 \le i \le n$.

Finally, we pass to homology by forming the quotient group

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}$$

where the n-cycles $Z_n(X) = \ker(\partial_n)$ contain the n-boundaries $B_n(X) = \mathrm{im}(\partial_{n+1})$ because $\partial_n \partial_{n+1} = 0$.

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \dots$$

1.4. **Cellular homology.** Suppose now that X is a CW complex, meaning that it comes with a skeleton filtration

$$\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^{n-1} \subset X^n \subset \dots X$$

such that X^n is obtained from X^{n-1} by attaching a set of n-cells

$$\begin{tikzcd}
\coprod_\alpha \partial D^n_\alpha \arrow[r] \arrow[d, "\coprod_\alpha \phi_\alpha = \phi"'] & \coprod_\alpha D^n_\alpha \arrow[d, "\Phi = \coprod_\alpha \Phi_\alpha"] \\
X^{n-1} \arrow[r] & X^n
\end{tikzcd}$$

and $X = \bigcup_n X^n$ has the weak topology. Then $H_*(X)$ can be calculated as the homology of a smaller complex, variously denoted $W_*(X) = C^{CW}_*(X) = \mathrm{Cell}_*(X)$. Here

$$W_n(X) = H_n(X^n, X^{n-1}) \cong \bigoplus_\alpha \mathbb{Z} \cong \mathbb{Z}\{n\text{-cells of } X\}$$

and

$$\partial_n \colon W_n(X) = \bigoplus_\alpha \mathbb{Z} \longrightarrow \bigoplus_\beta = W_{n-1}(X)$$

is given by a matrix $(d_{\beta,\alpha})$ of degrees of maps

$$S^{n-1} = \partial D^n_\alpha \xrightarrow{\phi_\alpha} X^{n-1} \longrightarrow X^{n-1}/X^{n-2} \cong \bigvee_\beta D^{n-1}_\beta/\partial D^{n-1}_\beta \longrightarrow D^{n-1}_\beta/\partial D^{n-1}_\beta \cong S^{n-1}\,.$$

There is a topological realization functor from simplicial sets back to CW complexes, with $W_*(|\operatorname{sing}_\bullet(X)|) \cong C_*(X)$, showing that $W_*(X)$ and $C_*(X)$ have the same homology groups. If X is a Δ-complex, the degree formula simplifies drastically to

$$\partial_n = \sum_{i=0}^n (-1)^i d_i\,.$$

1.5. **The orientation class.** The homology groups often contain geometric meaning, e.g., when the space X is a manifold. Consider the Lie group $X = SO(3)$ of rotations of Euclidean 3-space. It is a manifold and a topological group. There are homeomorphisms

$$SO(3) \cong \mathbb{R}P^3 \cong D^3/{\sim}$$

where $x \sim a(x) = -x$ for $x \in \partial D^3$. We map $x \in D^3$ to rotation by $\pi\|x\|$ radians around the axis $\mathbb{R}x \subset \mathbb{R}^3$, oriented in the direction of x. When $x = 0$, this is the identity rotation. Note that when $\|x\| = 1$, so that $x \in \partial D^3$, rotation by π around x equals rotation by π around $a(x) = -x$. The filtration

$$\emptyset \subset \mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \mathbb{R}P^3$$

makes $\mathbb{R}P^3$ a CW complex with one k-cell for each $0 \le k \le 3$. Hence the cellular chain complex $(W_*(\mathbb{R}P^3), \partial)$ has the form

$$0 \longleftarrow \mathbb{Z}\{\alpha_0\} \xleftarrow{0} \mathbb{Z}\{\alpha_1\} \xleftarrow{2} \mathbb{Z}\{\alpha_2\} \xleftarrow{0} \mathbb{Z}\{\alpha_3\} \longleftarrow 0 \longleftarrow \dots$$

with $\partial_1(\alpha_1) = 0$, $\partial_2(\alpha_2) = 2\alpha_1$ and $\partial_3(\alpha_3) = 0$. It follows that

$$H_*(SO(3)) \cong H_*(\mathbb{R}P^3) \cong (\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}, 0, \dots)\,.$$

With coefficients in $\mathbb{Z}/2$, we get

$$H_*(SO(3); \mathbb{Z}/2) \cong (\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, 0, \dots)$$

with $H_k(SO(3); \mathbb{Z}/2)$ generated by the class of α_k for each $0 \le k \le 3$.

For each point $x \in SO(3)$ we can map to the local homology at x:

$$H_3(SO(3)) \longrightarrow H_3(SO(3), SO(3) - \{x\}) \cong H_3(\mathbb{R}^3, \mathbb{R}^3 - \{0\}) \cong \mathbb{Z}$$

and $[\alpha_3]$ maps to a generator of $\mathbb{Z}$, for each x. This tells us that the generator $[\alpha_3]$ of $H_3(SO(3))$ is an orientation class of the (closed, oriented) 3-manifold $SO(3)$. We will show that for each closed, oriented n-manifold M there is such a generator $[M] \in H_n(M) \cong \mathbb{Z}$, whose restriction to $H_n(M, M - \{x\}) \cong \mathbb{Z}$ gives the local orientation at each point $x \in M$.

1.6. **Colimits.** The categories Top, Ch (of chain complexes) and grAb have some formal similarities. For instance, they all have all colimits and limits. Examples of colimits in topological spaces are the coproducts (= sums) $X \sqcup Y$ and $\coprod_i X_i$, and the pushouts

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & B \cup_A C \end{tikzcd}$$

where $B \cup_A C = (B \sqcup C)/{\sim}$. Here $f(a) \sim g(a)$ for all $a \in A$, where we interpret $f(a) \in B$ and $g(a) \in C$ as lying in $B \sqcup C$. Another example is the coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r] & \operatorname{coeq}(f, g) \end{tikzcd}$$

where $\operatorname{coeq}(f,g) = B/{\sim}$ with $f(a) \sim g(a)$ for $a \in A$. The pushout can be written as a coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "\iota_B f"] \arrow[r, shift right, "\iota_C g"'] & B \sqcup C \arrow[r] & B \cup_A C \end{tikzcd}$$

and the coequalizer can be written as a pushout

$$\begin{tikzcd} A \sqcup B \arrow[r, "f \sqcup 1"] \arrow[d, "g \sqcup 1"'] & B \arrow[d] \\ B \arrow[r] & \operatorname{coeq}(f,g)\,. \end{tikzcd}$$

A general diagram of spaces can be indexed on a (small) category I as a functor $F\colon I \to \mathrm{Top}$. For each object $i \in I$ we have a space $X_i = F(i)$ at the i-th vertex in the diagram, and for each morphism $\phi\colon i \to j$ in I we have a map $\phi_* = F(\phi)\colon X_i \to X_j$, giving an edge in the diagram from the i-th to the j-th vertex. A colimit of this diagram is then an initial space Y with structure maps $\iota_i\colon X_i \to Y$ for all $i \in I$ such that $\iota_i = \iota_j F(\phi)$ for each morphism $\phi\colon i \to j$ in I.

$$\begin{tikzcd} & & & & Y \arrow[dd, dashed, "h"] \\ \dots \arrow[r] & X_i \arrow[r, "F(\phi)"] \arrow[urrr, bend left, "\iota_i"] \arrow[drrr, bend right, "g_i"'] & X_j \arrow[r] \arrow[urr, "\iota_j"] \arrow[drr, "g_j"'] & \dots & \\ & & & & Z \end{tikzcd}$$

The word 'initial' means that if Z is any space with maps $g_i\colon X_i \to Z$ for all $i \in I$, satisfying $g_i = g_j F(\phi)$ for each $\phi\colon i \to j$ in I, then there is a unique map $h\colon Y \to Z$ such that $g_i = h\iota_i$ for each $i \in I$. We then write

$$Y = \operatorname*{colim}_{i \in I} X_i = \operatorname*{colim}_{I} F$$

and call h the canonical map (from the colimit). The coproduct, pushout and coequalizer correspond to diagrams indexed on the following three categories:

$$\bullet \qquad \bullet$$

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

We can also form sequential colimits, associated to diagrams

$$X_0 \xrightarrow{f_0} \dots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \dots$$

indexed on the category $I = (\mathbb{N}, \le)$:

$$(0) \longrightarrow \dots \longrightarrow (i-1) \longrightarrow (i) \longrightarrow (i+1) \longrightarrow \dots$$

Explicitly,

$$\operatorname*{colim}_{i \ge 0} X_i = \coprod_{i \ge 0} X_i/{\sim}$$

where $x_i \simeq f_i(x_i)$ for each $x_i \in X_i$ and $i \ge 1$. If each f_i is the inclusion of a closed subspace, then $\operatorname{colim}_i X_i = \bigcup_i X_i$ is the union of these spaces, with the weak topology.

The corresponding colimits in graded abelian groups are given degreewise. The coproduct of two abelian groups A and B is the direct sum $A \oplus B$, and the pushout $B \oplus_A C$ for two given homomorphisms $f\colon A \to B$

and $g\colon A \to C$ is the quotient group of $B \oplus C$ by the subgroup of elements of the form $(f(a), -g(a))$ with $a \in A$. Given a sequence of abelian groups and group homomorphisms

$$A_0 \xrightarrow{f_0} \dots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \dots$$

their colimit is given by

$$\operatorname*{colim}_{i \geq 0} A_i = \bigoplus_{i \geq 0} A_i / {\sim}$$

where $a_i \simeq f_i(a_i)$ for each $a_i \in A_i$ and $i \geq 0$.

For any diagram of spaces, there is a canonical homomorphism

$$h\colon \operatorname*{colim}_{i \in I} H_*(X_i) \longrightarrow H_*(\operatorname*{colim}_i X_i)\,.$$

This is an isomorphism in the case of coproducts, i.e., when I has only identity morphisms.

$$\bigoplus_i H_*(X_i) \cong H_*(\coprod_i X_i)$$

It is also an isomorphism in the case of sequential colimits, if we assume that each map $f_i\colon X_i \to X_{i+1}$ is a closed inclusion.

$$\operatorname*{colim}_i H_*(X_i) \cong H_*(\operatorname*{colim}_i X_i) = H_*(\bigcup_i X_i)$$

However, for pushouts and coequalizers the answer is more complicated. When $X = B \cup C$ and $A = B \cap C$, with $\{B, C\}$ an excisive (= good) pair, then we have a long exact Mayer–Vietoris sequence

$$\dots \to H_n(A) \longrightarrow H_n(B) \oplus H_n(C) \xrightarrow{\Psi_n} H_n(X) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{\Phi_{n-1}} H_{n-1}(B) \oplus H_{n-1}(C) \to \dots\,.$$

This shows that the canonical homomorphism

$$H_n(B) \oplus_{H_n(A)} H_n(C) \longrightarrow H_n(B \cup_A C)$$

induced by Ψ_n is injective, but in general it has a cokernel, given by the kernel of Φ_{n-1}. The topological hypothesis, of working with an excisive pair, can be bypassed through the use of mapping cylinders and similar homotopy colimits in place of the categorical colimits. The appearance of the kernel of Φ_{n-1} is a case of a (left) derived colimit, and can be handled through the use of homological algebra at the level of chain complexes, instead of working only at the level of graded abelian groups. For more complicated indexing categories I, the isomorphisms and exact sequences above must be replaced by the more general algebraic framework of spectral sequences.

2. August 26th lecture

2.1. **Limits.** Limits are categorically dual to colimits. Examples of limits in topological spaces are the products $X \times Y$ and $\prod_i X_i$, and the pullbacks (= fiber products)

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

where $X \times_Z Y = \{(x, y) \mid f(x) = g(y)\} \subset X \times Y$. Another example is the equalizer

$$\begin{tikzcd} \mathrm{eq}(f,g) \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

of two maps $f, g\colon X \to Y$, given by $\mathrm{eq}(f, g) = \{x \in X \mid f(x) = g(x)\}$. Pullbacks can be written in terms of products and equalizers, and equalizers can be written in terms of products and pullbacks. In general, a limit of an I-shaped diagram $F\colon I \to X$ of spaces is a terminal space Y with structure maps $\pi_i\colon Y \to X_i$ for each object $i \in I$ such that $F(\phi)\pi_i = \pi_j$ for each morphism $\phi\colon i \to j$ in I.

$$\begin{tikzcd}
Z \arrow[drr, "g_i"] \arrow[drrr, bend left, "g_j"] \arrow[dd, dashed, "h"] & & & & \\
& \dots \arrow[r] & X_i \arrow[r, "F(\phi)"] & X_j \arrow[r] & \dots \\
Y \arrow[urr, "\pi_i"] \arrow[urrr, bend right, "\pi_j"'] & & & &
\end{tikzcd}$$

The word ‘terminal’ means that if Z is any space with maps $g_i\colon Z \to X_i$ for all $i \in I$, satisfying $F(\phi)g_i = g_j$ for each $\phi\colon i \to j$ in I, then there is a unique map $h\colon Z \to Y$ such that $g_i = \pi_i h$ for each $i \in I$. We write

$$Y = \lim_{i \in I} X_i = \lim_I F$$

and call h the canonical map (to the limit). Given a tower

$$\dots \longrightarrow X_{i+1} \xrightarrow{f^{i+1}} X_i \xrightarrow{f^i} X^{i-1} \longrightarrow \dots \xrightarrow{f^1} X_0$$

indexed on the opposite category $I = (\mathbb{N}, \le)^{op}$, the sequential limit is given by

$$\lim_{i \ge 0} X_i = \{(x_i)_i \mid f^i(x_i) = x_{i-1} \text{ for all } i \ge 1\} \subset \prod_{i \ge 0} X_i \,.$$

The limits of abelian groups are given by the same formulas as for spaces. However, the canonical homomorphism

$$h\colon H_*(\lim_i X_i) \longrightarrow \lim_i H_*(X_i)$$

is hardly ever an isomorphism. This is in part due to the linearization implicit in the construction of singular homology, passing from the simplicial set $\mathrm{sing}_\bullet(X)$ to the chain complex $C_*(X)$.

There is a different sequence of functors from (based) spaces to (abelian) groups, called the higher homotopy groups:

$$\pi_n\colon \mathrm{Top}_* \longrightarrow \mathrm{Ab}$$

(for $n \ge 2$). We can define $\pi_n(X, x_0)$ as the set of homotopy classes of maps of pairs $f\colon (I^n, \partial I^n) \to (X, x_0)$, or of based maps $f\colon S^n \to X$, where I^n is the n-cube, and we identify $I^n/\partial I^n \cong S^n$. For these functors the canonical homomorphism

$$h\colon \pi_*(\lim_i X_i) \longrightarrow \lim_i \pi_*(X_i)$$

is an isomorphism in the case of products:

$$\pi_*(\prod_i X_i) \cong \prod_i \pi_*(X_i) \,.$$

In the case of pullbacks and equalizers, the situation is similar to that of the Mayer–Vietoris sequence for homology. If $f\colon X \to Y$ is a Serre fibration (= a good projection), then there is a long exact homotopy sequence

$$\dots \to \pi_{n+1}(X) \times \pi_{n+1}(Y) \xrightarrow{\Psi_{n+1}} \pi_{n+1}(Z) \xrightarrow{\partial} \pi_n(X \times_Z Y) \xrightarrow{\Phi_n} \pi_n(X) \times \pi_n(Y) \longrightarrow \pi_n(Z) \to \dots$$

This shows that the canonical homomorphism

$$\pi_n(X \times_Z Y) \longrightarrow \pi_n(X) \times_{\pi_n(Z)} \pi_n(Y)$$

induced by Φ_n is surjective, but in general it has a kernel, given by the cokernel of Ψ_{n+1}. The topological hypothesis, of working with a Serre fibration, can be bypassed by means of mapping path spaces and similar homotopy limits in place of categorical limits. The appearance of the cokernel of Ψ_{n+1} is a case of a (right) derived limit, and again heralds the role of homological algebra. Similarly, if

$$\dots \longrightarrow X_{i+1} \xrightarrow{f^{i+1}} X_i \xrightarrow{f^i} X^{i-1} \longrightarrow \dots \xrightarrow{f^1} X_0$$

is a tower of Serre fibrations, the canonical homomorphism

$$\pi_n(\lim_{i\geq 0} X_i) \longrightarrow \lim_{i\geq 0} \pi_n(X)$$

is always surjective, but has a kernel described by Jack Milnor (1962) in terms of the right derived sequential limit

$$\operatorname*{Rlim}_{i\geq 0} \pi_{n+1}(X_i)\,.$$

The latter groups are either trivial or uncountable. For more complicated indexing categories I, the isomorphisms and exact sequences above must, again, be replaced by the more general framework of spectral sequences.

It is, in general, very difficult to calculate the homotopy groups of a CW complex. There is a notion of a Postnikov tower

$$\dots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \dots \longrightarrow X_1 \to X_0 = *$$

for a (nice) connected space X, with $X = \lim_n X_n$, which is more-or-less dual to the skeleton filtration of a CW complex, and which makes it easy to describe the homotopy groups of X. However, given a CW complex X it is very difficult to determine the precise structure of its associated Postnikov tower. In practice one must therefore make calculations in homology, and then try to recover the homotopy groups from the homology groups, together with whatever extra structure is available.

2.2. **The Künneth theorem.** We return to homology in the case of products. If X and Y are CW complexes, then their product $X \times Y$ is a CW complex with n-skeleton

$$(X \times Y)^n = \bigcup_{i+j=n} X^i \times Y_j\,.$$

If $\Phi_\alpha \colon D^i \to X^i$ and $\Phi_\beta \colon D^j \to Y_j$ are characteristic maps, then the composite

$$D^{i+j} \cong D^i \times D^j \xrightarrow{\Phi_\alpha \times \Phi_\beta} X^i \times Y^j \subset (X \times Y)^{i+j}$$

is the characteristic map of an $n = i + j$-cell. We get a bijection

$$\{n\text{-cells in } X \times Y\} \cong \coprod_{i+j=n} \{i\text{-cells in } X\} \times \{j\text{-cells in } Y\}\,.$$

After linearizing, this gives an isomorphism

$$W_n(X \times Y) \cong \bigoplus_{i+j=n} W_i(X) \otimes W_j(Y)$$

where $\otimes$ denotes the tensor product of abelian groups. By construction, $A \otimes B$ is generated by symbols $a \otimes b$ with $a \in A$ and $b \in B$, subject to the bilinearity relations $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$ and $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$. There is a natural bijection between bilinear pairings $A \times B \to C$ and group homomorphisms $A \otimes B \to C$. The boundary homomorphism $\partial \colon W_n(X \times Y) \to W_{n-1}(X \times Y)$ is given by the (graded) Leibniz rule

$$\partial(\alpha \otimes \beta) = \partial\alpha \otimes \beta + (-1)^{|\alpha|} \alpha \otimes \partial\beta\,,$$

where $|\alpha|$ denotes the degree of α. This makes

$$W_*(X \times Y) \cong W_*(X) \otimes W_*(Y)\,,$$

where the right hand side is the tensor product of chain complexes. Hence

$$H_n(X \times Y) \cong H_n(W_*(X \times Y)) \cong H_n(W_*(X) \otimes W_*(Y))\,,$$

which leads us to the problem of calculating the homology of a tensor product $A_* \otimes B_*$ of two chain complexes (A_*, ∂) and (B_*, ∂). There is a homology cross product

$$H_i(A_*) \otimes H_j(B_*) \xrightarrow{\times} H_{i+j}(A_* \otimes B_*)$$

mapping $[\alpha] \otimes [\beta]$ to $[\alpha \otimes \beta]$, with α an i-cycle in A_* and β a j-cycle in B_*.

Proposition 2.1.

$$\bigoplus_{i+j=n} H_i(A_*) \otimes H_j(B_*) \xrightarrow{\times} H_n(A_* \otimes B_*)$$

is 'often' an isomorphism.

Theorem 2.2 (Hermann Künneth (1922))**.**

$$\bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \xrightarrow{\times} H_n(X \times Y)$$

is 'often' an isomorphism.

Here 'often' can be turned into 'always' by working with coefficients in a field, such as $\mathbb{Q}$ or $\mathbb{F}_p = \mathbb{Z}/p$. In general there are corrections terms that use homological algebra, i.e., the left derived functor(s) Tor of the tensor product. Originally this theorem was not stated in terms of homology groups, but in terms of their ranks, known as Betti numbers, so that

$$\beta_n(X \times Y) = \sum_{i+j=n} \beta_i(X) \cdot \beta_j(Y)\,.$$

Here $\beta_i(X) = \operatorname{rank} H_i(X)$. This rank equals the dimension over $\mathbb{Q}$ of $H_i(X;\mathbb{Q})$, so the Betti number formulation always holds, at least when all the numbers are finite.

2.3. **Cohomology.** Each topological space comes equipped with a canonical diagonal map

$$\Delta\colon X \longrightarrow X \times X$$

with $\Delta(x) = (x, x)$. If we can factor the induced homomorphism Δ_* through the cross product in the diagram

$$\begin{tikzcd}
 & H_n(X) \arrow[dl, dashed] \arrow[d, "\Delta_*"] \\
\bigoplus_{i+j=n} H_i(X) \otimes H_j(X) \arrow[r, "\times"] & H_n(X \times X)
\end{tikzcd}$$

then we obtain an algebraic structure internal to $H_*(X)$, not just a relation between $H_*(X)$ and $H_*(X \times X)$. In parallel work around 1935, James Alexander and Andrey Kolmogorov reformulated this idea in terms of cohomology. In each degree n let

$$C^n(X) = \operatorname{Hom}(C_n(X), \mathbb{Z}) \cong \{\operatorname{sing}_n(X) \longrightarrow \mathbb{Z}\}$$

be the abelian group of homomorphisms $\phi\colon C_n(X) \to \mathbb{Z}$, or equivalently, of functions $\operatorname{sing}_n(X) \to \mathbb{Z}$, mapping each singular n-simplex σ to an integer $\phi(\sigma)$. Composition with the boundary homomorphism $\partial_n\colon C_n(X) \to C_{n-1}(X)$ induces a coboundary homomorphism

$$\delta^{n-1}\colon C^{n-1}(X) \longrightarrow C^n(X)$$

given by

$$(\delta^{n-1}\phi)(\sigma) = \phi(\partial_n \sigma)$$

for $\sigma \in C_n(X)$ and $\phi \in C^{n-1}(X)$.

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\phi} \mathbb{Z}$$

In precise work it is better to use the formula

$$(\delta^{n-1}\phi)(\sigma) = (-1)^n \phi(\partial_n \sigma)\,,$$

but to simplify the exposition we will follow Hatcher's example and omit this sign. See also MacLane's "Homology", section II.3. Letting n vary we obtain a diagram

$$\cdots \to C^{n-1}(X) \xrightarrow{\delta^{n-1}} C^n(X) \xrightarrow{\delta^n} C^{n+1}(X) \to \dots$$

with $\delta^n\delta^{n-1} = 0$. This is the singular cochain complex (or cocomplex) of X. Let $Z^n(X) = \ker(\delta^n)$ be the subgroup of n-cocycles, and let $B^n(X) = \mathrm{im}(\delta^{n-1})$ be the subgroup of n-coboundaries. Their quotient group

$$H^n(X) = \frac{Z^n(X)}{B^n(X)} = \frac{\ker(\delta^n)}{\mathrm{im}(\delta^{n-1})}$$

is the n-th singular cohomology group of X. This defines a contravariant functor $H^n\colon \mathrm{Top}^{op} \to \mathrm{Ab}$, in the sense that any map $f\colon X \to Y$ induces a homomorphism $f^*\colon H^n(Y) \to H^n(X)$. Letting n vary we get a graded abelian group $H^*(X) = (H^n(X))_n$, and a contravariant functor $H^*\colon \mathrm{Top}^{op} \to \mathrm{grAb}$.

2.4. **The universal coefficient theorem.** The relation between homology and cohomology is given by the following special case of the universal coefficient theorem.

Theorem 2.3. *Eilenberg–MacLane (1941)*

$$H^n(X) \longrightarrow \mathrm{Hom}(H_n(X), \mathbb{Z})$$

is 'often' an isomorphism.

Here 'often' means 'always' when working with field coefficients, and in general there are correction terms that can be expressed in terms of homological algebra, namely the right derived functors Ext of Hom. The displayed homomorphism is adjoint to the Kronecker pairing

$$H^n(X) \otimes H_n(X) \xrightarrow{\langle -,- \rangle} \mathbb{Z}$$

mapping $[\phi] \otimes [\alpha]$ to $\phi(\alpha)$, for any n-cocycle ϕ and and n-cycle α.

2.5. **The cup product.** For cochain complexes (A^*, δ) and (B^*, δ) there is a pairing

$$H^i(A^*) \otimes H^j(B^*) \xrightarrow{\times} H^{i+j}(A^* \otimes B^*)$$

which, in the case of the singular cochain complexes of topological spaces (or of cellular cochains $W^n(X) = \mathrm{Hom}(W_n(X), \mathbb{Z})$ for CW complexes) specializes to a cohomology cross product

$$H^i(X) \otimes H^j(Y) \xrightarrow{\times} H^{i+j}(X \times Y)\,.$$

For $X = Y$ we can combine this with the homomorphism Δ^* induced by the diagonal map $\Delta\colon X \to X \times X$, which leads to a definition of the cohomology cup product

$$\begin{tikzcd}
H^i(X) \otimes H^j(X) \arrow[r, "\times"] \arrow[dr, "\cup"'] & H^{i+j}(X \times X) \arrow[d, "\Delta^*"] \\
& H^{i+j}(X)\,.
\end{tikzcd}$$

In symbols,

$$a \cup b = \Delta^*(a \times b)\,.$$

This construction turns the graded abelian group $H^*(X)$ into a graded ring, with product

$$H^*(X) \otimes H^*(X) \xrightarrow{\cup} H^*(X)\,.$$

The cocommutativity of Δ

$$\begin{tikzcd}
& X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
X \times X \arrow[rr, "\tau", "\cong"'] & & X \times X
\end{tikzcd}$$

leads to the graded commutativity of $H^*(X)$:

$$\begin{tikzcd}
H^*(X) \otimes H^*(X) \arrow[rr, "\tau", "\cong"'] \arrow[dr, "\cup"'] & & H^*(X) \otimes H^*(X) \arrow[dl, "\cup"] \\
& H^*(X)\,. &
\end{tikzcd}$$

Here the topological twist isomorphism τ induces the algebraic twist isomorphism $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$, where the sign ultimately derives from the fact that the twist map $S^2 \cong S^1 \wedge S^1 \to S^1 \wedge S^1 \cong S^2$ has degree -1.

2.6. **A monogenic cohomology ring.** For example, $H^*(SO(3))$ is the cohomology of $W^*(\mathbb{R}P^3)$:

$$0 \to \mathbb{Z}\{a_0\} \xrightarrow{\delta^0} \mathbb{Z}\{a_1\} \xrightarrow{\delta^1} \mathbb{Z}\{a_2\} \xrightarrow{\delta^2} \mathbb{Z}\{a_3\} \to 0$$

with $\delta^1(a_1) = 2a_2$. Hence

$$H^*(SO(3)) \cong H^*(\mathbb{R}P^3) = (\mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z}, 0, \ldots)\,.$$

With coefficients in $\mathbb{Z}/2$ we get

$$0 \to \mathbb{Z}/2\{a_0\} \xrightarrow{0} \mathbb{Z}\{a_1\} \xrightarrow{0} \mathbb{Z}/2\{a_2\} \xrightarrow{0} \mathbb{Z}/2\{a_3\} \to 0$$

and

$$H^*(SO(3); \mathbb{Z}/2) = (\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, 0, \ldots)$$

with $H^k(SO(3); \mathbb{Z}/2)$ generated by the class of a_k dual to α_k for each $0 \le k \le 3$. Letting $x \in H^1(SO(3); \mathbb{Z}/2)$ be the class of a_1, we can calculate that $x^2 = x \cup x \in H^2(SO(3); \mathbb{Z}/2)$ is the class of a_2, and $x^3 = x \cup x \cup x \in H^3(SO(3); \mathbb{Z}/2)$ is the class of a_3. Hence, as a graded ring, we can write

$$H^*(SO(3); \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^4)\,.$$

This exhibits $H^*(SO(3); \mathbb{Z}/2)$ as a much tighter (single-generator) algebraic structure than the graded abelian group $H_*(SO(3); \mathbb{Z}/2) = \mathbb{Z}/2\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$.

Conveniently, there is a quite direct definition of the cup product, due to Alexander and Hassler Whitney. At the cochain level they define a homomorphism

$$\cup\colon C^i(X) \otimes C^j(X) \longrightarrow C^{i+j}(X)$$

so that $\phi \cup \psi$ evaluated on an $(i+j)$-simplex σ is the product of ϕ evaluated on the 'front' i-face of σ and ψ evaluated on the 'back' j-face of σ. Then

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^i \phi \cup \delta\psi\,,$$

and this lets us define

$$\cup\colon H^i(X) \otimes H^j(X) \longrightarrow H^{i+j}(X)$$

directly by

$$[\phi] \cup [\psi] = [\phi \cup \psi]\,.$$

This will be the order in which we introduce these constructions. The cohomology cross product can be recovered from the cup product, since the composite map

$$X \otimes Y \xrightarrow{\Delta_{X \times Y}} (X \otimes Y) \times (X \otimes Y) \xrightarrow{\pi_X \times \pi_Y} X \times Y$$

is the identity. This shows that the composite

$$H^i(X) \otimes H^j(Y) \xrightarrow{\pi_X^* \otimes \pi_Y^*} H^i(X \times Y) \otimes H^j(X \times Y) \xrightarrow{\cup} H^{i+j}(X \times Y)$$

equals the cross product:

$$a \times b = \pi_X^*(a) \cup \pi_Y^*(b)\,.$$

2.7. **The Pontryagin product.** If G is a topological group, with multiplication $m\colon G \times G \to G$ mapping (g, h) to gh, we get an induced pairing

$$H_i(G) \otimes H_j(G) \xrightarrow{\times} H_{i+j}(G \times G) \xrightarrow{m_*} H_{i+j}(G)$$

making $H_*(G)$ a graded ring. This is called the Pontryagin product. In the cases where the cohomology cross product induce an isomorphism

$$\begin{tikzcd} & H^n(G) \arrow[dl, dashed] \arrow[d, "m^*"] \\ \bigoplus_{i+j=n} H^i(G) \otimes H^j(G) \arrow[r, "\times"] & H^n(G \times G) \end{tikzcd}$$

we can lift m^* to get a graded ring homomorphism

$$\psi\colon H^*(G) \longrightarrow H^*(G) \otimes H^*(G)$$

that makes $H^*(G)$ into a Hopf algebra. In other words, the cup product dual to Δ makes $H^*(G)$ an algebra, the coproduct dual to m makes $H^*(G)$ a coalgebra, and these two structures are compatible.

2.8. **Division algebras.** For example, when $G = SO(3)$ with the Lie group multiplication, we get a coproduct

$$\psi\colon H^*(SO(3); \mathbb{Z}/2) \longrightarrow H^*(SO(3); \mathbb{Z}/2) \otimes H^*(SO(3); \mathbb{Z}/2)$$

of the form

$$\mathbb{Z}/2[x]/(x^4) \longrightarrow \mathbb{Z}/2[x]/(x^4) \otimes \mathbb{Z}/2[x]/(x^4)\,.$$

By unitality of the product m, it follows that

$$\psi(x) = x \otimes 1 + 1 \otimes x\,.$$

This makes sense, because

$$\psi(x^4) = \psi(x)^4 = (x \otimes 1 + 1 \otimes x)^4 = x^4 \otimes 1 + 1 \otimes x^4 = 0$$

since we are working over $\mathbb{F}_2$, and this equals $\psi(0) = 0$, so the ring homomorphism ψ respects the relation $x^4 = 0$. This works because 4 is a power of 2.

Theorem 2.4. *If $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a bilinear pairing without zero-divisors, making $\mathbb{R}^n$ a division algebra, then n is a power of 2.*

Sketch proof. The induced map $m\colon \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$ of real projective spaces makes

$$H^*(\mathbb{R}P^{n-1}; \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^n)$$

a Hopf algebra, so that

$$(x \otimes 1 + 1 \otimes x)^n = \sum_{i=1}^{n-1} \binom{n}{i} x^i \otimes x^{n-i}$$

equals 0 in $\mathbb{Z}/2[x]/(x^n) \otimes \mathbb{Z}/2[x]/(x^n)$. This can only hold if n is a power of 2. □

The cases $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions) realize the first four powers of 2.

Theorem 2.5 (Kervaire, Bott–Milnor (1958))**.** *If $\mathbb{R}^n$ is a division algebra, then $n = 1$, 2, 4 or 8.*

3. August 31st lecture

3.1. **The cap product.** We can define a natural cap product pairing

$$H^j(X) \otimes H_{i+j}(X) \xrightarrow{\cap} H_i(X)$$

so that

$$\langle a \cup b, \gamma \rangle = \langle a, b \cap \gamma \rangle$$

for all $a \in H^i(X)$, $b \in H^j(X)$ and $\gamma \in H_{i+j}(X)$. We say that $b \cap -$ is 'adjoint' to $- \cup b$.

3.2. **Poincaré duality.** If M is a closed (= compact, with empty boundary) and oriented n-manifold, then there exists a unique fundamental class $[M] \in H_n(M)$ mapping to the orientation generator of

$$H_n(M, M - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$$

for each point $x \in M$.

Theorem 3.1 (Poincaré)**.** *The homomorphism*

$$D_M \colon H^j(M) \longrightarrow H_{n-j}(M)$$

given by $D_M(b) = b \cap [M]$ is an isomorphism.

3.3. **4-manifolds.** If M is a 1-connected 4-manifold, then $H_1(M) = 0$, so that $H^1(M) = 0$ by the universal coefficient theorem, and $H^3(M) = 0$ and $H_3(M) = 0$ by Poincaré duality. This implies that $H_2(M) \cong H^2(M) \cong \mathbb{Z}^r$ for some $r \geq 0$, by another application of the universal coefficient theorem. Choose a basis $(b_1, \dots, b_r)$ for $H^2(M)$. The composite

$$\cdot \colon H^2(M) \otimes H^2(M) \xrightarrow{\cup} H^4(M) \xrightarrow{\langle -,[M]\rangle} \mathbb{Z}$$

is then a perfect pairing, given by a symmetric unimodular integer $r \times r$-matrix

$$B = (b_i \cdot b_j)_{i,j=1}^r .$$

This establishes a strong connection

$$\{\text{simply-connected 4-manifolds}\} \longleftrightarrow \{\text{symmetric unimodular forms over } \mathbb{Z}\}$$

For example, $M = \mathbb{C}P^2$ with $H^*(\mathbb{C}P^2) = \mathbb{Z}[b]/(b^3)$ and $M = S^2 \times S^2$ with

$$H^*(S^2 \times S^2) = \mathbb{Z}[b_1]/(b_1^2) \otimes \mathbb{Z}[b_2]/(b_2^2)$$

correspond to the matrices

$$\begin{pmatrix} 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively. Mike Freedman (1982) constructed a simply-connected (and spin) topological 4-manifold with associated matrix

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

By a theorem of Rochlin (1952), this manifold admits no smooth (= differentiable) structure. More precisely, Rochlin proves that the signature of B, i.e., the difference between the number of positive eigenvalues and the number of negative eigenvalues (all of which are real and nonzero), is divisible by 16 if M is a differentiable 4-manifold that admits a spin structure. The signature of E_8 is 8.

3.4. **Homotopy theory.** The functors H_* and H^* are homotopy invariant, in the sense that if $f \simeq g \colon X \to Y$ then $f_* = g_* \colon H_*(X) \to H_*(Y)$ and $f^* = g^* \colon H^*(Y) \to H^*(X)$. Hence f_* only depends on the homotopy class $[f]$ of f. Let

$$[X, Y] = \{f \colon X \to Y\}/\simeq$$

be the set of such classes. We get a homotopy category Ho(Top), with the same objects as Top but with morphism sets

$$\mathrm{Ho(Top)}(X, Y) = [X, Y] .$$

A homotopy equivalence $f \colon X \to Y$ is an isomorphism in this category. We also get a factorizations

$$\mathrm{Top} \xrightarrow{f \mapsto [f]} \mathrm{Ho(Top)} \xrightarrow{H_*} \mathrm{grAb}$$

and

$$\mathrm{Top}^{op} \xrightarrow{f\mapsto[f]} \mathrm{Ho}(\mathrm{Top})^{op} \xrightarrow{H^*} \mathrm{grAb}$$

of the homology and cohomology functors.

Theorem 3.2 (Eilenberg–MacLane (ca. 1945?))**.** *The (reduced) cohomology functor $\tilde{H}^n$ is representable in* $\mathrm{Ho}(\mathrm{Top}_*)$*: There is a CW complex $K(\mathbb{Z}, n)$ and a natural isomorphism*

$$\tilde{H}^n(X) \cong [X, K(\mathbb{Z}, n)] .$$

This makes $X \mapsto H^n(X)$ internal to the category $\mathrm{Ho}(\mathrm{Top}_*)$, instead of being a transformation from Top_*^{op} to Ab. Note that

$$\pi_i K(\mathbb{Z}, n) = [S^i, K(\mathbb{Z}, n)] \cong \tilde{H}^n(S^i) \cong \begin{cases} \mathbb{Z} & \text{for } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

This characterizes the homotopy type of $K(\mathbb{Z}, n)$, which is known as an Eilenberg–MacLane complex of type $(\mathbb{Z}, n)$.

This is a first example of what Mike Hopkins calls a ‘designer homotopy type’: a space that has been constructed to have particular homotopy properties, rather than originating from a geometric source.

3.5. **Cohomology operations.** A cohomology operation θ is a natural transformation

$$\theta_X : H^n(X) \longrightarrow H^m(X)$$

for some fixed n and m. Naturality means that for each map $f : X \to Y$ the following square commutes.

$$\begin{tikzcd} H^n(Y) \arrow[r, "\theta_Y"] \arrow[d, "f^*"'] & H^m(Y) \arrow[d, "f^*"] \\ H^n(X) \arrow[r, "\theta_X"] & H^m(X) . \end{tikzcd}$$

One example is given by the cup square

$$\xi_X : H^n(X) \longrightarrow H^{2n}(X)$$

sending x to $x^2 = x \cup x$. By the representability of cohomology, and the Yoneda lemma from category theory, there is a one-to-one correspondence

$$\{\text{cohomology operations } H^n \to H^m\} \longleftrightarrow [K(\mathbb{Z}, n), K(\mathbb{Z}, m)] \cong H^m(K(\mathbb{Z}, n)) .$$

For $\theta : K(\mathbb{Z}, n) \to K(\mathbb{Z}, m)$ the homotopy class $[\theta]$ corresponds to the operation mapping $[a]$ to $[\theta a]$:

$$X \xrightarrow{a} K(\mathbb{Z}, n) \xrightarrow{\theta} K(\mathbb{Z}, m) .$$

Hence calculating the cohomology of the ‘designer homotopy type’ $K(\mathbb{Z}, n)$ lets us determine all cohomology operations with source H^n.

Working with coefficients in $\mathbb{F}_2 = \mathbb{Z}/2$, Norman Steenrod (1947) constructed ‘reduced squaring operations’

$$Sq^i : H^n(X; \mathbb{Z}/2) \longrightarrow H^{n+i}(X; \mathbb{Z}/2)$$

for all $n, i \geq 0$. Here $Sq^n(x) = x^2$, $Sq^0(x) = x$ and the operations $Sq^i(x)$ for $0 < i < n$ can be (and are often) nonzero. These correspond to classes

$$Sq^i \in H^{n+i}(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$$

Jean–Pierre Serre (1952) calculated the right hand side, showing that all cohomology operations

$$\theta_X : H^n(X; \mathbb{Z}/2) \longrightarrow H^m(X; \mathbb{Z}/2)$$

are generated by the cup product and the Steenrod operations. There are similar results for cohomology with coefficients in $\mathbb{F}_p = \mathbb{Z}/p$ for any prime p. The (non-commutative) algebra of operations generated by the Steenrod operations is called the Steenrod algebra, denoted A (or $\mathscr{A}$). Considering the mod 2 cohomology

$$X \longmapsto H^*(X; \mathbb{Z}/2)$$

as a graded-commutative, unstable A-module algebra provides a powerful algebraic image of the homotopy type of X. The unstable Adams spectral sequence gives a systematic way to try to recover information about $[X, Y]$ from the (derived) morphisms

$$H^*(Y; \mathbb{Z}/2) \longrightarrow H^*(X; \mathbb{Z}/2)$$

in this category.

3.6. **Classifying spaces.** A point in $\mathbb{R}P^3$ corresponds to a line L embedded in $\mathbb{R}^4$. By including $\mathbb{R}^4$ into $\mathbb{R}^\infty$, the choice of embedding is replaced by a contractible choice, so that the infinite projective space

$$\mathbb{R}P^\infty = \{\text{lines } L \subset \mathbb{R}P^\infty\}$$

is a classifying space for lines. Here

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x]$$

with $|x| = 1$. Hence the infinite sequence of additive generators for this cohomology is replaced by a single algebra generator. The Steenrod operations satisfy

$$Sq^i(x^j) = \binom{j}{i} x^{i+j} .$$

Similarly, for $k, n \geq 0$ the Grassmann manifold

$$\mathrm{Gr}_k(\mathbb{R}^{n+k}) = \{k\text{-dimensional subspaces } V \subset \mathbb{R}^{n+k}\}$$

corresponds to k-dimensional spaces V with a chosen embedding on $\mathbb{R}^{n+k}$. Letting the codimension n grow, the Grassmann space

$$\mathrm{Gr}_k(\mathbb{R}^\infty) = \{k\text{-dimensional subspaces } V \subset \mathbb{R}^\infty\}$$

classifies k-dimensional real vector spaces. This space is homotopy equivalent to the bar construction $BO(k)$ on the Lie group $O(k)$ is isometries $\mathbb{R}^k \to \mathbb{R}^k$:

$$\mathrm{Gr}_k(\mathbb{R}^\infty) \simeq BO(k) .$$

Theorem 3.3 (Eduard Stiefel, Whitney)**.**

$$H^*(\mathrm{Gr}_k(\mathbb{R}^\infty); \mathbb{Z}/2) = H^*(BO(k); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \dots, w_k]$$

where $|w_j| = j$.

Hence the infinite collection of additive generators is replaced by k algebra generators.

Theorem 3.4 (Wen Tsün Wu (1950))**.**

$$Sq^i(w_j) = w_i w_j + \binom{i-j}{1} w_{i-1} w_{j+1} + \cdots + \binom{i-j}{i} w_{i+j} .$$

For example, $Sq^1(w_2) = w_1 w_2 + w_3$, so the Steenrod operations tie the k generators of $H^*(BO(k); \mathbb{Z}/2)$ together.

3.7. **Bordism.** We turn to applications of algebraic topology to geometric topology, i.e., manifolds. In Poincaré's initial work, an n-cycle in a space X was viewed as a map

$$f \colon M^n \longrightarrow X$$

from a (suitable) closed n-manifold M. Two n-cycles $f \colon M \to X$ and $g \colon N \to X$ were viewed as equivalent if they were cobordant, i.e., if there is a (suitable) compact $n+1$-manifold W with a map

$$F \colon W^{n+1} \longrightarrow X$$

such that $\partial W \cong M \sqcup N$ and $F|\partial W = f \sqcup g$. Already the case $X = *$ is interesting, when we can ignore the maps to X. We say that two closed n-manifolds M and N are cobordant if there is a compact $(n+1)$-manifold W with $\partial W = M \sqcup N$. This defines an equivalence relation on the collection of closed n-manifolds. Let $\mathscr{N}_n$ be the set of such cobordism classes. The pairings taking (M, N) to $M \sqcup N$ and $M \times N$ define a sum and a product, making

$$\mathscr{N}_* = \bigoplus_{n \geq 0} \mathscr{N}_n$$

a graded commutative ring, known as the bordism ring.

Theorem 3.5 (René Thom (1953))**.**

$$\mathscr{N}_* = \mathbb{Z}/2[x_n \mid n \neq 2^i - 1]$$

is the polynomial $\mathbb{Z}/2$*-algebra on generators* $x_2, x_4, x_5, x_6, x_8, \dots$*.*

This classifies compact manifolds up to cobordism, and is an important first step on the way to the (finer, and much harder) surgery classification of manifolds up to homeomorphism, or up to diffeomorphism in the smooth case. For even n we can take $\mathbb{R}P^n$ to represent x_n, while for n odd Albrecht Dold (1956) constructed explicit manifolds representing x_n. In particular, $\mathbb{R}P^1$ and $\mathbb{R}P^3$ are boundaries (Exercise: Why?), while $\mathbb{R}P^2$ and $\mathbb{R}P^4$ are not.

3.8. **The Pontryagin–Thom construction.** We associate to a closed, smooth n-manifold M a homotopy class of maps

$$gc\colon S^{n+k} \longrightarrow MO(k)$$

where

$$MO(k) = D(\gamma^k)/S(\gamma^k)$$

is the Thom complex of the tautological k-plane bundle

$$E(\gamma^k) \longrightarrow \mathrm{Gr}_k(\mathbb{R}^\infty) \simeq BO(k)$$

with fiber over a point $V \in \mathrm{Gr}_k(\mathbb{R}^\infty)$ equal to the vector space V. We can assume that these vector spaces are equipped with an Euclidean inner product, and then the sphere and disc bundles $S(\gamma^k) \subset D(\gamma^k)$ in $E(\gamma)$ consist of the vectors $v \in V$ with $\|v\| = 1$ and $\|v\| \le 1$, respectively.

Given M as above, we can find a smooth embedding $M^n \subset \mathbb{R}P^{n+k}$ for sufficiently large k, and the choice of such an embedding becomes a contractible choice in the colimit over k. The tangent spaces $T_xM \subset \mathbb{R}P^{n+k}$ for $x \in M$ combine to the tangent $\mathbb{R}^n$-bundle

$$TM = E(\tau_M) \longrightarrow M\,.$$

Their orthogonal (= normal) complements $N_xM = T_xM^\perp \subset \mathbb{R}P^{n+k}$ combine to the normal $\mathbb{R}^k$-bundle

$$NM = E(\nu_M) \longrightarrow M\,.$$

By the tubular neighborhood theorem, there is an embedding $D(\nu_M) \subset \mathbb{R}P^{n+k}$ of the disc bundle of the normal bundle, extending the embedding of M viewed as the 0-section in $D(\nu_M)$, so that the sphere bundle $S(\nu_M)$ is the topological boundary of $D(\nu_M)$ in $\mathbb{R}P^{n+k}$. There is then a collapse map

$$c\colon S^{n+k} \longrightarrow S^{n+k}/(S^{n+k} - \operatorname{int} D(\nu_M)) \cong D(\nu_M)/S(\nu_M)\,.$$

Next, there is a Gauss map

$$g\colon M \longrightarrow \mathrm{Gr}_k(\mathbb{R}^{n+k}) \subset \mathrm{Gr}_k(\mathbb{R}^\infty)$$

taking $x \in X$ to the k-dimensional subspace $N_xM \subset \mathbb{R}^{n+k} \subset \mathbb{R}^\infty$ viewed as a point in the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^\infty) \simeq BO(k)$. This is covered by a bundle map

$$\begin{tikzcd} E(\nu_M) \arrow[r, "g"] \arrow[d] & E(\gamma^k) \arrow[d] \\ M \arrow[r, "g"] & \mathrm{Gr}_k(\mathbb{R}^\infty) \end{tikzcd}$$

inducing a map

$$g\colon D(\nu_M)/S(\nu_M) \longrightarrow D(\gamma^k)/S(\gamma^k) = MO(k)$$

of Thom complexes. The composite gc is the promised map, well-defined up to homotopy.

Cobordant n-manifolds produce homotopic maps $S^{n+k} \to MO(k)$, for k sufficiently large. Using transversality for smooth maps, Thom proved that this construction gives an isomorphism

$$\pi_{n+k} MO(k) \xleftrightarrow{\cong} \mathscr{N}_n\,.$$

Hence the manifold-geometric problem of classifying manifolds up to cobordism has been turned into a homotopy-theoretic problem of determining the homotopy groups of the Thom complex $MO(k)$, for k large with respect to n.

There are structure maps

$$\sigma\colon MO(k) \wedge S^1 \longrightarrow MO(k+1)$$

implicit in the formation of the colimit above. The sequence of spaces $MO(k)$ and maps σ, for $k \geq 0$, is an example of a spectrum in the sense of algebraic topology, usually denoted MO. By definition

$$\pi_n(MO) = \operatorname*{colim}_k \pi_{n+k} MO(k)\,,$$

so that $\pi_*(MO) \cong \mathscr{N}_*$. These homotopy groups are always abelian, recovering the graded abelian group structure on $\mathscr{N}_*$. To recover the multiplicative structure given by the Cartesian product of manifolds, we need to know that MO is a (commutative) ring spectrum. This generalizes the notion of a ring from algebra, and such objects were called 'brave new rings' by Friedhelm Waldhausen (ca. 1980).

3.9. **Thom isomorphism.** A generalization of the Künneth theorem (for the cohomology of a product) gives an isomorphism

$$H^*(BO(k); \mathbb{Z}/2) \cong H^{*+k}(MO(k); \mathbb{Z}/2)$$

for each $k \geq 0$. The calculation of $H^*(BO(k); \mathbb{Z}/2)$ therefore gives a calculation of $H^{*+k}(MO(k); \mathbb{Z}/2)$, which in the limit over k shows that

$$H^*(MO; \mathbb{Z}/2) = \mathbb{Z}/2[w_j \mid j \geq 1]\,.$$

Knowledge of the Steenrod operations in $H^*(MO; \mathbb{Z}/2)$, and the (stable) Adams spectral sequence, then leads to Thom's conclusion, that

$$\pi_*(MO) \cong \mathbb{Z}/2[x_n \mid n \neq 2^i - 1]\,.$$

Roughly speaking, there are composite Steenrod operations

$$Sq^{2^{i-1}} \circ \cdots \circ Sq^2 \circ Sq^1$$

that eliminate the generators in degrees $2^i - 1$, for all $i \geq 1$.

3.10. **Summary.**

- Chapter 3
  - Section 3.1: Cohomology Groups
  - Section 3.2: Cup Product
  - Section 3.3: Poincaré Duality
- Chapter 4
  - Section 4.1: Homotopy Groups
  - Section 4.2: Elementary Methods of Calculation
  - Section 4.3: Connections with Cohomology
  - Section 4.L: Steenrod Squares and Powers

Part 2. Cohomology

4. September 2nd lecture

4.1. **The universal coefficient theorem.** We discuss Sections 3.1 and 3.A together.

Theorem 4.1 (Eilenberg–MacLane (1941))**.** *Let (C_*, ∂) be a chain complex of free abelian groups, and let G be any abelian group. Let $G \otimes C_*$ denote the chain complex with boundary $1 \otimes \partial$, and let* $\operatorname{Hom}(C_*, G)$ *denote the cochain complex with boundary* $\operatorname{Hom}(\partial, 1)$*. There are natural short exact sequences*

$$0 \to G \otimes H_n(C_*) \xrightarrow{i} H_n(G \otimes C_*) \longrightarrow \operatorname{Tor}(G, H_{n-1}(C_*)) \to 0$$

and

$$0 \to \operatorname{Ext}(H_{n-1}(C_*), G) \longrightarrow H^n(\operatorname{Hom}(C_*, G)) \xrightarrow{h} \operatorname{Hom}(H_n(C_*), G) \to 0$$

for each integer n. Here $i\colon g \otimes [\alpha] = [g \otimes \alpha]$ for each n-cycle $\alpha \in C_n$, and $h([\phi])\colon [\alpha] \mapsto \phi(\alpha)$ for each n-cocycle $\phi\colon C_n \to G$ and n-cycle $\alpha \in C_n$. Each of these short exact sequences admits a (non-natural) splitting.

When applied to $C_* = C_*(X)$ this algebraic theorem has the following topological consequence.

Theorem 4.2 (Eilenberg–MacLane (1941))**.** *Let X be any topological space and let G be any abelian group. There are natural short exact sequences*

$$0 \to G \otimes H_n(X) \xrightarrow{i} H_n(X; G) \longrightarrow \mathrm{Tor}(G, H_{n-1}(X)) \to 0$$

and

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \xrightarrow{h} \mathrm{Hom}(H_n(X), G) \to 0$$

for each integer n. Each of these admits a (non-natural) splitting.

Similar results apply for relative (co-)homology, reduced (co-)homology and, when the spaces are CW complexes, for the corresponding forms of cellular (co-)homology.

4.2. **Exact sequences.** To prove these results we need to make sense of the derived functors Tor of $\otimes$ and Ext of Hom. We do this in the greater generality of (left) modules over a ring R.

A diagram of left R-modules

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact (at B) if $\mathrm{im}(f) = \ker(g)$. This implies that $gf = 0$, so that the diagram is a complex, and we can talk about its homology $H = \ker(g)/\mathrm{im}(f)$. The complex is exact if and only if $H = 0$. By the Noether isomorphism theorem the canonical homomorphism

$$B/\ker(g) \xrightarrow{\cong} \mathrm{im}(g)$$

is an isomorphism. Hence exactness at B tells us that

$$\mathrm{cok}(f) = B/\mathrm{im}(f) = B/\ker(g) \cong \mathrm{im}(g)\,.$$

If a diagram

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

is exact (at B and C) then

$$\mathrm{cok}(f) \cong \mathrm{im}(g) = \ker(h)\,.$$

A diagram

$$0 \to A \xrightarrow{i} B$$

is exact (at A) if and only if i is injective (= one-to-one). A diagram

$$B \xrightarrow{j} C \to 0$$

is exact (at C) if and only if j is surjective (= onto). A diagram

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

is exact (at A, B and C) if and only if $A \cong \mathrm{im}(i) = \ker(j)$ and $\mathrm{im}(j) = C$, so that $A \cong i(A)$ and $B/i(A) \cong C$. We call this a short exact sequence (SES), and say that C is an extension of C by A.

Lemma 4.3 (Splitting lemma, Hatcher p. 147)**.** *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

be a short exact sequence. The following are equivalent.

1. *There is a section $s\colon C \to B$ with $js = 1_C$.*
2. *There is a retraction $r\colon B \to A$ with $ri = 1_A$.*
3. *There is a retraction $r\colon B \to A$ and a section $s\colon C \to B$ with $ri = 1_A$, $ir + sj = 1_B$, $js = 1_C$ and $rs = 0$.*

Proof. Given s with $js = 1_C$, define $r\colon B \to A$ so that $ir(b) = b - sj(b)$ for all $b \in B$. This is possible because $j(b - sj(b)) = j(b) - j(b) = 0$.

Given r with $ri = 1_A$, define $s\colon C \to B$ so that $sj(b) = b - ir(b)$ for all $b \in B$. This is possible because $i(a) - ir(i(a))) = 0$ for all $a \in A$. □

An R-module F is free if it admits a basis, i.e., a set S such that there is a natural bijection

$$\mathrm{Hom}_R(F, G) \cong \{\text{functions } S \to G\}$$

for all R-modules G. Equivalently,

$$F \cong R\{S\} = \bigoplus_{s \in S} R\,.$$

Lemma 4.4. *If F is free and $j\colon B \to C$ is surjective then*

$$j\colon\ \mathrm{Hom}_R(F, B) \longrightarrow \mathrm{Hom}_R(F, C)$$

(mapping $h\colon F \to B$ to $jh\colon F \to C$) is surjective. Hence for any $g\colon F \to C$ there exists a factorization $g = jh$ with $h\colon F \to B$.

We think of g as a lift of h over j, and express the surjectivity of j by an arrow $C \to 0$, with implied exactness at C.

$$\begin{tikzcd}
& F \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\
B \arrow[r, "j"] & C \arrow[r] & 0
\end{tikzcd}$$

An R-module satisfying this lifting property is said to be projective. The lemma then tells us that free modules are projective.

Proof. Let S be a basis for F. The function

$$j\colon \{\text{functions } S \to B\} \longrightarrow \{\text{functions } S \to C\}$$

is surjective. □

Lemma 4.5. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

be a short exact sequence, with C a free module. The equivalent conditions of the splitting lemma are then satisfied.

Proof. The identity map $1_C\colon C \to C$ admits a lift s over j, which is a section to j. □

Lemma 4.6. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

be a short exact sequence of left R-modules, with C free. Then

$$0 \to G \otimes_R A \xrightarrow{1\otimes i} G \otimes_R B \xrightarrow{1\otimes j} G \otimes_R C \to 0$$

is (split) short exact for any right R-module G, and

$$0 \to \mathrm{Hom}_R(C, G) \xrightarrow{\mathrm{Hom}(j,1)} \mathrm{Hom}_R(B, G) \xrightarrow{\mathrm{Hom}(i,1)} \mathrm{Hom}_R(A, G) \to 0$$

is (split) short exact for any left R-module G.

Proof. Choose a splitting $s\colon C \to B$. Then

$$i \oplus s\colon A \oplus C \xrightarrow{\cong} B$$

is an isomorphism, and induces isomorphisms

$$1 \otimes (i \oplus s)\colon G \otimes_R (A \oplus C) \xrightarrow{\cong} G \otimes_R C$$

and

$$\mathrm{Hom}(i \oplus s, 1)\colon\ \mathrm{Hom}_R(B, G) \xrightarrow{\cong} \mathrm{Hom}_R(A \oplus C, G)\,.$$

These readily induce splittings

$$G \otimes_R A \oplus G \otimes_R C \xrightarrow{\cong} G \otimes_R B$$

and

$$\mathrm{Hom}_R(B, G) \xrightarrow{\cong} \mathrm{Hom}_R(A, G) \times \mathrm{Hom}_R(C, G) \cong \mathrm{Hom}_R(A, G) \oplus \mathrm{Hom}_R(C, G)\,,$$

which are suitably compatible with the displayed maps $1 \otimes i$, $1 \otimes j$, $\mathrm{Hom}(j, 1)$ and $\mathrm{Hom}(i, 1)$. □

4.3. **Cycle and boundary complexes.** Suppose that (C_*, ∂) is a chain complex of free R-modules.

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \dots .$$

Let $Z_n = \ker(\partial_n)$ and $B_n = \mathrm{im}(\partial_{n+1})$. We view Z_* and B_* as chain complexes with trivial (= zero) boundary homomorphisms:

$$\cdots \to Z_{n+1} \xrightarrow{0} Z_n \xrightarrow{0} Z_{n-1} \to \dots$$

$$\cdots \to B_{n+1} \xrightarrow{0} B_n \xrightarrow{0} B_{n-1} \to \dots$$

We then have a short exact sequence of chain complexes

$$0 \to (Z_*, 0) \longrightarrow (C_*, \partial) \xrightarrow{\partial} (B_{*-1}, 0) \to 0$$

given in degree n by the short exact sequence

(1) $$0 \to Z_n \longrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \to 0\,.$$

The displayed maps are chain maps, because the diagram

$$\begin{tikzcd}
Z_n \arrow[r] \arrow[d, "0"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[d, "0"] \\
Z_{n-1} \arrow[r] & C_{n-1} \arrow[r, "\partial_{n-1}"] & B_{n-2}
\end{tikzcd}$$

commutes, for each n.

Suppose now that R is a 'free ideal ring', in the sense that each submodule of a free module is itself free. (This is not quite standard terminology, but is convenient for our discussion.) This condition is satisfied for each principal ideal domain (PID), including the ring of integers $R = \mathbb{Z}$. In view of the inclusions

$$B_n \subset Z_n \subset C_n$$

it then follows that each B_n and Z_n is free. In particular, each short exact sequence (1) admits a splitting.

It follows that

$$0 \to G \otimes_R Z_n \longrightarrow G \otimes_R C_n \xrightarrow{1 \otimes \partial_n} G \otimes_R B_{n-1} \to 0$$

and

$$0 \to \mathrm{Hom}_R(B_{n-1}, G) \xrightarrow{\mathrm{Hom}(\partial, 1)} \mathrm{Hom}_R(C_n, G) \longrightarrow \mathrm{Hom}_R(Z_n, G) \to 0$$

are (split) short exact, for each n. Hence

$$0 \to (G \otimes_R Z_*, 0) \longrightarrow (G \otimes_R C_*, \partial) \xrightarrow{1 \otimes \partial} (G \otimes_R B_{*-1}, 0) \to 0$$

and

$$0 \to (\mathrm{Hom}_R(B_{*-1}, G), 0) \xrightarrow{\mathrm{Hom}(\partial, 1)} (\mathrm{Hom}_R(C_*, G), \mathrm{Hom}(\partial, 1)) \longrightarrow (\mathrm{Hom}_R(Z_*, G), 0) \to 0$$

are both short exact sequences of chain and cochain complexes. Note that the splittings are usually not compatible with the boundary homomorphisms, so these diagrams are not split short exact sequences of (co-)chain complexes.

The associated long exact sequences in homology take the form

(2) $$\cdots \to G \otimes_R B_n \xrightarrow{1 \otimes \iota_n} G \otimes_R Z_n \longrightarrow H_n(G \otimes_R C_*) \longrightarrow G \otimes_R B_{n-1} \xrightarrow{1 \otimes \iota_{n-1}} G \otimes_R Z_{n-1} \to \dots$$

and

$$(3)\quad \cdots \to \operatorname{Hom}_R(Z_{n-1}, G) \overset{\operatorname{Hom}(\iota_{n-1},1)}{\longrightarrow} \operatorname{Hom}_R(B_{n-1}, G)$$

$$\longrightarrow H^n(\operatorname{Hom}_R(C_*, G)) \longrightarrow \operatorname{Hom}_R(Z_n, G) \overset{\operatorname{Hom}(\iota_n,1)}{\longrightarrow} \operatorname{Hom}_R(B_n, G) \to \dots$$

where $\iota_n \colon B_n \to Z_n$ denotes the inclusion. This uses that the homology of a chain complex with zero boundaries is given by the underlying graded module. The expressions given for the connecting homomorphisms are verified by inspection of the definitions.

We thus want to identify the kernel and cokernel of the homomorphisms

$$1 \otimes \iota_n \colon G \otimes_R B_n \longrightarrow G \otimes_R Z_n$$

and

$$\operatorname{Hom}(\iota_n, 1) \colon \operatorname{Hom}_R(Z_n, G) \longrightarrow \operatorname{Hom}_R(B_n, G)\,.$$

Note that we have a short exact sequence

$$(4)\qquad 0 \to B_n \xrightarrow{\iota_n} Z_n \xrightarrow{\pi_n} H_n(C_*) \to 0$$

for each n, where B_n and Z_n are free R-modules, and π_n denotes the canonical surjection mapping an n-cycle α to its homology class $[\alpha]$.

Proposition 4.7. *Let R be a free ideal ring. There are functors $M \mapsto \operatorname{Tor}^R(G, M)$ and $M \mapsto \operatorname{Ext}_R(M, G)$, defined for left R-modules M, and natural exact sequences*

$$0 \to \operatorname{Tor}^R(G, H_n(C_*)) \longrightarrow G \otimes_R B_n \xrightarrow{1\otimes\iota_n} G \otimes_R Z_n \xrightarrow{1\otimes\pi_n} G \otimes_R H_n(C_*) \to 0$$

and

$$0 \to \operatorname{Hom}_R(H_n(C_*), G) \overset{\operatorname{Hom}(\pi_n,1)}{\longrightarrow} \operatorname{Hom}_R(Z_n, G) \overset{\operatorname{Hom}(\iota_n,1)}{\longrightarrow} \operatorname{Hom}_R(B_n, G) \longrightarrow \operatorname{Ext}_R(H_n(C_*), G) \to 0\,.$$

We postpone the proof of this proposition.

4.4. **The universal coefficient theorem for chain complexes.**

Theorem 4.8 (Algebraic UCT)**.** *Let R be a free ideal ring, let (C_*, ∂) be a chain complex of free R-modules, and let G be any R-module. Let $G \otimes_R C_*$ denote the chain complex with boundary $1 \otimes \partial$, and let $\operatorname{Hom}_R(C_*, G)$ denote the cochain complex with boundary $\operatorname{Hom}(\partial, 1)$. There are natural short exact sequences*

$$0 \to G \otimes_R H_n(C_*) \xrightarrow{i} H_n(G \otimes_R C_*) \longrightarrow \operatorname{Tor}_1^R(G, H_{n-1}(C_*)) \to 0$$

and

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(C_*), G) \longrightarrow H^n(\operatorname{Hom}_R(C_*, G)) \xrightarrow{h} \operatorname{Hom}_R(H_n(C_*), G) \to 0$$

for each integer n. Here $i \colon g \otimes [\alpha] = [g \otimes \alpha]$ for each n-cycle $\alpha \in C_n$, and $h([\phi]) \colon [\alpha] \mapsto \phi(\alpha)$ for each n-cocycle $\phi \colon C_n \to G$ and n-cycle $\alpha \in C_n$. Each of these short exact sequences admits a (non-natural) splitting.

Proof. By exactness of (2) we have a natural short exact sequence

$$0 \to \operatorname{cok}(1 \otimes \iota_n) \longrightarrow H_n(G \otimes_R C_*) \longrightarrow \ker(1 \otimes \iota_{n-1}) \to 0\,.$$

Under the (easy) isomorphism

$$\operatorname{cok}(1 \otimes \iota_n) \cong G \otimes_R H_n(C_*)$$

the left hand homomorphism corresponds to

$$i \colon G \otimes_R H_n(C_*) \longrightarrow H_n(G \otimes_R C_*)$$

mapping $g \otimes [\alpha]$ to $[g \otimes \alpha]$. By Proposition 4.7, there is also an isomorphism

$$\operatorname{Tor}^R(G, H_{n-1}(C_*)) \cong \ker(1 \otimes \iota_{n-1})\,.$$

In combination, these give the asserted short exact sequence. ((ETC: Discuss splitting.))

By exactness of (3) we have a natural short exact sequence

$$0 \to \operatorname{cok}(\operatorname{Hom}(\iota_{n-1}, 1)) \longrightarrow H_n(\operatorname{Hom}_R(C_*, G)) \longrightarrow \ker(\operatorname{Hom}(\iota_n, 1)) \to 0\,.$$

Under the (easy) isomorphism

$$\mathrm{Hom}_R(H_n(C_*), G) \cong \ker(\mathrm{Hom}(\iota_n, 1))$$

the right hand homomorphism corresponds to

$$h\colon H_n(\mathrm{Hom}_R(C_*, G)) \longrightarrow \mathrm{Hom}_R(H_n(C_*), G)$$

mapping the class $[\phi]$ of $\phi\colon C_n \to G$ to $h([\phi])\colon H_n(C_*) \to G$ taking $[\alpha]$ to $\phi(\alpha)$. By the proposition, there is also an isomorphism

$$\mathrm{cok}(\mathrm{Hom}(\iota_{n-1}, 1)) \cong \mathrm{Ext}_R(H_{n-1}(C_*), G)\,.$$

In combination, these give the asserted short exact sequence. ((ETC: Discuss splitting.)) □

We often write $H_n(C_*; G) = H_n(C_* \otimes_R G)$ and $H^n(C_*; G) = H^n(\mathrm{Hom}_R(C_*, G))$ for the homology and cohomology groups of C_* with coefficients in G.

4.5. **Free resolutions.** Let R be a general ring (associative, with unit). A chain map

$$f\colon (C_*, \partial) \longrightarrow (D_*, \partial)$$

is called a quasi-isomorphism (or a homology equivalence) if it induces an isomorphism of homology groups

$$f_*\colon H_n(C_*) \longrightarrow H_n(D_*)$$

in each degree n. In this case we can think of C_* and D_* as equivalent models for the common sequence of homology groups.

An R-module M can be viewed as a chain complex given by M in degree 0, with zero modules in all other degrees. Its boundary homomorphisms are then necessarily trivial. Let (C_*, ∂) be a non-negatively graded chain complex, meaning that $C_n = 0$ for $n < 0$. A chain map

$$\epsilon\colon C_* \longrightarrow M$$

can then be pictured as below:

$$\begin{tikzcd}
\dots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \dots \arrow[r] & C_1 \arrow[r, "\partial_1"] \arrow[d] & C_0 \arrow[r] \arrow[d, "\epsilon"] & 0 \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

More efficiently, we can picture this as an augmented chain complex

$$\dots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} M \to 0\,.$$

Lemma 4.9. *The chain map $\epsilon\colon C_* \to M$ is a quasi-isomorphism if and only if the augmented chain complex is exact (at all points).*

In the situation of the lemma, we may say that $\epsilon\colon C_* \to M$ is a resolution of M. If each C_n is free, then we say that $\epsilon\colon C_* \to M$ is a free resolution of M. We often say that C_* is a (free) resolution, leaving the augmentation ϵ implicit.

Example 4.10. (1) When M is free, the very short exact sequence

$$0 \to M \xrightarrow{1_M} M \to 0$$

is a free resolution, with $C_0 = M$ and $C_n = 0$ for $n > 0$.

(2) For $R = \mathbb{Z}$ and $M = \mathbb{Z}/m$ with $m \geq 1$, the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m \to 0$$

is a free resolution, with $C_0 = C_1 = \mathbb{Z}$ and $C_n = 0$ for $n > 1$.

(3) For $R = \mathbb{Z}[x]$ and $M = \mathbb{Z}/m$, the exact sequence

$$0 \to R \xrightarrow{(\, x \ -m\,)} R \oplus R \xrightarrow{\binom{m}{x}} R \longrightarrow M \to 0$$

is a free resolution, with $C_n = 0$ for $n > 2$.

(4) When R is a free ideal ring the short exact sequence (4) is a free resolution of $H_n(C_*)$.

Lemma 4.11. *Each R-module M admits a free resolution.*

Proof. Let C_0 be any free R-module with a surjection $\epsilon\colon C_0 \to M$. For example, we may take $C_0 = R\{M\}$ equal to the free module on the underlying set of M, with $\epsilon(m) = m$ for each $m \in M$. Inductively, assume that we have constructed an exact complex

$$C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} M \to 0\,.$$

Let C_{n+1} be any free R-module with a surjection $\epsilon_{n+1}\colon C_{n+1} \to \ker(\partial_n)$. For example, we may take $C_{n+1} = R\{\ker(\partial_n)\}$. Let ∂_{n+1} be the composite

$$C_{n+1} \xrightarrow{\epsilon_{n+1}} \ker(\partial_n) \subset C_n\,.$$

Then $\mathrm{im}(\partial_{n+1}) = \ker(\partial_n)$, and this completes the inductive step. □

The free resolution $\epsilon\colon C_* \to M$ obtained in the previous proof is sometimes called the canonical free resolution (associated to the free–forgetful adjunction between sets and R-modules). It is functorial, in the sense that for any homomorphism $h\colon M \to N$, if $\epsilon\colon D_* \to N$ is the canonical free resolution of N, then there is a preferred chain map $f\colon C_* \to D_*$ covering h, and this construction is compatible with composition.

4.6. **The comparison theorem.** Free resolutions are not unique, so we need to compare different free resolutions of the same R-module. This will be an application of the following fundamental result in homological algebra.

Theorem 4.12 (Comparison Theorem). *Let $h\colon M \to N$ be an R-module homomorphism, let F_* and E_* be non-negatively graded chain complexes, let $\epsilon\colon F_* \to M$ be a chain map with F_* free, and let $\epsilon\colon E_* \to N$ be a quasi-isomorphism. Then there exists a chain map*

$$f\colon F_* \longrightarrow E_*$$

covering h, in the sense that

$$\begin{tikzcd} F_* \arrow[r, "\epsilon"] \arrow[d, dashed, "f"'] & M \arrow[d, "h"] \\ E_* \arrow[r, "\epsilon"] & N \end{tikzcd}$$

commutes. Furthermore, f is well-defined up to chain homotopy, in the sense that if $g\colon F_* \to E_*$ is a second chain map lifting h, then there is a chain homotopy $P\colon F_* \to E_{*+1}$ from f to g.

Proof. We construct the components f_n of a chain map f covering h by induction on $n \geq 0$. To start, f_0 is defined to be a lift in the diagram

$$\begin{tikzcd} & F_0 \arrow[dl, dashed, "f_0"'] \arrow[d, "h\epsilon"] & \\ E_0 \arrow[r, "\epsilon"] & N \arrow[r] & 0\,, \end{tikzcd}$$

and this lift exists because F_0 is free and ϵ is surjective. Next, f_1 is taken to be a lift in the diagram

$$\begin{tikzcd} & F_1 \arrow[dl, dashed, "f_1"'] \arrow[d, "f_0\partial_1"] & \\ E_1 \arrow[r, "\partial_1"] & \ker(\epsilon) \arrow[r] & 0\,. \end{tikzcd}$$

Here $f_0\partial_1\colon E_1 \to F_0$ factors through $\ker(\epsilon)$ because $\epsilon(f_0\partial_1) = h\epsilon\partial_1 = h0 = 0$, and ∂_1 maps E_1 surjectively to $\ker(\epsilon)$ by exactness of $E_* \to N$ at E_0. We can also draw this step as follows.

$$\begin{tikzcd} F_1 \arrow[r, "\partial_1"] \arrow[d, dashed, "f_1"'] & F_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"] & M \arrow[d, "h"] \\ E_1 \arrow[r, "\partial_1"] & E_0 \arrow[r, "\epsilon"] & N \end{tikzcd}$$

For $n \geq 2$ we suppose, by induction, that we have chosen f_{n-1} and f_{n-2} making the solid arrow part of the following diagram commute.

$$\begin{tikzcd}
F_n \arrow[r, "\partial_n"] \arrow[d, dashed, "f_n"'] & F_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & F_{n-2} \arrow[d, "f_{n-2}"] \\
E_n \arrow[r, "\partial_n"] & E_{n-1} \arrow[r, "\partial_{n-1}"] & E_{n-2}
\end{tikzcd}$$

Then $\partial_{n-1} f_{n-1} \partial_n = f_{n-2} \partial_{n-1} \partial_n = f_{n-2} 0 = 0$, so $f_{n-1}\partial_n \colon F_n \to E_{n-1}$ factors through $\ker(\partial_{n-1})$. By exactness, this equals the image of $\partial_n \colon E_n \to E_{n-1}$, so we can define f_n as a lift in the following diagram.

$$\begin{tikzcd}
& F_n \arrow[dl, dashed, "f_n"'] \arrow[d, "f_{n-1}\partial_n"] & \\
E_n \arrow[r, "\partial_n"] & \ker(\partial_{n-1}) \arrow[r] & 0\,.
\end{tikzcd}$$

This completes the inductive step, and proves the existence of a chain map $f \colon F_* \to E_*$ covering h.

For uniqueness up to chain homotopy, suppose that $g \colon F_* \to E_*$ is second chain map covering h. We construct a chain homotopy $P \colon F_* \to E_{*+1}$ with

$$\partial_{n+1} P_n + P_{n-1} \partial_n = g_n - f_n$$

for each $n \geq 0$.

To start, P_0 is defined to be a lift in the diagram

$$\begin{tikzcd}
& F_0 \arrow[dl, dashed, "P_0"'] \arrow[d, "g_0 - f_0"] & \\
E_1 \arrow[r, "\partial_1"] & \ker(\epsilon) \arrow[r] & 0\,.
\end{tikzcd}$$

This works because $\epsilon(g_0 - f_0) = h\epsilon - h\epsilon = 0$.

Next, P_1 is defined to make the following diagram commute.

$$\begin{tikzcd}
& F_1 \arrow[dl, dashed, "P_1"'] \arrow[d, "g_1 - f_1 - P_0\partial_1"] & \\
E_2 \arrow[r, "\partial_2"] & \ker(\partial_1) \arrow[r] & 0\,.
\end{tikzcd}$$

This is possible, because

$$\partial_1(g_1 - f_1 - P_0\partial_1) = g_0\partial_1 - f_0\partial_1 - (g_0 - f_0)\partial_1 = 0\,.$$

For $n \geq 2$ we suppose that we have chosen $P_{n-1} \colon F_{n-1} \to E_n$ and $P_{n-2} \colon F_{n-2} \to E_{n-1}$ with $\partial_n P_{n-1} + P_{n-2}\partial_{n-1} = g_{n-1} - f_{n-1}$.

$$\begin{tikzcd}
& F_n \arrow[r, "\partial_n"] \arrow[dl, dashed, "P_n"'] \arrow[d, "g_n - f_n"] & F_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, "P_{n-1}"'] \arrow[d, "g_{n-1} - f_{n-1}"] & F_{n-2} \arrow[dl, "P_{n-2}"] \\
E_{n+1} \arrow[r, "\partial_{n+1}"'] & E_n \arrow[r, "\partial_n"'] & E_{n-1} &
\end{tikzcd}$$

Then

$$\begin{aligned}
\partial_n(g_n - f_n - P_{n-1}\partial_n) &= g_{n-1}\partial_{n-1} - f_{n-1}\partial_{n-1} - (\partial_n P_{n-1})\partial_n \\
&= g_{n-1}\partial_{n-1} - f_{n-1}\partial_{n-1} - (g_{n-1} - f_{n-1} - P_{n-2}\partial_{n-1})\partial_n \\
&= g_{n-1}\partial_{n-1} - f_{n-1}\partial_{n-1} - g_{n-1}\partial_{n-1} + f_{n-1}\partial_{n-1} + P_{n-2}0 \\
&= 0\,.
\end{aligned}$$

Hence we can define P_n as the following lift, which exists because F_n is free and $\operatorname{im}(\partial_{n+1}) = \ker(\partial_n)$ in E_n.

$$\begin{tikzcd} & F_n \arrow[dl, dashed, "P_n"'] \arrow[d, "g_n - f_n - P_{n-1}\partial_n"] & \\ E_{n+1} \arrow[r, "\partial_{n+1}"'] & \ker(\partial_n) \arrow[r] & 0\,. \end{tikzcd}$$

□

This leads to the following uniqueness result.

Corollary 4.13. *Let $\epsilon\colon C_* \to M$ and $\epsilon\colon D_* \to M$ be two free resolutions of the same R-module M. Then there exists a chain map $f\colon C_* \to D_*$ covering the identity of M, which is unique up to chain homotopy.*

$$\begin{tikzcd} C_* \arrow[rr, dashed, "f", "\simeq"'] \arrow[dr, "\epsilon"'] & & D_* \arrow[dl, "\epsilon"] \\ & M & \end{tikzcd}$$

Furthermore, f admits a chain homotopy inverse $g\colon D_* \to C_*$. In particular, f is a chain homotopy equivalence.

Proof. The comparison theorem applies with $F_* = C_*$, $E_* = D_*$ and $h = 1_M$, since C_* is free and $D_* \to M \to 0$ is exact. Hence there is a unique chain homotopy class of chain maps $f\colon C_* \to D_*$ covering 1_M.

The comparison theorem also applies with $F_* = D_*$, $E_* = C_*$ and $h = 1_M$, since D_* is free and $C_* \to M \to 0$ is exact. Hence there is a unique chain homotopy class of chain maps $g\colon D_* \to C_*$ covering 1_M.

The composite $gf\colon C_* \to C_*$ is a chain map covering the identity of M. The identity map $1_{C_*}\colon C_* \to C_*$ has the same property. By the comparison theorem, there is a chain homotopy P from gf to 1_{C_*}.

Finally, the composite $fg\colon D_* \to D_*$ is a chain map covering the identity of M. The identity map $1_{D_*}\colon D_* \to D_*$ has the same property. Hence there is a chain homotopy Q from fg to 1_{D_*}. This shows that f and g are chain homotopy inverses, so that f is a chain homotopy equivalence. □

5. September 7th lecture

5.1. The derived functors Tor and Ext.

Definition 5.1. Let G be a right R-module. For each left R-module M with canonical free resolution $\epsilon\colon C_* \to M$ and each $q \ge 0$ let

$$\operatorname{Tor}_q^R(G, M) = H_q(G \otimes_R C_*, 1 \otimes \partial)$$

be the q-th homology group of the chain complex

$$\cdots \to G \otimes_R C_{q+1} \xrightarrow{1\otimes\partial_{q+1}} G \otimes_R C_q \xrightarrow{1\otimes\partial_q} G \otimes_R C_{q-1} \longrightarrow \dots \longrightarrow G \otimes_R C_1 \xrightarrow{1\otimes\partial_1} G \otimes_R C_0 \to 0\,.$$

In other words, to calculate $\operatorname{Tor}_*^R(G, M)$ we replace M with C_* in $G \otimes_R M$, and compute the homology of the resulting chain complex. Functoriality of the canonical free resolution proves that $\operatorname{Tor}_q^R(G, M)$ is functorial in M.

Definition 5.2. Let G be a left R-module. For each left R-module M with canonical free resolution $\epsilon\colon C_* \to M$ and each $q \ge 0$ let

$$\operatorname{Ext}_R^q(M, G) = H^q(\operatorname{Hom}_R(C_*, G), \operatorname{Hom}(\partial, 1))$$

be the q-th cohomology group of the cochain complex

$$0 \to \operatorname{Hom}_R(C_0, G) \xrightarrow{\operatorname{Hom}(\partial_1,1)} \operatorname{Hom}_R(C_1, G) \longrightarrow$$
$$\dots \longrightarrow \operatorname{Hom}_R(C_{q-1}, G) \xrightarrow{\operatorname{Hom}(\partial_q,1)} \operatorname{Hom}_R(C_q, G) \xrightarrow{\operatorname{Hom}(\partial_{q+1},1)} \operatorname{Hom}_R(C_{q+1}, G) \longrightarrow \dots$$

In other words, to calculate $\operatorname{Ext}_R^*(M,G)$ we replace M with C_* in $\operatorname{Hom}_R(M,G)$, and compute the cohomology of the resulting cochain complex. Functoriality of the canonical free resolution proves that $\operatorname{Ext}^q_R(M,G)$ is contravariantly functorial in M.

Lemma 5.3. *There are natural isomorphisms*

$$\operatorname{Tor}_0^R(G,M) \cong G \otimes_R M$$

and

$$\operatorname{Hom}_R(M,G) \cong \operatorname{Ext}^0_R(G,M)\,.$$

Proof. The exactness of

$$C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} M \to 0$$

implies the exactness of

$$G \otimes_R C_1 \xrightarrow{1\otimes\partial_1} G \otimes_R C_0 \xrightarrow{1\otimes\epsilon} G \otimes_R M \to 0$$

and

$$0 \to \operatorname{Hom}_R(M,G) \xrightarrow{\operatorname{Hom}(\epsilon,1)} \operatorname{Hom}_R(C_0,G) \xrightarrow{\operatorname{Hom}(\partial_1,1)} \operatorname{Hom}_R(C_1,G)\,.$$

By definition, $\operatorname{Tor}_0^R(G,M)$ is the cokernel of $1\otimes\partial_1$, which $1\otimes\epsilon$ maps isomorphically to $G\otimes_R M$. Likewise, $\operatorname{Ext}^0_R(G,M)$ is the kernel of $\operatorname{Hom}(\partial_1,1)$, which receives the isomorphism $\operatorname{Hom}(\epsilon,1)$ from $\operatorname{Hom}_R(M,G)$. □

We call $\operatorname{Tor}_q^R(G,-)$ the q-th left derived functor of $G\otimes_R(-)$, and call $\operatorname{Ext}^q_R(G,-)$ the q-th right derived functor of $\operatorname{Hom}_R(-,G)$. By the uniqueness up to chain homotopy equivalence of free resolutions, we can also use any other free resolution (other than the canonical one) to calculate these functors.

Proposition 5.4. *Let $\epsilon\colon D_* \to M$ be any free resolution of a left R-module M. There are canonical isomorphisms*

$$\operatorname{Tor}_q^R(G,M) \cong H_q(G\otimes_R D_*)$$

and

$$\operatorname{Ext}^q_R(M,G) \cong H^q(\operatorname{Hom}_R(D_*,G))$$

for each $q \ge 0$.

Proof. There is a chain homotopy equivalence $f\colon C_* \to D_*$ covering M, which is unique up to chain homotopy. Let $g\colon D_* \to C_*$ denote a chain homotopy inverse. The induced chain homomorphism

$$1\otimes f\colon G\otimes_R C_* \longrightarrow G\otimes_R D_*$$

is then a chain homotopy equivalence, with chain homotopy inverse $1\otimes g$. In particular, $1\otimes f$ induces an isomorphism

$$(1\otimes f)_*\colon H_*(G\otimes_R C_*) \xrightarrow{\cong} H_*(G\otimes_R D_*)$$

and any two choices of f induce the same isomorphism. This is the first canonical isomorphism.

The induced chain homomorphism

$$\operatorname{Hom}(f,1)\colon \operatorname{Hom}_R(D_*,G) \longrightarrow \operatorname{Hom}_R(C_*,G)$$

is also a chain homotopy equivalence, with chain homotopy inverse $\operatorname{Hom}(g,1)$. In particular, $\operatorname{Hom}(f,1)$ induces an isomorphism

$$\operatorname{Hom}(f,1)^*\colon H^*(\operatorname{Hom}_R(D_*,G)) \xrightarrow{\cong} H^*(\operatorname{Hom}_R(C_*,G))$$

and any two choices of f induce the same isomorphism. This is the inverse of the second canonical isomorphism. □

Example 5.5. (1) When M is free, the very short resolution $C_* = M$ with $C_q = 0$ for $q > 0$ shows that $\operatorname{Tor}_q^R(G,M) = 0$ and $\operatorname{Ext}^q_R(M,G) = 0$ for $q > 0$.

(2) For $R = \mathbb{Z}$ and $M = \mathbb{Z}/m$ with $m \geq 1$, the short exact resolution with $C_0 = C_1 = \mathbb{Z}$ and $C_q = 0$ for $q > 1$ shows that

$$\mathrm{Tor}_1^{\mathbb{Z}}(G, \mathbb{Z}/m) \cong \ker(m\colon G \longrightarrow G)$$

and

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, G) \cong \mathrm{cok}(m\colon G \longrightarrow G)\,,$$

while $\mathrm{Tor}_q^{\mathbb{Z}}(G, \mathbb{Z}/m) = 0$ and $\mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/m, G) = 0$ for all $q > 1$.

(3) For $R = \mathbb{Z}[x]$ and $M = \mathbb{Z}/m$, the resolution with $C_0 = C_2 = R$, $C_1 = R \oplus R$ and $C_q = 0$ for $q > 2$ shows that $\mathrm{Tor}_2^R(\mathbb{Z}/m, \mathbb{Z}/m) \cong \mathbb{Z}/m$ and $\mathrm{Ext}^2_R(\mathbb{Z}/m, \mathbb{Z}/m) \cong \mathbb{Z}/m$, while $\mathrm{Tor}_q^R(\mathbb{Z}/m, M) = 0$ and $\mathrm{Ext}^q_R(\mathbb{Z}/m, M) = 0$ for $q > 2$.

(4) When R is a free ideal ring the short exact sequence (4) is a free resolution of $H_n(C_*)$. Hence, in these cases $\mathrm{Tor}_q^R(G, H_n(C_*)) = 0$ and $\mathrm{Ext}^q_R(H_n(C_*), G) = 0$ for all $q > 1$.

Lemma 5.6. *When F is a field, $\mathrm{Tor}_q^F(G, M) = 0$ and $\mathrm{Ext}^q_F(M, G) = 0$ for all $q > 0$.*

Proof. Each module over a field is free. □

Lemma 5.7. *When R is a free ideal ring, $\mathrm{Tor}_q^R(G, M) = 0$ and $\mathrm{Ext}^q_R(M, G) = 0$ for all $q > 1$.*

Proof. For these rings each R-module M admits a short exact resolution

$$0 \to D_1 \xrightarrow{\partial_1} D_0 \xrightarrow{\epsilon} M \to 0\,.$$

Here D_0 can be any free R-module with a surjection ϵ to M. We let $D_1 = \ker(\epsilon)$, with ∂_1 the inclusion. Hence D_1 is a submodule of a free module, which, by hypothesis, is always free. This proves that we can calculate the groups $\mathrm{Tor}_*^R(G, M)$ as the homology groups of the complex

$$0 \to G \otimes_R D_1 \xrightarrow{1\otimes\partial_1} G \otimes_R D_0 \to 0$$

concentrated in degrees 0 and 1. In particular, these groups vanish for $q > 1$. Likewise, we can calculate the groups $\mathrm{Ext}^*_R(M, G)$ as the cohomology groups of the complex

$$0 \to \mathrm{Hom}_R(D_0, G) \xrightarrow{\mathrm{Hom}(\partial_1,1)} \mathrm{Hom}_R(D_1, G) \to 0$$

concentrated in degrees 0 and 1. In particular, these groups vanish for $q > 1$. □

For these rings, which include PIDs, hence also the integers, we often simplify the notation and write $\mathrm{Tor}^R(G, M) = \mathrm{Tor}_1^R(G, M)$ and $\mathrm{Ext}_R(M, G) = \mathrm{Ext}^1_R(M, G)$. When $R = \mathbb{Z}$, or is otherwise implicitly understood, we can simplify the notation further and write $\mathrm{Tor}(G, M) = \mathrm{Tor}_1^{\mathbb{Z}}(G, M)$ and $\mathrm{Ext}(M, G) = \mathrm{Ext}^1_{\mathbb{Z}}(M, G)$. The former is sometimes called the torsion product, and denoted $G * M$. The latter classifies extensions of M by G (up to a suitable notion of isomorphism).

Proof of Proposition 4.7. We can calculate $\mathrm{Tor}_q^R(G, H_n(C_*))$ and $\mathrm{Ext}^q_R(H_n(C_*), G)$ for $q \in \{0, 1\}$ using the free resolution

$$0 \to B_n \xrightarrow{\iota_n} Z_n \xrightarrow{\pi_n} H_n(C_*) \to 0$$

of $H_n(C_*)$, as explained in the proof of the previous lemma. □

5.2. **Field coefficients.**

Proposition 5.8. *When F is a field, and C_* is any chain complex of F-modules (= F-vector spaces), the natural homomorphism*

$$h\colon H^n(\mathrm{Hom}_F(C_*, F)) \xrightarrow{\cong} \mathrm{Hom}_F(H_n(C_*), F)$$

is an isomorphism.

Proof. This is now a special case of Theorem 4.8. □

5.3. **Integer coefficients.** Let $R = \mathbb{Z}$ and consider an abelian group M. Let $T \subset M$ be the torsion subgroup, consisting of the elements of finite order. The quotient group $F = M/T$ is then torsion-free. If M is finitely generated then T is a finite group, and F is free. In this case the short exact sequence

$$0 \to T \longrightarrow M \longrightarrow F \to 0$$

splits, so that $M \cong T \oplus F$.

Lemma 5.9. (1) *If T is finite, then* $\operatorname{Hom}(T, \mathbb{Z}) = 0$ *and* $\operatorname{Ext}(T, \mathbb{Z}) \cong T$.
(2) *If F is finitely generated and free, then* $\operatorname{Hom}(F, \mathbb{Z}) \cong F$ *and* $\operatorname{Ext}(F, \mathbb{Z}) = 0$.
(3) *If $M \cong T \oplus F$, then* $\operatorname{Hom}(M, \mathbb{Z}) \cong F$ *and* $\operatorname{Ext}(M, \mathbb{Z}) \cong T$.

(This will be clear from the coming week's homework.)

Proposition 5.10. *Let C_* be a chain complex of free abelian groups, with finitely generated homology groups $H_m(C_*)$ for $m \in \{n-1, n\}$. Let $T_m \subset H_m(C_*)$ be the torsion subgroup, and let $F_m = H_m(C_*)/T_m$ be the (torsion-)free quotient group. Then*

$$H^n(C_*) = H^n(\operatorname{Hom}(C_*, \mathbb{Z})) \cong T_{n-1} \oplus F_n\,.$$

Example 5.11. If $H_*(C_*) = (\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}, 0, ...)$ then $H^*(C_*) = (\mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z}, 0, ...)$.

Similar results apply for R any principal ideal domain (PID), where $T \subset M$ is taken to be the submodule of R-torsion elements, i.e., the elements $x \in M$ with $rx = 0$ for some nonzero $r \in R$.

5.4. **Preservation.**

Proposition 5.12. *Let R be a free ideal ring, let (C_*, ∂) and (D_*, ∂) be complexes of free R-modules, and let $f \colon C_* \to D_*$ be a quasi-isomorphism. Then the induced homomorphisms*

$$f_* \colon H_n(C_* \otimes_R G) \longrightarrow H_n(D_* \otimes_R G)$$

and

$$f^* \colon H^n(\operatorname{Hom}_R(D_*, G)) \longrightarrow H^n(\operatorname{Hom}_R(C_*, G))$$

are isomorphisms, for every R-module G.

Proof. The short exact sequences in the universal coefficient theorem are natural, meaning that the diagrams

$$\begin{tikzcd}
0 \arrow[r] & G \otimes_R H_n(C_*) \arrow[r, "i"] \arrow[d, "1\otimes f_*"', "\cong"] & H_n(G \otimes_R C_*) \arrow[r] \arrow[d, "f_*"'] & \operatorname{Tor}_1^R(G, H_{n-1}(C_*)) \arrow[r] \arrow[d, "\operatorname{Tor}(1,f_*)"', "\cong"] & 0 \\
0 \arrow[r] & G \otimes_R H_n(D_*) \arrow[r, "i"] & H_n(G \otimes_R D_*) \arrow[r] & \operatorname{Tor}_1^R(G, H_{n-1}(D_*)) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1_R(H_{n-1}(D_*), G) \arrow[r] \arrow[d, "\operatorname{Ext}(f_*,1)"', "\cong"] & H^n(\operatorname{Hom}_R(D_*, G)) \arrow[r, "h"] \arrow[d, "f^*"'] & \operatorname{Hom}(H_n(D_*), G) \arrow[r] \arrow[d, "\operatorname{Hom}(f_*,1)"', "\cong"] & 0 \\
0 \arrow[r] & \operatorname{Ext}^1_R(H_{n-1}(C_*), G) \arrow[r] & H^n(\operatorname{Hom}_R(C_*, G)) \arrow[r, "h"] & \operatorname{Hom}(H_n(C_*), G) \arrow[r] & 0
\end{tikzcd}$$

commute. By assumption, $f_* \colon H_m(C_*) \to H_m(D_*)$ is an isomorphism for $m \in \{n-1, n\}$, which, by functoriality, implies that $1 \otimes f_*$, $\operatorname{Tor}(1, f_*)$, $\operatorname{Hom}(f_*, 1)$ and $\operatorname{Ext}(f_*, 1)$ are isomorphisms. It follows, by a special case of the Five Lemma, that $f_* \colon H_n(G \otimes_R C_*) \to H_n(G \otimes_R D_*)$ and $f^* \colon H^n(\operatorname{Hom}_R(D_*, G)) \to H^n(\operatorname{Hom}_R(C_*, G))$ are isomorphisms. $\square$

6.1. **Cohomology of spaces.** ((ETC: See Hatcher, pages 197–204.))

For $n \ge 0$ let
$$\Delta^n = \{(x_0, \dots, x_n) \mid x_i \ge 0, \sum_{i=0}^n x_i = 1\}$$
be the standard n-simplex in $\bigoplus_{i=0}^n \mathbb{R}$. This is the convex hull of the $(n+1)$ vertices $v_0, \dots, v_n$, where v_i has coordinates $(v_i)_i = 1$ and $(v_i)_j = 0$ for $i \ne j$. We write $\Delta^n = [v_0, \dots, v_n]$ for this convex hull. For any space X, a singular n-simplex in X is a map
$$\sigma \colon \Delta^n \longrightarrow X$$
and the singular n-chains is the free abelian group
$$C_n(X) = \mathbb{Z}\{\sigma \colon \Delta^n \to X\} \cong \bigoplus_\sigma \mathbb{Z}$$
consisting of finite linear combinations $\alpha = \sum_\sigma g_\sigma \sigma$ of n-simplices, with integer coefficients $g_\sigma \in \mathbb{Z}$.

For each $0 \le i \le n$ we identify the face
$$[v_0, \dots, \widehat{v_i}, \dots, v_n] \subset \Delta^n ,$$
opposite to the i-th vertex, with $\Delta^{n-1} = [v_0, \dots, v_{n-1}]$ by the unique affine linear isomorphism that preserves the ordering of the vertices. In this way we view the restricted map
$$\sigma|[v_0, \dots, \widehat{v_i}, \dots, v_n]$$
as a singular $(n-1)$-simplex in X, and form the boundary chain
$$\partial_n \sigma = \sum_{i=0}^n (-1)^i \sigma|[v_0, \dots, \widehat{v_i}, \dots, v_n]$$
as the alternating sum. This definition extends to linear homomorphisms
$$\dots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \dots$$
and one can check that $\partial_n \partial_{n+1} = 0$, since each term $\sigma|[v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_{n+1}]$ appears twice, but with opposite signs. This defines the singular chain complex $(C_*(X), \partial)$ of X.

For any abelian group G, we can form
$$C_n(X; G) = G \otimes C_n(X) \cong \bigoplus_{\sigma \colon \Delta^n \to X} G$$
consisting of finite linear combinations $\sum_\sigma g_\sigma \sigma$, now with coefficients $g_\sigma \in G$. The resulting chain complex
$$\dots \to G \otimes C_{n+1}(X) \xrightarrow{1 \otimes \partial_{n+1}} G \otimes C_n(X) \xrightarrow{1 \otimes \partial_n} G \otimes C_{n-1}(X) \to \dots$$
can also be written
$$\dots \to C_{n+1}(X; G) \xrightarrow{\partial_{n+1}} C_n(X; G) \xrightarrow{\partial_n} C_{n-1}(X; G) \to \dots .$$
By definition,
$$H_n(X; G) = H_n(G \otimes C_*(X), 1 \otimes \partial) = H_n(C_*(X; G), \partial)$$
is the n-th singular homology group of X with coefficients in G. This theory was discussed in Section 2.2 of Hatcher's book, on pages 153–155.

Likewise, for any abelian group G we can form
$$C^n(X; G) = \mathrm{Hom}(C_n(X), G) \cong \prod_{\sigma \colon \Delta^n \to X} G$$
consisting of linear homomorphisms $\phi \colon C_n(X) \to G$, or equivalently, of functions
$$\phi \colon \{\sigma \colon \Delta^n \to X\} \longrightarrow G .$$
We call ϕ an n-cochain on X with values in G. The induced coboundary map
$$\delta^n \colon C^n(X; G) \longrightarrow C^{n+1}(X; G)$$

is $\mathrm{Hom}(\partial_{n+1}, 1)$, sending $\phi\colon C_n(X) \to G$ to the composite

$$\delta^n\phi = \phi\partial_{n+1}\colon C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\phi} G\,.$$

Explicitly,

$$(\delta\phi)(\sigma) = \sum_{i=0}^{n+1}(-1)^i\phi(\sigma|[v_0, \dots, \widehat{v_i}, \dots, v_{n+1}])$$

for each singular $(n+1)$-simplex σ in X.

Lemma 6.1. $\delta^n\delta^{n-1} = 0$.

Proof. For each $(n-1)$-cochain ϕ, the composite

$$\delta^n\delta^{n-1}\phi = \phi\partial_n\partial_{n+1}\colon C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\phi} G$$

is zero, since $\partial_n\partial_{n+1} = 0$. □

Hence we obtain a cochain complex

$$\dots \to \mathrm{Hom}(C_{n-1}(X), G) \xrightarrow{\mathrm{Hom}(\partial_n,1)} \mathrm{Hom}(C_n(X), G) \xrightarrow{\mathrm{Hom}(\partial_{n+1},1)} \mathrm{Hom}(C_{n+1}(X), G) \to \dots$$

which can also be written

$$\dots \to C^{n-1}(X; G) \xrightarrow{\delta^{n-1}} C^n(X; G) \xrightarrow{\delta^n} C^{n+1}(X; G) \to \dots\,.$$

By definition,

$$H^n(X; G) = H^n(\mathrm{Hom}(C_*(X), G), \mathrm{Hom}(\partial, 1)) = H^n(C^*(X; G), \delta)$$

is the n-th singular homology group of X with coefficients in G. More explicitly

$$H^n(X; G) = \frac{\ker(\delta^n\colon C^n(X; G) \to C^{n+1}(X; G))}{\mathrm{im}(\delta^{n-1}\colon C^{n-1}(X; G) \to C^n(X; G))}\,.$$

The elements of

$$Z^n(X; G) = \ker(\delta^n\colon C^n(X; G) \to C^{n+1}(X; G))$$

are called n-cocycles, and the elements of

$$B^n(X; G) = \mathrm{im}(\delta^{n-1}\colon C^{n-1}(X; G) \to C^n(X; G))$$

are called n-coboundaries. We have $B^n(X; G) \subset Z^n(X; G) \subset C^n(G; G)$ and $Z^n(X; G)/B^n(X; G) = H^n(X; G)$. An n-cocycle is a homomorphism $\phi\colon C_n(X) \to G$ with $\delta\phi = \phi\partial = 0$, i.e., which vanishes on the boundaries $B_n(X) \subset C_n(X)$. An n-coboundary is a homomorphism $\phi\colon C_n(X) \to G$ of the form $\phi = \delta\psi = \psi\partial$ for some $(n-1)$-cochain $\psi\colon C_{n-1}(X) \to G$. In other words, this is a cochain whose value on a simplex σ only depends on its boundary $\partial\sigma$.

Example 6.2. $H^0(*; G) \cong G$ and $H^n(*; G) = 0$ for $n \neq 0$.

6.2. **Functoriality.** Let $f\colon X \to Y$ be a map of spaces. The induced chain map

$$f_\#\colon C_*(X) \longrightarrow C_*(Y)$$

is given in degree n by mapping $\sigma\colon \Delta^n \to X$ to $f\sigma\colon \Delta^n \to Y$. The dual cochain map

$$f^\#\colon C^*(Y; G) \longrightarrow C^*(X; G)$$

is given in degree n by mapping $\phi\colon C_n(Y) \to G$ to $\phi f\colon C_n(X) \to G$. Note how the source and target are interchanged. Passing to cohomology we obtain an induced homomorphism

$$f^*\colon H^n(Y; G) \longrightarrow H^n(X; G)$$

in each degree n. This makes $H^n(X; G)$ a contravariant functor in X, meaning that, if $g\colon Y \to Z$ is another map, then

$$(gf)^* = f^*g^*\colon H^*(Z; G) \longrightarrow H^*(X; G)$$

and if $f = 1_X$, then f^* is the identity homomorphism.

Proposition 6.3. *Let $X = \coprod_i X_i$ be a coproduct of spaces, with structure maps $\iota_i \colon X_i \to X$. The product*

$$\prod_i \iota_i^* \colon H^n(X;G) \xrightarrow{\cong} \prod_i H^n(X_i;G)$$

is an isomorphism, for each n and G.

Proof. Applying $\mathrm{Hom}(-,G)$ turns coproducts into products. Hence the chain isomorphism $\bigoplus_i C_*(X_i) \cong C_*(X)$ induces a cochain isomorphism $C^*(X;G) \cong \prod_i C^*(X_i;G)$. Furthermore, forming the homology of complexes commutes with products (and coproducts). □

6.3. **Homotopy invariance.**

Theorem 6.4. *If $f \simeq g \colon X \to Y$, then $f^* = g^* \colon H^n(Y;G) \to H^n(X;G)$, for each n and G.*

Proof. Write the prism $[0,1] \times \Delta^n$ as the convex hull of points $u_i = (v_i, 0)$ and $w_i = (v_i, 1)$ in $\mathbb{R} \times \mathbb{R}^\infty$. (Hatcher writes v_i in place of u_i here.) This is the union of the $(n+1)$-simplices

$$[u_0, \dots, u_i, w_i, \dots, w_n]$$

for $0 \le i \le n$, each of which is identified with

$$[v_0, \dots, v_{n+1}]$$

by the affine linear isomorphism that preserves the ordering of the vertices. Let

$$P_n(\Delta^n) = \sum_{i=0}^{n} (-1)^i [u_0, \dots, u_i, w_i, \dots, w_n] \,.$$

Let $F \colon [0,1] \times X \to Y$ be a homotopy from f to g, so that $F(0,x) = f(x)$ and $F(1,x) = g(x)$. The prism operator $P_n \colon C_n(X) \to C_{n+1}(Y)$ maps $\sigma \colon \Delta^n \to X$ to

$$P_n(\sigma) = \sum_{i=0}^{n} (-1)^i F(1 \times \sigma) | [u_0, \dots, u_i, w_i, \dots, w_n]$$

Here $F(1 \times \sigma) \colon [0,1] \times \Delta^n \to Y$. Then

$$\partial_{n+1} P_n + P_{n-1} \partial_n = g_n - f_n$$

for all n. Let

$$P^{n+1} = \mathrm{Hom}(P_n, 1) \colon C^{n+1}(Y;G) \to C^n(X;G)$$

be the dual homomorphism, given by

$$P^{n+1}(\phi)(\sigma) = \sum_{i=0}^{n} (-1)^i \phi(F(1 \times \sigma) | [u_0, \dots, u_i, w_i, \dots, w_n]) \,.$$

Then

$$P^{n+1} \delta^n + \delta^{n-1} P^n = g^n - f^n$$

for all n, so that $P^\# \colon C^{*+1}(Y;G) \to C^*(X;G)$ is a cochain homotopy from $g^\#$ to $f^\#$. It follows, as in the case for homology, that the induced homomorphisms g^* and f^* are equal. □

6.4. **The universal coefficient theorem for spaces.** Given any ring R, the chain complex $C_*(X;R) = R \otimes C_*(X)$ consists of free R-modules. For any R-module G there are natural isomorphisms

$$C_*(X;G) \cong G \otimes_R C_*(X;R)$$

and

$$C^*(X;G) \cong \mathrm{Hom}_R(C_*(X;R), G) \,.$$

Applying Theorem 4.8 to $C_*(X;R)$ we obtain the topological version of the universal coefficient theorem.

Theorem 6.5 (Topological UCT)**.** *Let R be a free ideal ring, let X be any space, and let G be an R-module. There are natural short exact sequences*

$$0 \to G \otimes_R H_n(X;R) \xrightarrow{i} H_n(X;G) \longrightarrow \mathrm{Tor}_1^R(G, H_{n-1}(X;R)) \to 0$$

and

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R), G) \longrightarrow H^n(X;G) \xrightarrow{h} \mathrm{Hom}_R(H_n(X;R), G) \to 0\,.$$

Each of these admits a splitting.

Corollary 6.6. *Let F be a field, let X be any space, and let G be an F-module (= F-vector space). The natural homomorphisms*

$$i\colon G \otimes_F H_n(X;F) \longrightarrow H_n(X;G)$$

and

$$h\colon H^n(X;G) \longrightarrow \mathrm{Hom}_F(H_n(X;F), G)$$

are isomorphisms. In particular,

$$H^n(X;F) \cong \mathrm{Hom}_F(H_n(X;F), F)$$

is the functional dual of $H_n(X;F)$.

Example 6.7. $H^0(X;G) \cong \mathrm{Hom}(H_0(X), G) \cong \prod_{\pi_0(X)} G$ and $H^1(X;G) \cong \mathrm{Hom}(H_1(X), G)$. For path-connected (X, x_0) this is isomorphic to $\mathrm{Hom}(\pi_1(X, x_0), G)$ in the category Gp.

6.5. **Relative cohomology.** By a pair (X, A) of spaces, we mean a space X and a subspace $A \subset X$. The inclusion defines a chain map $i\colon C_*(A) \to C_*(X)$. We define the relative chain complex $C_*(X, A)$ as the quotient complex, so that there is a short exact sequence of chain complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{n+1}(A) \arrow[r, "i"] \arrow[d, "\partial_{n+1}"'] & C_{n+1}(X) \arrow[r, "j"] \arrow[d, "\partial_{n+1}"'] & C_{n+1}(X,A) \arrow[r] \arrow[d, "\partial_{n+1}"'] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\partial_n"'] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial_n"'] & C_n(X,A) \arrow[r] \arrow[d, "\partial_n"'] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i"] \arrow[d] & C_{n-1}(X) \arrow[r, "j"] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We let $C^n(X, A; G) = \mathrm{Hom}(C_n(X, A), G)$. Note that the free abelian group on the n-simplices in X that do not lie in A is a subgroup of $C_n(X)$ that maps isomorphically to $C_n(X, A)$. Hence the latter group is free, and each row in the diagram above admits a splitting, but the splittings do not form a chain map. It follows that applying $\mathrm{Hom}(-, G)$ to the diagram above gives a short exact sequence of cochain complexes

(5)
$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C^{n-1}(X, A; G) \arrow[r, "j^*"] \arrow[d, "\delta^{n-1}"] & C^{n-1}(X; G) \arrow[r, "i^*"] \arrow[d, "\delta^{n-1}"] & C^{n-1}(A; G) \arrow[r] \arrow[d, "\delta^{n-1}"] & 0 \\
0 \arrow[r] & C^{n}(X, A; G) \arrow[r, "j^*"] \arrow[d, "\delta^{n}"] & C^{n}(X; G) \arrow[r, "i^*"] \arrow[d, "\delta^{n}"] & C^{n}(A; G) \arrow[r] \arrow[d, "\delta^{n}"] & 0 \\
0 \arrow[r] & C^{n+1}(X, A; G) \arrow[r, "j^*"] \arrow[d] & C^{n+1}(X; G) \arrow[r, "i^*"] \arrow[d] & C^{n+1}(A; G) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Again, this splits in each degree n, but the splittings do not form a chain map. The left hand column is thus a subcomplex of $C^*(X; G)$. Its cohomology of defines the relative cohomology groups

$$H^n(X, A; G) = H^n(C^*(X, A; G), \delta) = \frac{\ker(\delta^n \colon C^n(X, A; G) \to C^{n+1}(X, A; G))}{\mathrm{im}(\delta^{n-1} \colon C^{n-1}(X, A; G) \to C^n(X, A; G))}\,.$$

Here the relative n-cocycles $Z^n(X, A; G) = \ker(\delta^n \colon C^n(X, A; G) \to C^{n+1}(X, A; G))$ are the n-cochains $\phi \colon C_n(X) \to G$ that vanish on boundaries and on chains in A. The relative n-coboundaries $B^n(X, A; G) = \mathrm{im}(\delta^{n-1} \colon C^{n-1}(X, A; G) \to C^n(X, A; G))$ have the form $\phi = \psi\partial_n$ where $\psi \colon C_{n-1}(X) \to G$ vanishes on chains in A.

A map $f \colon (X, A) \to (Y, B)$ of pairs is a map $f \colon X \to Y$ such that $f(A) \subset B$. We may write $f|A \colon A \to B$ for the restricted map of subspaces. The chain map $f_\# \colon C_*(X) \to C_*(Y)$ takes $C_*(A)$ to $C_*(B)$ and induces a chain map $f_\# \colon C_*(X, A) \to C_*(Y, B)$. Dually, we obtain a cochain map

$$f^\# \colon C^*(Y, B; G) \longrightarrow C^*(X, A; G)$$

and induced homomorphisms

$$f^* \colon H^n(Y, B; G) \longrightarrow H^n(X, A; G)$$

making the relative cohomology groups $H^n(X, A; G)$ contravariantly functorial in (X, A).

6.6. **The long exact sequence of a pair.** The long exact sequence in cohomology associated to

$$0 \to C^*(X, A; G) \xrightarrow{j^*} C^*(X; G) \xrightarrow{i^*} C^*(A; G) \to 0$$

takes the form

$$\cdots \to H^{n-1}(A) \xrightarrow{\delta} H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \to \dots$$

where the connecting homomorphism δ increases cohomological degree by 1. It maps the cohomology class $[\phi]$ of an n-cocycle $\phi \colon C_n(A) \to G$ to the class of $\psi\partial_{n+1} \colon C_{n+1}(X) \to G$, where $\psi \colon C_n(X) \to G$ is any n-cochain extending ϕ (so that $\phi = \psi i = i^*\psi$). Then $\psi\partial_{n+1}$ vanishes on chains in A, hence lies in $C^{n+1}(X, A; G) \subset C^{n+1}(X; G)$. It is a relative cocycle, and its cohomology class is

$$\delta[\phi] = [\psi\partial_{n+1}]\,.$$

There is a variant of the universal coefficient theorem for relative (co-)homology, and the following compatibility result.

Lemma 6.8. *The square*

$$\begin{array}{ccc} H^n(A;G) & \xrightarrow{\delta} & H^{n+1}(X,A;G) \\ \downarrow h & & \downarrow h \\ \mathrm{Hom}(H_n(A),G) & \xrightarrow{\partial^*} & \mathrm{Hom}(H_{n+1}(X,A),G) \end{array}$$

commutes.

((ETC: See Hatcher, pages 200-201 for a proof.))

6.7. **Excision.** The computability of (co-)homology largely comes from the following fact.

Proposition 6.9. *For "good pairs" (X,A) the quotient map $q\colon (X,A) \to (X/A,*)$ induces isomorphisms*

$$q_*\colon H_n(X,A) \cong H_n(X/A,*)$$

and

$$q^*\colon H^n(X/A,*;G) \cong H^n(X,A;G)$$

for each n and G. Here $*$ denotes the point A/A in X/A.

Here "good pairs" include the cases where X is a CW complex and A is a subcomplex of X, and, more generally, for pairs satisfying the homotopy extension property (cf. pages 14–17 in Chapter 0). In combination with the long exact sequence for the pair, this lets us reduce questions about X to questions about A and X/A, which may be simpler spaces.

A closely related result is the excision theorem.

Theorem 6.10. *Let $Z \subset A \subset X$, where the closure of Z is contained in the interior of X. Equivalently, with $B = X - Z$, X is the union of the interior of A and the interior of B. Then the inclusion $(X-Z, A-Z) \to (X,A)$ induces an isomorphism*

$$\mathrm{exc}\colon H^n(X,A;G) \xrightarrow{\cong} H^n(X-Z, A-Z;G)$$

for each n and G.

The following is equivalent, with $B = X - Z$.

Theorem 6.11. *Let $A, B \subset X$, where X is the union of the interior of A and the interior of B. Then the inclusion $(B, A\cap B) \to (X,A)$ induces an isomorphism*

$$\mathrm{exc}\colon H^n(X,A;G) \xrightarrow{\cong} H^n(B, A\cap B;G)$$

for each n and G.

Both statements follow from the corresponding statements in integral homology, using the naturality of the universal coefficient theorem.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X,A),G) & \longrightarrow & H^n(X,A;G) & \xrightarrow{h} & \mathrm{Hom}(H_n(X,A),G) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\mathrm{exc} & & \downarrow\cong & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(B,A\cap B),G) & \longrightarrow & H^n(B,A\cap B;G) & \xrightarrow{h} & \mathrm{Hom}(H_n(B,A\cap B),G) & \longrightarrow & 0 \end{array}$$

Proof of Proposition 6.9. For a pair (X,A), let the cone on A be

$$CA = \frac{[0,1]\times A}{\{0\}\times A}$$

with vertex v. This is a contractible space. There is an embedding

$$\iota\colon A \longrightarrow CA$$

taking a to $(a,1)$. By the mapping cone of $i\colon A \to X$ we mean the pushout $Ci = X \cup_A CA$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\iota"'] & X \arrow[d] \\ CA \arrow[r] & X \cup_A CA\,, \end{tikzcd}$$

with $X/A \cong (X \cup_A CA)/CA$. The closure of $\{v\}$ lies in the interior of CA within $X \cup_A CA$, so there is an excision isomorphism

$$H_n(X \cup_A CA - \{v\}, CA - \{v\}) \xrightarrow{\cong} H_n(X \cup_A CA, CA),.$$

Furthermore, (X, A) is a deformation retract of $(X \cup_A CA - \{v\}, CA - \{v\})$, so the inclusion induces an isomorphism

$$H_n(X, A) \xrightarrow{\cong} H_n(X \cup_A CA - \{v\}, CA - \{v\})\,.$$

If (X, A) has the homotopy extension property, then it follows formally that $(X \cup_A CA, CA)$ also has the homotopy extension property. Since CA is contractible, it follows (Hatcher, Proposition 0.17) that the quotient map $X \cup_A CA \to X \cup_A CA/CA \cong X/A$ is a homotopy equivalence. This restricts to a homotopy equivalence $CA \to CA/CA \cong A/A$. Hence there is an isomorphism

$$H_n(X \cup_A CA, CA) \xrightarrow{\cong} H_n(X/A, A/A)\,.$$

The proof for cohomology follows the same lines. □

6.8. **Reduced cohomology.** For each space X we define an augmentation $\epsilon\colon C_0(X) \to \mathbb{Z}$ by $\epsilon(\sigma) = 1$ for each 0-simplex $\sigma\colon \Delta^0 \to X$. For each 1-simplex $\tau\colon \Delta^1 \to X$ we have $\partial_1(\tau) = \tau|[v_1] - \tau|[v_0]$, so $\epsilon\partial_1 = 1 - 1 = 0$. Hence

$$\dots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is a chain complex, with n-th homology the reduced homology group $\tilde{H}_n(X)$. When $X = *$ is a point, $\tilde{H}_n(X) = 0$ for all n. For non-empty spaces X we have $\tilde{H}_n(X) \cong H_n(X)$ when $n \neq 0$, and $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. For based spaces (X, x_0) the composite $\tilde{H}_n(X) \to H_n(X) \to H_n(X, \{x_0\})$ is an isomorphism, for all n.

Applying $\mathrm{Hom}(-, G)$ we obtain a cochain complex

$$0 \to G \xrightarrow{\eta} C^0(X; G) \xrightarrow{\delta^0} C^1(X; G) \xrightarrow{\delta^1} C^2(X; G) \to \dots$$

where the coaugmentation η maps $g \in G$ to the 0-cocycle $\eta(g)$ given by $\sigma \mapsto g$ for each $\sigma\colon \Delta^0 \to X$. The n-th cohomology of this complex is the reduced cohomology group $\tilde{H}^n(X; G)$. When $X = *$ is a point, $\tilde{H}^n(X; G) = 0$ for all n. For non-empty X we have $H^n(X; G) \cong \tilde{H}^n(X; G)$ when $n \neq 0$, and $H^0(X; G) \cong G \oplus \tilde{H}^0(X; G)$. In the based case, the composite $H^n(X, \{x_0\}; G) \to H^n(X; G) \to \tilde{H}^n(X; G)$ is an isomorphism.

The augmented chain complex $\epsilon\colon C_*(X) \to \mathbb{Z}$ is functorial in X, so the coaugmented cochain complex $\eta\colon G \to C^*(X; G)$ is contravariantly functorial in X, and so are the reduced cohomology groups $\tilde{H}^n(X; G)$.

There is a variant of the long exact sequence of a pair for reduced cohomology, inserting the row

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & G \arrow[r, "1"] \arrow[d, "\eta"'] & G \arrow[r] \arrow[d, "\eta"'] & 0 \\ 0 \arrow[r] & C^0(X, A; G) \arrow[r, "j^*"] \arrow[d, "\delta^0"'] & C^0(X; G) \arrow[r, "i^*"] \arrow[d, "\delta^0"'] & C^0(A; G) \arrow[r] \arrow[d, "\delta^0"'] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

at the top of diagram (5). This replaces $H^0(X;G)$ and $H^0(A;G)$ with $\tilde{H}^0(X;G)$ and $\tilde{H}^0(A;G)$, respectively, but does not alter the other groups. The long exact sequence in reduced cohomology thus begins

$$0 \to H^0(X,A;G) \xrightarrow{j^*} \tilde{H}^0(X;G) \xrightarrow{i^*} \tilde{H}^0(A;G) \xrightarrow{\delta} H^1(X,A;G) \to \dots .$$

There is a variant of the universal coefficient theorem for reduced (co-)homology.

Proposition 6.12. *For each "good pair" of spaces (X,A) there is a natural long exact sequence*

$$\dots \to \tilde{H}^{n-1}(A;G) \xrightarrow{\delta} \tilde{H}^n(X/A;G) \xrightarrow{q^*} \tilde{H}^n(X;G) \xrightarrow{i^*} \tilde{H}^n(A;G) \xrightarrow{\delta} \tilde{H}^{n+1}(X/A;G) \to \dots .$$

6.9. **The long exact sequence of a triple.** By a triple (X,A,B) of spaces we mean a pair (X,A) with another subspace $B \subset A$. There is a short exact sequence of chain complexes

$$0 \to C_*(A,B) \xrightarrow{i} C_*(X,B) \xrightarrow{j} C_*(X,A) \to 0$$

and each $C_n(X,A)$ is free, so in each degree the short exact sequence of abelian groups is split. Hence the sequence of cochain complexes

$$0 \to C^*(X,A;G) \xrightarrow{j^*} C^*(X,B;G) \xrightarrow{i^*} C^*(A,B;G) \to 0$$

remains short exact. The associated long exact sequence in cohomology takes the following form.

$$\dots \to H^{n-1}(A,B) \xrightarrow{\delta} H^n(X,A;G) \xrightarrow{j^*} H^n(X,B;G) \xrightarrow{i^*} H^n(A,B;G) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \dots$$

Lemma 6.13. *The connecting homomorphism*

$$\delta\colon H^n(A,B;G) \longrightarrow H^{n+1}(X,A;G)$$

of the triple (X,A,B) factors as the composite

$$H^n(A,B;G) \xrightarrow{j^*} H^n(A;G) \xrightarrow{\delta} H^{n+1}(X,A;G) .$$

Proof. This is a consequence of naturality of the connecting homomorphism, for the following map of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & C^*(X,A;G) \arrow[r, "j^*"] \arrow[d, "1"'] & C^*(X,B;G) \arrow[r, "i^*"] \arrow[d] & C^*(A,B;G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^*(X,A;G) \arrow[r, "j^*"] & C^*(X;G) \arrow[r, "i^*"] & C^*(A;G) \arrow[r] & 0
\end{tikzcd}$$

□

6.10. **Cellular cohomology.** For a CW complex X, with n-skeleton X^n, the cellular n-chains are given by

$$W_n(X) = H_n(X^n, X^{n-1}) ,$$

which is a free abelian group generated by the set of n-cells in X. The cellular boundary

$$\partial_n\colon W_n(X) \longrightarrow W_{n-1}(X)$$

is the connecting homomorphism in the long exact sequence in homology for the triple (X^n, X^{n-1}, X^{n-2}). There is an isomorphism

$$H_n(W_*(X), \partial) \cong H_n(X)$$

for all n, which is natural for cellular (= skeleton-preserving) maps of CW-complexes.

Dually, the cellular n-cochains with coefficients in G are given by

$$W^n(X;G) = H^n(X^n, X^{n-1};G) \xrightarrow{\cong} \mathrm{Hom}(W_n(X), G)$$

and the cellular coboundary $\delta^{n-1}\colon W^{n-1}(X;G) \to W^n(X;G)$ is the connecting homomorphism in the long exact sequence in cohomology for (X^n, X^{n-1}, X^{n-2}). This means that

$$W^*(X;G) = \mathrm{Hom}(W_*(X), G)$$

as cochain complexes. The same proof as for homology shows that there is a natural isomorphism

$$H^n(W^*(X;G), \delta) \cong H^n(X;G)$$

for all n and G.

7. September 14th lecture

7.1. **Simplicial cohomology.** A Δ-complex X is a special kind of CW-complex, with n-cells ($=$ n-simplices) given by attaching copies of Δ^n

$$\begin{tikzcd} \Delta^{n-1} \arrow[r] \arrow[dr, "\Phi_\beta"'] & \partial\Delta^n \arrow[r] \arrow[d, "\phi_\alpha"] & \Delta^n \arrow[d, "\Phi_\alpha"] \\ & X^{n-1} \arrow[r] & X^n \end{tikzcd}$$

in such a way that each restriction of the attaching map ϕ_α to each $(n-1)$-face of $\partial\Delta^n$ is the characteristic map Φ_β of some $(n-1)$-cell ($= (n-1)$-simplex).

The cellular complex $W_*(X)$ of a Δ-complex is written $\Delta_*(X)$. Dually, the cellular n-cochains $W^n(X;G)$ are

$$\Delta^n(X;G) = \mathrm{Hom}(\Delta_n(X), G)$$

and $\delta\colon \Delta^{n-1}(X;G) \to \Delta^n(X;G)$ is $\mathrm{Hom}(\partial_n, 1)$.

The characteristic map $\Phi\colon \Delta^n \to X^n \subset X$ of each n-cell in X is a singular n-simplex, which defines an inclusion of chain complexes

$$0 \to \Delta_*(X) \longrightarrow C_*(X)$$

and this induces an isomorphism in homology:

$$H_n^\Delta(X) \cong H_n(X)$$

for each n. See Theorem 2.27 in Hatcher, where it is proved by induction on k that this is an isomorphism when X is replaced by its k-skeleton X^k. The conclusion then follows by passage to sequential colimits, since the canonical homomorphisms $\mathrm{colim}_k\, H_n^\Delta(X^k) \to H_n^\Delta(X)$ and $\mathrm{colim}_k\, H_n(X^k) \to H_n(X)$ are isomorphisms. The last statement uses that each compact subspace of X lies in some X^k.

Dually we can restrict each singular cochain to the cells in the Δ-complex structure, and obtain a projection of cochain complexes

$$C^*(X;G) \longrightarrow \Delta^*(X;G) \to 0$$

which induces an isomorphism between singular and simplicial cohomology

$$H^n(X;G) \cong H^n_\Delta(X;G)$$

for each n and G.

7.2. **The cochain cup product.** Let R be a ring and X a space. We introduce a cup product

$$\cup\colon C^p(X;R) \times C^q(X;R) \longrightarrow C^{p+q}(X;R)$$

satisfying the Leibniz rule

$$\delta^{p+q}(\phi \cup \psi) = \delta^p\phi \cup \psi + (-1)^p \phi \cup \delta^q\psi\,.$$

The rule $[\phi] \cup [\psi] = [\phi \cup \psi]$ then defines a cohomology cup product

$$\cup\colon H^p(X;R) \times H^q(X;R) \longrightarrow H^{p+q}(X;R)\,.$$

This makes $C^*(X;R)$ a differential graded ring. It also makes $H^*(X;R)$ a graded ring, called the cohomology ring of X. If R is commutative, this is a graded-commutative R-algebra.

For any $(p+q)$-simplex $\sigma\colon \Delta^{p+q} \to X$ let the front p-face be

$$\sigma|[v_0, \dots, v_p]\colon \Delta^p \longrightarrow X$$

and let the back q-face be

$$\sigma|[v_p, \dots, v_{p+q}]\colon \Delta^q \longrightarrow X\,.$$

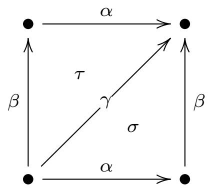
Definition 7.1. Let $\phi \in C^p(X;R)$ and $\psi \in C^q(X;R)$. The cup product $\phi \cup \psi \in C^{p+q}(X;R)$ is given by

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_p]) \cdot \psi(\sigma|[v_p, \dots, v_{p+q}])\,.$$

For a Δ-complex X, this bilinear pairing descends to define a cup product of simplicial cochains:

$$\begin{tikzcd} C^p(X;R) \times C^q(X;R) \arrow[r, "\cup"] \arrow[d] & C^{p+q}(X;R) \arrow[d] \\ \Delta^p(X;R) \times \Delta^q(X;R) \arrow[r, "\cup"] & \Delta^{p+q}(X;R)\,. \end{tikzcd}$$

Example 7.2. The torus $X = T^2$ can be triangulated as follows:



The simplicial chain complex

$$0 \to \mathbb{Z}\{\sigma, \tau\} \xrightarrow{\partial} \mathbb{Z}\{\alpha, \beta, \gamma\} \xrightarrow{\partial} \mathbb{Z} \to 0$$

has boundaries $\partial(\sigma) = \beta - \gamma + \alpha$, $\partial(\tau) = \alpha - \gamma + \beta$ and $\partial(\alpha) = \partial(\beta) = \partial(\gamma) = 0$.

Let $\phi\colon \Delta_1(T^2) \to \mathbb{Z}$ be given by $\phi(\alpha) = 1$, $\phi(\beta) = 0$ and $\phi(\gamma) = 1$, and let $\psi\colon \Delta_1(T^2) \to \mathbb{Z}$ be given by $\phi(\alpha) = 0$, $\phi(\beta) = 1$ and $\phi(\gamma) = 1$. These are both simplicial 1-cocycles.

By definition, $\phi \cup \psi\colon \Delta_2(T^2) \to \mathbb{Z}$ is given by

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, v_1]) \cdot \psi(\sigma|[v_1, v_2]) = \phi(\alpha) \cdot \psi(\beta) = 1 \cdot 1 = 1$$
$$(\phi \cup \psi)(\tau) = \phi(\tau|[v_0, v_1]) \cdot \psi(\tau|[v_1, v_2]) = \phi(\beta) \cdot \psi(\alpha) = 0 \cdot 0 = 0\,.$$

On the other hand, $\psi \cup \phi\colon \Delta_2(T^2) \to \mathbb{Z}$ is given by

$$(\psi \cup \phi)(\sigma) = \psi(\sigma|[v_0, v_1]) \cdot \phi(\sigma|[v_1, v_2]) = \psi(\alpha) \cdot \phi(\beta) = 0 \cdot 0 = 0$$
$$(\psi \cup \phi)(\tau) = \psi(\tau|[v_0, v_1]) \cdot \phi(\tau|[v_1, v_2]) = \psi(\beta) \cdot \phi(\alpha) = 1 \cdot 1 = 1\,.$$

Furthermore, $\phi \cup \phi\colon \Delta_2(T^2) \to \mathbb{Z}$ is given by

$$(\phi \cup \phi)(\sigma) = \phi(\sigma|[v_0, v_1]) \cdot \phi(\sigma|[v_1, v_2]) = \phi(\alpha) \cdot \phi(\beta) = 1 \cdot 0 = 0$$
$$(\phi \cup \phi)(\tau) = \phi(\tau|[v_0, v_1]) \cdot \phi(\tau|[v_1, v_2]) = \phi(\beta) \cdot \phi(\alpha) = 0 \cdot 0 = 0$$

and $\psi \cup \psi\colon \Delta_2(T^2) \to \mathbb{Z}$ is given by

$$(\psi \cup \psi)(\sigma) = \psi(\sigma|[v_0, v_1]) \cdot \psi(\sigma|[v_1, v_2]) = \psi(\alpha) \cdot \psi(\beta) = 0 \cdot 1 = 0$$
$$(\psi \cup \psi)(\tau) = \psi(\tau|[v_0, v_1]) \cdot \psi(\tau|[v_1, v_2]) = \psi(\beta) \cdot \psi(\alpha) = 1 \cdot 0 = 0\,.$$

Note that the cochain level cup product is not commutative.

Lemma 7.3. *The cup product is associative and unital, with unit $1 \in C^0(X;R)$ corresponding to the augmentation $\epsilon\colon C_0(X) \to \mathbb{Z} \to R$.*

Proof. Let $\phi \in C^p(X;R)$, $\psi \in C^q(X;R)$ and $\theta \in C^r(X;R)$. Then $((\phi \cup \psi) \cup \theta)$ and $(\phi \cup (\psi \cup \theta))$ both evaluate to

$$\phi(\sigma|[v_0, \dots, v_p]) \cdot \psi(\sigma|[v_p, \dots, v_{p+q}]) \cdot \theta(\sigma|[v_{p+q}, \dots, v_{p+q+r}])$$

on $\sigma\colon \Delta^{p+q+r} \to X$. Similarly, $\phi \cup 1$ evaluates to

$$\phi(\sigma|[v_0, \dots, v_p]) \cdot \epsilon(\sigma|[v_p]) = \phi(\sigma)$$

on $\sigma\colon \Delta^p \to X$. The proof that $1 \cup \psi = \psi$ is similar. $\square$

Lemma 7.4 (Leibniz rule)**.** *Let $\phi \in C^p(X;R)$ and $\psi \in C^q(X;R)$. Then*

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^p \phi \cup \delta\psi$$

in $C^{p+q+1}(X;R)$.

Proof. Let $\sigma\colon \Delta^{p+q+1} \to X$ be any $(p+q+1)$-simplex. Then

$$\begin{aligned}
(\delta\phi \cup \psi)(\sigma) &= (\delta\phi)(\sigma|[v_0,\dots,v_{p+1}]) \cdot \psi(\sigma|[v_{p+1},\dots,v_{p+q+1}]) \\
&= \phi(\partial(\sigma|[v_0,\dots,v_{p+1}])) \cdot \psi(\sigma|[v_{p+1},\dots,v_{p+q+1}]) \\
&= \sum_{i=0}^{p+1} (-1)^i \phi(\sigma|[v_0,\dots,\widehat{v_i},\dots,v_{p+1}]) \cdot \psi(\sigma|[v_{p+1},\dots,v_{p+q+1}]) \\
&= \sum_{i=0}^{p} (-1)^i \phi(\sigma|[v_0,\dots,\widehat{v_i},\dots,v_{p+1}]) \cdot \psi(\sigma|[v_{p+1},\dots,v_{p+q+1}]) \\
&\quad + (-1)^{p+1} \phi(\sigma|[v_0,\dots,v_p]) \cdot \psi(\sigma|[v_{p+1},\dots,v_{p+q+1}])\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\phi \cup \delta\psi)(\sigma) &= \phi(\sigma|[v_0,\dots,v_p]) \cdot (\delta\psi)(\sigma|[v_p,\dots,v_{p+q+1}]) \\
&= \phi(\sigma|[v_0,\dots,v_p]) \cdot \psi(\partial(\sigma|[v_p,\dots,v_{p+q+1}])) \\
&= \sum_{j=0}^{q+1} (-1)^j \phi(\sigma|[v_0,\dots,v_p]) \cdot \psi(\sigma|[v_p,\dots,\widehat{v_{p+j}},\dots v_{p+q+1}]) \\
&= \sum_{j=1}^{q+1} (-1)^j \phi(\sigma|[v_0,\dots,v_p]) \cdot \psi(\sigma|[v_p,\dots,\widehat{v_{p+j}},\dots v_{p+q+1}]) \\
&\quad + \phi(\sigma|[v_0,\dots,v_p]) \cdot \psi(\sigma|[v_{p+1},\dots v_{p+q+1}])
\end{aligned}$$

Hence

$$\begin{aligned}
(\delta\phi \cup \psi)(\sigma) + (-1)^p(\phi \cup \delta\psi)(\sigma) &= \sum_{i=0}^{p} (-1)^i \phi(\sigma|[v_0,\dots,\widehat{v_i},\dots,v_{p+1}]) \cdot \psi(\sigma|[v_{p+1},\dots,v_{p+q+1}]) \\
&\quad + \sum_{j=1}^{q+1} (-1)^{p+j} \phi(\sigma|[v_0,\dots,v_p]) \cdot \psi(\sigma|[v_p,\dots,\widehat{v_{p+j}},\dots v_{p+q+1}])
\end{aligned}$$

since the last two summands cancel. On the other hand,

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma|[v_0,\dots,\widehat{v_i},\dots,v_{p+q+1}] + \sum_{j=1}^{q+1} (-1)^{p+j} \sigma|[v_0,\dots,\widehat{v_{p+j}},\dots,v_{p+q+1}]$$

so

$$\begin{aligned}
(\delta(\phi \cup \psi))(\sigma) &= (\phi \cup \psi)(\partial\sigma) \\
&= \sum_{i=0}^{p} (-1)^i (\phi \cup \psi)(\sigma|[v_0,\dots,\widehat{v_i},\dots,v_{p+q+1}]) \\
&\quad + \sum_{j=1}^{q+1} (-1)^{p+j} (\phi \cup \psi)(\sigma|[v_0,\dots,\widehat{v_{p+j}},\dots,v_{p+q+1}])
\end{aligned}$$

and this equals to expression above. □

Lemma 7.5. $\cup\colon C^p(X;R) \times C^q(X;R) \to C^{p+q}(X;R)$ *restricts to pairings*

$$\begin{aligned}
Z^p(X;R) \times Z^q(X;R) &\longrightarrow Z^{p+q}(X;R) \\
B^p(X;R) \times Z^q(X;R) &\longrightarrow B^{p+q}(X;R) \\
Z^p(X;R) \times B^q(X;R) &\longrightarrow B^{p+q}(X;R)\,.
\end{aligned}$$

Proof. The follows from the Leibniz rule. If ϕ and ψ are cocycles, then $\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^p \psi \cup \delta\psi = 0 \cup \psi \pm \psi \cup 0 = 0$, so $\phi \cup \psi$ is a cocycle.

If $\phi = \delta\xi$ is a coboundary and ψ is a cocycle, then $\delta(\xi \cup \psi) = \delta\xi \cup \psi + (-1)^{p-1} \xi \cup \delta\psi = \phi \cup \psi + (-1)^{p-1} \xi \cup 0 = \phi \cup \psi \pm 0 = \phi \cup \psi$ is a coboundary. The third case is similar. □

7.3. The cohomology cup product.

Definition 7.6. Let $x = [\phi] \in H^p(X;R)$ and $y = [\psi] \in H^q(X;R)$, where ϕ and ψ are cocycles. Then
$$x \cup y = [\phi \cup \psi] \in H^{p+q}(X;R)\,.$$

This is well-defined by the previous lemma, and is clearly bilinear, associative and unital, since the cochain level cup product has these properties.

When X is a Δ-complex, the cup product in singular cohomology descends to simplicial cohomology.

$$\begin{tikzcd}
H^p(X;R) \times H^q(X;R) \arrow[r, "\cup"] \arrow[d, "\cong"'] & H^{p+q}(X;R) \arrow[d, "\cong"] \\
H^p_\Delta(X;R) \times H^q_\Delta(X;R) \arrow[r, "\cup"] & H^{p+q}_\Delta(X;R)\,.
\end{tikzcd}$$

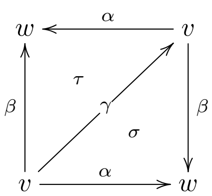
Example 7.7. Let $X = T^2$ as in the previous example. We have $H_0^\Delta(T^2) = \mathbb{Z}$, $H_1^\Delta(T^2) \cong \mathbb{Z}^2$ generated by the classes $[\alpha]$ and $[\beta]$, and $H_2^\Delta(T^2) \cong \mathbb{Z}$ generated by $[\sigma - \tau]$. Note that $[\gamma] = [\alpha] + [\beta]$.

It follows that $H^0_\Delta(T^2;R) = R$ is generated by the unit 1, $H^1_\Delta(T^2;R) \cong R^2$ is generated by classes $a = [\phi]$ and $b = [\psi]$. The cup product $\phi \cup \psi$ evaluates to
$$(\phi \cup \psi)(\sigma - \tau) = 1 - 0 = 1$$
on the generating 2-cycle. Hence $a \cup b = [\phi \cup \psi]$ evaluates to 1 on $[\sigma - \tau]$, and is therefore a generator of $H^2_\Delta(T^2;R) \cong R$. Hence
$$H^*(T^2;R) \cong H^*_\Delta(T^2;R) \cong R\{1, a, b, a \cup b\}$$
with a and b in degree 1. Furthermore, $\phi \cup \phi$ and $\psi \cup \psi$ evaluate to 0 on $\sigma - \tau$, so $a \cup a = 0$ and $b \cup b = 0$. Finally,
$$(\psi \cup \phi)(\sigma - \tau) = 0 - 1 = -1$$
so $b \cup a$ is the negative of $a \cup b$.

It follows that if $f\colon T^2 \to T^2$ induces $f_*\colon \mathbb{Z}^2 \cong H_1(T^2) \to H_1(T^2) \cong \mathbb{Z}^2$ with (2×2 integer) matrix A, then $f^*\colon \mathbb{Z}^2 \cong H^1(T^2) \to H^1(T^2) \cong \mathbb{Z}^2$ has matrix A^t, and $f^*\colon H^2(T^2) \to H^2(T^2)$ and $f_*\colon H_2(T^2) \to H_2(T^2)$ are given by multiplication by $\det(A)$. Hence f has (Hopf) degree $\det(A)$.

8. September 16th lecture

Example 8.1. The projective plane $X = \mathbb{R}P^2$ can be triangulated as follows:



The simplicial chain complex
$$0 \to \mathbb{Z}\{\sigma,\tau\} \xrightarrow{\partial} \mathbb{Z}\{\alpha,\beta,\gamma\} \xrightarrow{\partial} \mathbb{Z}\{v,w\} \to 0$$
has boundaries $\partial(\sigma) = \beta - \alpha + \gamma$, $\partial(\tau) = \alpha - \beta + \gamma$, $\partial(\alpha) = \partial(\beta) = w - v$ and $\partial(\gamma) = 0$. With $R = \mathbb{Z}/2$ we get $H_0^\Delta(\mathbb{R}P^2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ generated by $[v] = [w]$, $H_1^\Delta(\mathbb{R}P^2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ generated by $[\gamma] = [\alpha + \beta]$, and $H_2^\Delta(\mathbb{R}P^2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ generated by $[\sigma + \tau]$.

Let $\phi\colon \Delta_1(\mathbb{R}P^2) \to \mathbb{Z}/2$ be given by $\phi(\alpha) = 1$, $\phi(\beta) = 0$ and $\phi(\gamma) = 1$. Then ϕ is a simplicial 1-cocycle.

By definition $\phi \cup \phi\colon \Delta_2(\mathbb{R}P^2) \to \mathbb{Z}/2$ is given by
$$(\phi \cup \phi)(\sigma) = \phi(\sigma|[v_0,v_1]) \cdot \phi(\sigma|[v_1,v_2]) = \phi(\gamma) \cdot \phi(\beta) = 1 \cdot 0 = 0$$
$$(\phi \cup \phi)(\tau) = \phi(\tau|[v_0,v_1]) \cdot \phi(\tau|[v_1,v_2]) = \phi(\gamma) \cdot \phi(\alpha) = 1 \cdot 1 = 1\,.$$

It follows that $H^0_\Delta(\mathbb{R}P^2;\mathbb{Z}/2) = \mathbb{Z}/2$ is generated by the unit 1, and $H^1_\Delta(\mathbb{R}P^2;\mathbb{Z}/2$ is generated by $a = [\phi]$, which evaluates to 1 on $[\gamma]$. Furthermore, $a \cup a = [\phi \cup \phi]$ evaluates to 1 on $[\sigma + \tau]$. Hence $a \cup a$ generates $H^2_\Delta(\mathbb{R}P^2;\mathbb{Z}/2)$, so that

$$H^*(\mathbb{R}P^2;\mathbb{Z}/2) \cong H^*_\Delta(\mathbb{R}P^2;\mathbb{Z}/2) \cong \mathbb{Z}/2\{1, a, a^2\}$$

with a in degree 1.

8.1. **Relative cup products.** Consider a triple $(X; A, B)$. If $\phi \in C^p(X;R)$ vanishes on simplices in A and $\psi \in C^q(X;R)$ vanishes on simplices in B, then $\phi \cup \psi \in C^{p+q}(X;R)$ is zero on simplices in A and on simplices in B. Hence the cochain cup product restricts to a pairing

$$\cup\colon C^p(X, A; R) \times C^q(X, B; R) \longrightarrow C^{p+q}(X; A + B; R)$$

where $C^*(X, A+B; R) = \mathrm{Hom}(C_*(X, A+B), R)$ with $C_*(X, A+B) = C_*(X)/(C_*(A)+C_*(B))$. We obtain an induced birelative cup product in cohomology:

$$\cup\colon H^p(X, A; R) \times H^q(X, B; R) \longrightarrow H^{p+q}(X; A + B; R)$$

In many cases the inclusion $C_*(A) + C_*(B) \subset C_*(A \cup B)$ is a quasi-isomorphism. For instance, this follows from the Excision Theorem if A and B are open in $A \cup B$. Then $C_*(X, A + B) \to C_*(X, A \cup B)$ and $C^*(X, A \cup B; R) \to C^*(X, A + B; R)$ are quasi-isomorphisms, so that we can rewrite the birelative cup product as

$$\cup\colon H^p(X, A; R) \times H^q(X, B; R) \longrightarrow H^{p+q}(X; A \cup B; R)\,.$$

In particular, there are always relative cup products

$$\begin{aligned} H^p(X, A; R) \times H^q(X; R) &\longrightarrow H^{p+q}(X; A; R) \\ H^p(X; R) \times H^q(X, B; R) &\longrightarrow H^{p+q}(X; B; R) \\ H^p(X, A; R) \times H^q(X, A; R) &\longrightarrow H^{p+q}(X; A; R)\,. \end{aligned}$$

8.2. **Naturality.**

Lemma 8.2. *The cochain cup product is natural.*

Proof. For any map $f\colon X \to Y$ and ring R, the assertion is that the following diagram commutes.

$$\begin{tikzcd} C^p(Y;R) \times C^q(Y;R) \arrow[r, "\cup"] \arrow[d, "f^\# \times f^\#"'] & C^{p+q}(Y;R) \arrow[d, "f^\#"] \\ C^p(X;R) \times C^q(X;R) \arrow[r, "\cup"] & C^{p+q}(X;R) \end{tikzcd}$$

so that

$$f^\#(\phi \cup \psi) = f^\#\phi \cup f^\#\psi\,.$$

This follows by direct computation, for any $p + q$-simplex σ in X:

$$\begin{aligned} f^\#(\phi \cup \psi)(\sigma) &= (\phi \cup \psi)(f\sigma) \\ &= \phi(f\sigma|[v_0, \dots, v_p]) \cdot \psi(f|\sigma|[v_p, \dots, v_{p+q}]) \\ &= (f^\#\phi)(\sigma|[v_0, \dots, v_p]) \cdot (f^\#\psi)(\sigma|[v_p, \dots, v_{p+q}]) \\ &= (f^\#\phi \cup f^\#\psi)(\sigma)\,. \end{aligned}$$

The following diagram may be equally clear.

$$\begin{tikzcd} \Delta^p \arrow[dr, "\text{front}"'] \arrow[drrr, bend left, "\text{apply } \phi"] & & & \\ & \Delta^{p+q} \arrow[r, "\sigma"] & X \arrow[r, "f"] & Y \\ \Delta^q \arrow[ur, "\text{back}"] \arrow[urrr, bend right, "\text{apply } \psi"'] & & & \end{tikzcd}$$

□

Proposition 8.3. *The cohomology cup product is natural, so that*

$$f^*(x \cup y) = f^*x \cup f^*y$$

in $H^*(X;R)$ for $f\colon X \to Y$ and $x, y \in H^*(Y;R)$.

Proof. The assertion is that the following diagram commutes.

$$\begin{tikzcd}
H^p(Y;R) \times H^q(Y;R) \arrow[r, "\cup"] \arrow[d, "f^\# \times f^\#"'] & H^{p+q}(Y;R) \arrow[d, "f^\#"] \\
H^p(X;R) \times H^q(X;R) \arrow[r, "\cup"] & H^{p+q}(X;R) \,.
\end{tikzcd}$$

This follows by passage to cohomology classes from the cochain statement. □

8.3. **The Alexander–Whitney diagonal approximation.** With good sign conventions, there is an Alexander–Whitney chain map

$$AW\colon C_*(X) \longrightarrow C_*(X) \otimes C_*(X)$$

given on an n-simplex σ by

$$AW(\sigma) = \sum_{p+q=n} \sigma|[v_0, \dots, v_p] \otimes \sigma|[v_p, \dots, v_{p+q}]\,.$$

Under the Eilenberg–Zilber chain homotopy equivalence $C_*(X) \otimes C_*(X) \simeq C_*(X \times X)$, this corresponds to the chain map $\Delta_\#\colon C_*(X) \to C_*(X \times X)$ induced by the diagonal $\Delta\colon X \to X \times X$. Given cochains $\phi\colon C_p(X) \to R$ and $\psi\colon C_q(X) \to R$ we can then form the composite

$$C_{p+q}(X) \overset{AW_{p,q}}{\longrightarrow} C_p(X) \otimes C_q(X) \overset{\phi\otimes\psi}{\longrightarrow} R \otimes R \overset{\cdot}{\longrightarrow} R$$

and this defines $\phi \cup \psi$.

The cocommutativity of the diagonal is reflected in the graded cocommutativity of the cup product.

8.4. **Graded commutativity.** A 0-cocycle $\phi\colon \Delta_0(X) \to R$ is the same as a function $X \to R$. For another 0-cocycle ψ the cup product $\phi \cup \psi$ equals the product of these two functions, since $(\phi \cup \psi)(\sigma) = \phi(\sigma) \cdot \psi(\sigma)$ in this case. As for the rings of functions on spaces, this product is commutative when R is commutative. We have seen in examples that the cup product of cocycles of higher degree needs not be commutative. However, this problem disappears when we pass to cohomology classes.

Theorem 8.4. *Let R be commutative. Then*

$$x \cup y = (-1)^{pq} y \cup x$$

in $H^{p+q}(X;R)$, for $x \in H^p(X;R)$ and $y \in H^q(X;R)$.

Proof. The permutation of $\{0, 1, \dots, n\}$ sending i to $(n-i)$ has sign $\epsilon_n = (-1)^{n(n+1)/2}$. We note that $\epsilon_p \epsilon_q = (-1)^{pq} \epsilon_{p+q}$. Let

$$\rho\colon C_n(X) \longrightarrow C_n(X)$$

be given by

$$\rho(\sigma) = \epsilon_n \sigma|[v_n, \dots, v_0]\,.$$

Here $\sigma|[v_n, \dots, v_0]$ denotes the composite

$$\Delta^n \overset{\cong}{\longrightarrow} \Delta^n \overset{\sigma}{\longrightarrow} X$$

where the first affine linear isomorphism sends v_i to v_{n-i} for each $0 \le i \le n$.

We first show that $\rho\colon C_*(X) \to C_*(X)$ is a chain map. Then we construct a chain homotopy $Q\colon C_*(X) \to C_{*+1}(X)$ from ρ to the identity of $C_*(X)$. We obtain a chain map $\rho^*\colon C^*(X;R) \to C^*(X;R)$ and a chain homotopy Q^* from ρ^* to the identity of $C^*(X;R)$.

For cocycles $\phi \in C^p(X;R)$ and $\psi \in C^q(X;R)$ it then follows that $\phi \cup \psi$ is cohomologous to $\rho^*(\phi) \cup \rho^*(\psi)$. Furthermore,

$$\begin{aligned}
(\rho^*(\phi) \cup \rho^*(\psi))(\sigma) &= \rho^*(\phi)(\sigma|[v_0, \dots, v_p]) \cdot \rho^*(\psi)(\sigma|[v_p, \dots, v_{p+q}]) \\
&= \phi(\rho(\sigma|[v_0, \dots, v_p])) \cdot \psi(\rho(\sigma|[v_p, \dots, v_{p+q}])) \\
&= \epsilon_p \phi(\sigma|[v_p, \dots, v_0]) \cdot \epsilon_q \psi(\sigma|[v_{p+q}, \dots, v_p]) \\
&= (-1)^{pq} \epsilon_{p+q} \phi(\sigma|[v_p, \dots, v_0]) \cdot \psi(\sigma|[v_{p+q}, \dots, v_p]) \\
&= (-1)^{pq} \epsilon_{p+q} \psi(\sigma|[v_{p+q}, \dots, v_p]) \cdot \phi(\sigma|[v_p, \dots, v_0]) \\
&= (-1)^{pq} \epsilon_{p+q} (\psi \cup \phi)(\sigma|[v_{p+q}, \dots, v_0]) \\
&= (-1)^{pq} (\psi \cup \phi)(\rho(\sigma)) \\
&= (-1)^{pq} (\rho^*(\psi \cup \phi))(\sigma) \,.
\end{aligned}$$

Hence $\rho^*(\phi) \cup \rho^*(\psi) = (-1)^{pq} \rho^*(\psi \cup \phi)$, which is cohomologous to $(-1)^{pq} \psi \cup \phi$. If $x = [\phi]$ and $y = [\psi]$ we can conclude that

$$x \cup y = [\phi \cup \psi] = [(-1)^{pq} \psi \cup \phi] = (-1)^{pq} y \cup x \,,$$

as claimed.

To check that ρ is a chain map, we see for any n-simplex σ that

$$\partial(\rho(\sigma)) = \sum_{i=0}^{n} (-1)^i \epsilon_n \sigma|[v_n, \dots, \widehat{v_{n-i}}, \dots, v_0]$$

is equal to

$$\rho(\partial(\sigma)) = \epsilon_{n-1} \sum_{i=0}^{n} (-1)^i \sigma|[v_n, \dots, \widehat{v_i}, \dots, v_0]$$

since $\epsilon_n = (-1)^n \epsilon_{n-1}$.

To check that ρ is chain homotopic to the identity, we use the twisted prism operator $Q \colon C_*(X) \to C_{*+1}(X)$, given by

$$Q\sigma = \sum_{i=0}^{n} (-1)^i \epsilon_{n-i} \pi(1 \times \sigma)|[u_0, \dots, u_i, w_n, \dots, w_i]$$

where $\pi \colon I \times X \to X$ is the projection. Direct calculation shows that $\partial Q + Q\partial = \rho - 1$. See Hatcher, pages 211–212. □

8.5. **The cohomology ring.** We view the cohomology groups $H^n(X;G)$ for varying integers n as a graded abelian group. This can be interpreted as a sequence

$$H^*(X;G) = (H^n(X;G))_n$$

of abelian groups, with $H^n(X;G)$ in degree (or dimension) n. Alternatively, it can be considered as a single abelian group

$$\bigoplus_n H^n(X;G)$$

together with the data specifying this sum decomposition, so that a general element a is given as a finite sum

$$a = \sum_n a_n$$

with $a_n \in H^n(X;G)$. Sometimes it is more convenient to consider the single abelian group

$$\prod_n H^n(X;G)$$

together with the data specifying this product factorization, so that a general element a is given as sequence $(a_n)_n$ or a formal sum

$$a = \sum_n a_n$$

with $a_n \in H^n(X;G)$. Let us concentrate on the first point of view.

For any ring R we can view each $H^n(X;R)$ as a left (or right) R-module. The cup products

$$\cup\colon H^p(X;R)\times H^q(X;R)\longrightarrow H^{p+q}(X;R)$$

are then R-bilinear, and factor through the tensor product

$$\cup\colon H^p(X;R)\otimes_R H^q(X;R)\longrightarrow H^{p+q}(X;R)\,.$$

We define the tensor (= convolution) product of the two graded R-modules M^* and N^* to be the graded abelian group $M^*\otimes_R N^*=(M\otimes_R N)^*$ with

$$(M\otimes_R N)^n=\bigoplus_{p+q=n} M^p\otimes_R N^q\,.$$

The cup product then corresponds to a homomorphism

$$\cup\colon H^*(X;R)\otimes_R H^*(X;R)\longrightarrow H^*(X;R)$$

given in degree n as the sum

$$\cup\colon \bigoplus_{p+q=n} H^p(X;R)\otimes_R H^q(X;R)\longrightarrow H^n(X;R)$$

of the cup products from degree p and q to $p+q=n$.

If R is commutative then these are R-modules and R-linear homomorphisms. The tensor product of graded R-modules is then symmetric monoidal, in the sense that there are isomorphisms

$$\begin{aligned}(L^*\otimes M^*)\otimes N^* &\cong L^*\otimes(M^*\otimes N^*)\\ R\otimes_R M^*&\cong M^*\cong M^*\otimes_R R\\ \tau\colon M^*\otimes_R N^*&\cong N^*\otimes_R M^*\end{aligned}$$

satisfying some standard coherence compatibilities, including the Mac Lane pentagon and hexagon. Here we focus on the graded symmetric structure, where τ is given by

$$\tau(x\otimes y)=(-1)^{pq}y\otimes x$$

for $x\in M^p$ and $y\in N^q$. We suppress the other two isomorphisms from the notation, treating them as identities. We have shown that the following diagrams commute.

$$\begin{tikzcd}
H^*(X;R)\otimes_R H^*(X;R)\otimes_R H^*(X;R) \arrow[r, "\cup\otimes 1"] \arrow[d, "1\otimes\cup"'] & H^*(X;R)\otimes_R H^*(X;R) \arrow[d, "\cup"] \\
H^*(X;R)\otimes_R H^*(X;R) \arrow[r, "\cup"] & H^*(X;R)
\end{tikzcd}$$

$$\begin{tikzcd}
R\otimes_R H^*(X;R) \arrow[r, "\eta\otimes 1"] \arrow[dr, "1"'] & H^*(X;R)\otimes_R H^*(X;R) \arrow[d, "\cup"] & H^*(X;R)\otimes_R R \arrow[l, "1\otimes\eta"'] \arrow[dl, "1"] \\
& H^*(X;R) &
\end{tikzcd}$$

$$\begin{tikzcd}
H^*(X;R)\otimes_R H^*(X;R) \arrow[rr, "\tau"] \arrow[dr, "\cup"'] & & H^*(X;R)\otimes_R H^*(X;R) \arrow[dl, "\cup"] \\
& H^*(X;R) &
\end{tikzcd}$$

This shows that the cup product makes $H^*(X;R)$ a commutative monoid in the category of graded R-modules, which is what we call a graded commutative R-algebra. Furthermore, this structure is natural in X.

Example 9.1. We will show that $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]$ with x in degree 1, and $H^*(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$. These are the polynomial $\mathbb{Z}/2$-algebra and truncated polynomial $\mathbb{Z}/2$-algebra of height $n+1$, respectively, on one generator x in degree 1.

Similarly, $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[y]$ with y in degree 2, and $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}/2[y]/(y^{n+1})$. These are the polynomial $\mathbb{Z}$-algebra and truncated polynomial $\mathbb{Z}$-algebra of height $n+1$, respectively, on one generator in degree 2.

Furthermore, $H^*(S^n;\mathbb{Z}) \cong \mathbb{Z}[\iota_n]/(\iota_n^2) = \Lambda_{\mathbb{Z}}[\iota_n]$ with ι_n in degree n. This is the exterior algebra on one generator in degree n.

9.1. **Tensor product of graded algebras.** Let R be commutative. If A^* and B^* are graded R-algebra, we define their tensor product to be $A^* \otimes_R B^* = (A \otimes_R B)^*$, given in degree n by

$$(A \otimes_R B)^n = \bigoplus_{p+q=n} A^p \otimes_R B^q\,.$$

The multiplication on $A^* \otimes_R B^*$ is given by

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (-1)^{p_2 q_1} x_1 x_2 \otimes y_1 y_2$$

where $x_2 \in A^{p_2}$ and $y_1 \in B^{q_1}$. This is the composite homomorphism

$$A^* \otimes_R B^* \otimes_R A^* \otimes_R B^* \overset{1\otimes\tau\otimes 1}{\longrightarrow} A^* \otimes_R A^* \otimes_R B^* \otimes_R B^* \overset{\phi\otimes\phi}{\longrightarrow} A^* \otimes_R B^*\,,$$

where now $\phi\colon A^* \otimes_R A^* \to A^*$ denotes the product in A^*, and likewise for B^*.

Example 9.2. The tensor product

$$R[x_1] \otimes_R \cdots \otimes_R R[x_n] \cong R[x_1, \dots, x_n]$$

of n polynomial algebras on generators $x_1, \dots, x_n$, all in even degrees, is the polynomial algebra on n generators. It has a basis given by the monomials

$$x_1^{i_1} \cdots x_n^{i_n}$$

with $i_1, \dots, i_n \geq 0$. We shall see that there is an isomorphism of graded algebras

$$H^*((\mathbb{R}P^\infty)^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, \dots, x_n]$$

with each x_i of degree 1, and an isomorphism

$$H^*((\mathbb{C}P^\infty)^n) \cong \mathbb{Z}[y_1, \dots, y_n]$$

with each y_i of degree 2.

The tensor product

$$\Lambda_R[x_1] \otimes_R \cdots \otimes_R \Lambda_R[x_n] \cong \Lambda_R[x_1, \dots, x_n]$$

of n exterior algebras on generators $x_1, \dots, x_n$, all in odd degrees, is the exterior algebra on n generators. It has a basis given by the 2^n monomials

$$x_1^{i_1} \cdots x_n^{i_n}$$

with $i_1, \dots, i_n \in \{0, 1\}$. Equivalently, these are the monomials

$$x_{j_1} \cdots x_{j_k}$$

with $1 \leq j_1 < \cdots < j_k \leq n$, for $0 \leq k \leq n$. Note that $x_i^2 = 0$ and $x_i x_j = -x_j x_i$, for all i and j. We shall see that there is an isomorphism of graded algebras

$$H^*(S^{d_1} \times \cdots \times S^{d_n}) \cong \Lambda_{\mathbb{Z}}[\iota_{d_1}, \dots, \iota_{d_n}]$$

when each d_i is odd.

Lemma 9.3. *If A^* and B^* are graded commutative R-algebras, then so is $A^* \otimes_R B^*$.*

Proof. We need to check that

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (-1)^{p_2 q_1} x_1 x_2 \otimes y_1 y_2$$

equals

$$\begin{aligned}(-1)^{(p_1+q_1)(p_2+q_2)}(x_2 \otimes y_2) \cdot (x_1 \otimes y_1) &= (-1)^{(p_1+q_1)(p_2+q_2)}(-1)^{p_1 q_2} x_2 x_1 \otimes y_1 y_2 \\ &= (-1)^{(p_1+q_1)(p_2+q_2)}(-1)^{p_1 q_2}(-1)^{p_1 p_2}(-1)^{q_1 q_2} x_1 x_2 \otimes y_1 y_2 \,,\end{aligned}$$

which is clear. □

9.2. **Tensor product of chain complexes.** We make a quick survey of results of Section 3.B. Let R be a commutative ring. If (A_*, ∂) and (B_*, ∂) are chain complexes of R-modules, we define their tensor product $(A_* \otimes_R B_*, \partial)$ to be given in degree n by

$$(A \otimes_R B)_n = (A_* \otimes_R B_*)_n = \bigoplus_{p+q=n} A_p \otimes_R B_q$$

with boundary $\partial \colon (A \otimes_R B)_n \to (A \otimes_R B)_{n-1}$ given by the Leibniz rule

$$\partial(a \otimes b) = \partial a \otimes b + (-1)^p a \otimes \partial b$$

for $a \in A_p$, $b \in B_q$. Then $\partial^2 = 0$, so this defines a chain complex.

If X and Y are CW complexes, then $X \times Y$ is a CW complex with n-skeleton

$$(X \times Y)^n = \bigcup_{p+q=n} X^p \times Y^q \,.$$

We build $(X \times Y)^n$ from $(X \times Y)^{n-1}$ by attaching $D^p \times D^q$ along

$$\partial(D^p \times D^q) = \partial D^p \times D^q \cup D^p \times \partial D^q$$

for each p-cell e^p_α in X and each q-cell e^q_β in Y, where $p + q = n$.

$$\begin{tikzcd} \partial(D^p \times D^q) \arrow[r] \arrow[d, "\phi_\alpha \times \Phi_\beta \cup \Phi_\alpha \times \phi_\beta"'] & D^p \times D^q \arrow[d, "\Phi_\alpha \times \Phi_\beta"] \\ (X \times Y)^{n-1} \arrow[r] & (X \times Y)^n \,. \end{tikzcd}$$

We get an isomorphism of cellular chain complexes

$$\times \colon W_*(X) \otimes W_*(Y) \cong W_*(X \times Y) \,.$$

(Here $R = \mathbb{Z}$.) In degree n this is

$$\bigoplus_{p+q=n} \mathbb{Z}\{p\text{-cells in } X\} \otimes \mathbb{Z}\{q\text{-cells in } Y\} \cong \mathbb{Z}\{n\text{-cells in } X \times Y\}$$

taking e^p_α and e^q_β to the cell $e^n_{(\alpha,\beta)}$ with characteristic map $\Phi_\alpha \times \Phi_\beta$. The boundary on the left hand side is defined to correspond to the boundary on the right hand side, so that this becomes an isomorphism of chain complexes.

9.3. **The algebraic Künneth formula.** The universal coefficient theorem gave a split short exact sequence

$$0 \to G \otimes_R H_n(C_*) \xrightarrow{i} H_n(G \otimes C_*) \longrightarrow \mathrm{Tor}^R(G, H_{n-1}(C_*)) \to 0$$

for free ideal rings R, arbitrary R-modules G and free R-module complexes (C_*, ∂).

A similar proof establishes the algebraic Künneth theorem.

Theorem 9.4. *Let R be a free ideal ring, and let (A_*, ∂) and (B_*, ∂) be free R-module chain complexes. There is a natural short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(A_*) \otimes_R H_q(B_*) \xrightarrow{i} H_n(A_* \otimes_R B_*) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}^R(H_p(A_*), H_q(B_*)) \to 0$$

in each degree n. Here i maps $[x] \otimes [y]$ to $[x \otimes y]$, for cycles $x \in A_p$ and $y \in B_q$. The sequence splits, but not naturally.

If the Tor-term vanishes we get a Künneth isomorphism. This always happens if R is a field.

Corollary 9.5. *Let F be a field, and let (A_*, ∂) and (B_*, ∂) be F-module chain complexes. There is a natural isomorphism*

$$i\colon H_*(A_*) \otimes_F H_*(B_*) \xrightarrow{\cong} H_*(A_* \otimes_F B_*)\,.$$

9.4. **The topological Künneth formula.** Applying the algebraic theorem to cellular chain complexes gives the following topological theorem.

Theorem 9.6. *Let R be a free ideal ring, and let X and Y be CW complexes. There is a natural short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(X;R) \otimes_R H_q(Y;R) \xrightarrow{\times} H_n(X \times Y;R) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}^R(H_p(X;R), H_q(Y;R)) \to 0$$

in each degree n. The sequence splits, but not naturally.

We call $\times$ the homology cross product.

Corollary 9.7. *Let F be a field, and let X and Y be CW complexes. There is a natural isomorphism*

$$\times\colon H_*(X;F) \otimes_F H_*(Y;F) \xrightarrow{\cong} H_*(X \times Y;F)\,.$$

9.5. **The Eilenberg–Zilber theorem.** For singular homology, it is not true that $C_*(X) \otimes C_*(Y)$ is isomorphic to $C_*(X \times Y)$. However, each n-simplex in $X \times Y$ has the form

$$\theta = (\sigma, \tau)\colon \Delta^n \longrightarrow X \times Y$$

where $\sigma = \pi_1\theta\colon \Delta^n \to X$ and $\tau = \pi_2\theta\colon \Delta^n \to Y$. Here $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are the projections, i.e., the structure maps of the product. We can define an Alexander–Whitney chain homomorphism

$$AW\colon C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$$

mapping $C_n(X \times Y)$ to

$$\bigoplus_{p+q=n} C_p(X) \otimes C_q(Y)$$

by

$$AW(\theta) = \sum_{p+q=n} \sigma|[v_0, \dots, v_p] \otimes \tau|[v_p, \dots, v_{p+q}]\,.$$

We can also define an Eilenberg–Zilber (or Eilenberg–MacLane) chain homomorphism

$$EZ\colon C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$$

given by

$$EZ(\sigma \otimes \tau) = \sum_{\pi} (-1)^{\pi} (\sigma \times \tau)\ell_{\pi}$$

for $\sigma\colon \Delta^p \to X$ and $\tau\colon \Delta^q \to Y$. Here π ranges over the (p,q)-shuffles, which are the permutations of $1, 2, \dots, n$ that preserve the ordering of $1, \dots, p$ and of $p+1, \dots, p+q = n$. We write $(-1)^{\pi}$ for the sign of this permutation. The notation ℓ_{π} refers to an affine linear map

$$\ell_{\pi}\colon \Delta^n \longrightarrow \Delta^p \times \Delta^q$$

and $(\sigma \times \tau)\ell_{\pi}\colon \Delta^n \to X \times Y$. As π varies the ℓ_{π} give the n-simplices of a Δ-complex structure on $\Delta^p \times \Delta^q$.

Theorem 9.8 (Eilenberg–Zilber (1953))**.** *AW and EZ are chain homotopy inverses, inducing isomorphisms*

$$AW_*\colon H_*(C_*(X \times Y)) \cong H_*(C_*(X) \otimes C_*(Y))\colon EZ_*\,.$$

It follows that the topological Künneth formula applies to all spaces X and Y, not only CW complexes.

Remark 9.9. For each prime p and natural number n, Jack Morava constructed a generalized homology theory $K(n)_*(-)$, called the n-th Morava K-theory, with coefficient ring $K(n)_* = K(n)_*(\text{point}) = \mathbb{F}_p[v_n^{\pm 1}]$, where $|v_n| = 2(p^n - 1)$. The implicit prime p is not shown in the notation. This is a graded field, and the Künneth isomorphism also holds for this theory:

$$\times\colon K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y) \cong K(n)_*(X \times Y)\,.$$

The corresponding cohomology theory is complex oriented, and has an associated formal group law of height n. This is the origin of chromatic homotopy theory. As a consequence of the nilpotence theorem of Ethan Devinatz, Mike Hopkins and Jeff Smith, these are essentially the only examples of homology theories with a Künneth isomorphism for all spaces X and Y. Together with the ordinary homology theories $K(0)_*(X) = H_*(X;\mathbb{Q})$ and $K(\infty)_*(X) = H_*(X;\mathbb{F}_p)$, these are the primes or points of a stable homotopy theoretic "thickening"

$$\operatorname{Spec} S \longleftarrow \operatorname{Spec} \mathbb{Z}$$

of the algebraic prime ideal spectrum of the integers.

10. September 23rd lecture

10.1. **The cohomology cross product.** Instead of working out the combinatorics of the Eilenberg–Zilber shuffle pairing, we follow Hatcher's Section 3.2 and give a proof of a Künneth theorem in cohomology that emphasizes the axiomatic description of cohomology theories. This sidesteps some issues that would arise from the (over-)simplified sign conventions we use for the coboundary δ and the evaluation of tensor products of cochains on tensor products of chains, i.e., $\delta\phi = \phi\partial$ and $(\phi \otimes \psi)(\alpha \otimes \beta) = \phi(\alpha) \otimes \psi(\beta)$.

Let X and Y be spaces, and R any ring. Our aim is to analyze the cohomology of $X \times Y$. We can use the cup product in $X \times Y$ to create a cohomology cross product

$$\times\colon H^p(X;R) \times H^q(Y;R) \longrightarrow H^{p+q}(X \times Y;R)\,.$$

Letting $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ be the projections given by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$, this is defined by the formula

$$x \times y = \pi_1^*(x) \cup \pi_2^*(y)\,.$$

Equivalently, this is the composite

$$H^p(X;R) \times H^q(Y;R) \overset{\pi_1^* \times \pi_2^*}{\longrightarrow} H^p(X \times Y;R) \times H^q(X \times Y;R) \overset{\cup}{\longrightarrow} H^p(X \times Y;R)\,.$$

More explicitly, for $p + q = n$ an n-simplex in $X \times Y$ has the form

$$(\sigma,\tau)\colon \Delta^n \longrightarrow X \times Y$$

with $\sigma = \pi_1(\sigma,\tau)$ and $\tau = \pi_2(\sigma,\tau)$. For $\phi \in C^p(X;R)$ and $\psi \in C^q(Y;R)$ the cochain cross product $\phi \times \psi \in C^{p+q}(X \times Y;R)$ is given by

$$(\phi \times \psi)(\sigma,\tau) = \phi(\sigma|[v_0,\dots,v_p]) \cdot \psi(\tau|[v_p,\dots,v_{p+q}])\,.$$

This can be illustrated as follows.

$$\begin{tikzcd}
\Delta^p \arrow[rrr, "\text{apply } \phi"] \arrow[dr, "\text{front}"'] & & & X \\
& \Delta^{p+q} \arrow[r, "{(\sigma,\tau)}"] & X \times Y \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\
\Delta^q \arrow[rrr, "\text{apply } \psi"'] \arrow[ur, "\text{back}"] & & & Y
\end{tikzcd}$$

Since the cross product is R-bilinear, we can equally well give it as a homomorphism

$$\times\colon H^p(X;R) \otimes_R H^q(Y;R) \longrightarrow H^{p+q}(X \times Y;R)$$

and to combine these for $p + q = n$ to a homomorphism

$$\times\colon H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R)$$

of graded abelian groups.

Lemma 10.1. *When R is commutative,*

$$\times \colon H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R)$$

is a homomorphism of graded commutative R-algebras.

Proof. This follows by direct calculation from the definition of the product on $H^*(X;R) \otimes_R H^*(Y;R)$, and the graded commutativity of the cup product in $H^*(X \times Y;R)$. See page 216 in (the revised Section 3.2 of) Hatcher. □

10.2. **The cohomology Künneth theorem.**

Theorem 10.2. *Let R be any ring and let X and Y be CW complexes. Assume that $H^q(Y;R)$ is a finitely generated and free R-module, for each q. Then*

$$\times \colon H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R)$$

is an isomorphism.

This confirms the claims in Example 9.2. It suffices that $H^q(Y;R)$ is a finitely generated and projective for each q, and it is not necessary that X and Y are CW complexes.

To prove this formula, we view Y as fixed, and view

$$h^n(X) = \bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(Y;R)$$

and

$$k^n(X) = H^n(X \times Y;R)$$

as contravariant functors in X. The cross product then defines a natural transformation

$$\mu_X \colon h^n(X) \longrightarrow k^n(X)$$

that we wish to show is an isomorphism. This is clear when X is a point. The plan is to proceed by induction over the terms in the skeleton filtration of X. To facilitate the inductive step, it is best to formulate a relative version of the result, for arbitrary CW pairs (X, A). Hence, let

$$h^n(X, A) = \bigoplus_{p+q=n} H^p(X, A;R) \otimes_R H^q(Y;R)$$

and

$$k^n(X, A) = H^n(X \times Y, A \times Y;R) .$$

Definition 10.3. A cohomology theory E^* on CW pairs is a sequence of contravariant functors

$$E^n \colon (X, A) \mapsto E^n(X, A)$$

(with $E^n(X) = E^n(X, \emptyset)$) and natural transformations

$$\delta_{(X,A)} \colon E^n(A) \longrightarrow E^{n+1}(X, A)$$

satisfying homotopy invariance, excision, exactness and the sum axiom.

(1) If $f \simeq g \colon (X, A) \to (Y, B)$ are homotopic maps then $f^* = g^* \colon E^*(Y, B) \to E^*(X, A)$.
(2) If $X = A \cup B$ is a union of subcomplexes, then $(B, A \cap B) \to (X, A)$ induces an isomorphism $E^*(X, A) \cong E^*(B, A \cap B)$.
(3) The sequence

$$E^{n-1}(A) \xrightarrow{\delta} E^n(X, A) \xrightarrow{j^*} E^n(X) \xrightarrow{i^*} E^n(A) \xrightarrow{\delta} E^{n+1}(X, A)$$

(with $i \colon (A, \emptyset) \to (X, \emptyset)$ and $j \colon (X, \emptyset) \to (X, A)$ the inclusions) is exact for all n.
(4) The inclusions $X_\alpha \to \coprod_\alpha X_\alpha$ induce an isomorphism

$$E^*(\coprod_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha E^*(X_\alpha) .$$

Let

$$\delta\colon h^n(A) = \bigoplus_{p+q=n} H^p(A;R)\otimes_R H^q(Y;R) \xrightarrow{\bigoplus \delta\otimes 1} \bigoplus_{p+q=n} H^{p+1}(X,A;R)\otimes_R H^q(Y;R) = h^{n+1}(X,A)$$

be induced by the connecting homomorphisms $\delta\colon H^p(A;R)\to H^{p+1}(X,A;R)$ and let

$$\delta\colon H^n(A\times Y;R) \longrightarrow H^{n+1}(X\times Y, A\times Y;R)\,.$$

be the connecting homomorphism of the pair $(X\times Y, A\times Y)$.

Proposition 10.4. *h^n and k^n are cohomology theories.*

Proof. This is mostly straightforward; see pages 217–218 in Hatcher. The hypothesis on $H^*(Y;R)$ ensures that

$$\begin{aligned} H^{p-1}(A;R)\otimes_R H^q(Y;R) &\xrightarrow{\delta\otimes 1} H^p(X,A;R)\otimes_R H^q(Y;R) \xrightarrow{j^*\otimes 1} H^p(X;R)\otimes_R H^q(Y;R)\\ &\xrightarrow{i^*\otimes 1} H^p(A;R)\otimes_R H^q(Y;R) \xrightarrow{\delta\otimes 1} H^{p+1}(X,A;R)\otimes_R H^q(Y;R)\end{aligned}$$

is exact, and that the canonical morphism

$$\Big(\prod_\alpha H^p(X_\alpha;R)\Big)\otimes_R H^q(Y;R) \longrightarrow \prod_\alpha H^p(X_\alpha;R)\otimes_R H^q(Y;R)$$

is an isomorphism. □

Definition 10.5. A morphism $\mu\colon E^*\to F^*$ of cohomology theories on CW pairs is a natural transformation

$$\mu_{(X,A)}\colon E^*(X,A)\longrightarrow F^*(X,A)$$

that is compatible with the coboundaries, in the sense that the square

$$\begin{tikzcd} E^n(A) \arrow[r, "\delta"] \arrow[d, "\mu_A"'] & E^{n+1}(X,A) \arrow[d, "\mu_{(X,A)}"] \\ F^n(A) \arrow[r, "\delta"] & F^{n+1}(X,A) \end{tikzcd}$$

commutes for each n.

The relative cross product

$$\begin{aligned} H^p(X,A;R)\otimes_R H^q(Y;R) &\xrightarrow{\pi_1^*\otimes\pi_2^*} H^p(X\times Y, A\times Y;R)\otimes_R H^q(X\times Y, A\times Y;R)\\ &\xrightarrow{\cup} H^{p+q}(X\times Y, A\times Y;R)\end{aligned}$$

defines a natural transformation

$$\mu_{(X,A)}\colon h^n(X,A)\longrightarrow k^n(X,A)\,.$$

Proposition 10.6. *$\mu\colon h^*\to k^*$ is a morphism of cohomology theories.*

Proof. This amounts to checking the commutativity of the following square.

$$\begin{tikzcd} H^p(A;R)\otimes_R H^q(Y;R) \arrow[r, "\delta"] \arrow[d, "\times"'] & H^{p+1}(X,A;R)\otimes_R H^q(Y;R) \arrow[d, "\times"] \\ H^{p+q}(A\times Y;R) \arrow[r, "\delta"] & H^{p+q+1}(X\times Y, A\times Y;R) \end{tikzcd}$$

□

Proposition 10.7. *If a morphism $\mu\colon E^*\to F^*$ of cohomology theories is an isomorphism for $(X,A) = (point, \emptyset)$, then it is an isomorphism for all CW pairs (X,A).*

Proof. By the diagram

$$\begin{tikzcd}
E^{n-1}(X) \arrow[r, "i^*"] \arrow[d, "\mu_X"'] & E^{n-1}(A) \arrow[r, "\delta"] \arrow[d, "\mu_A"'] & E^n(X,A) \arrow[r, "j^*"] \arrow[d, "\mu_{(X,A)}"'] & E^n(X) \arrow[r, "i^*"] \arrow[d, "\mu_X"'] & E^n(A) \arrow[d, "\mu_A"'] \\
F^{n-1}(X) \arrow[r, "i^*"] & F^{n-1}(A) \arrow[r, "\delta"] & F^n(X,A) \arrow[r, "j^*"] & F^n(X) \arrow[r, "i^*"] & F^n(A)
\end{tikzcd}$$

and the five lemma it suffices to prove this for $A = \emptyset$. We next prove the case of m-dimensional CW complexes $X = X^m$, by induction on m. By the diagram

$$\begin{tikzcd}
E^{n-1}(X^{m-1}) \arrow[r, "\delta"] \arrow[d, "\mu_{X^{m-1}}"'] & E^n(X^m, X^{m-1}) \arrow[r, "j^*"] \arrow[d, "\mu_{(X^m,X^{m-1})}"'] & E^n(X^m) \arrow[r, "i^*"] \arrow[d, "\mu_{X^m}"'] & E^n(X^{m-1}) \arrow[r, "\delta"] \arrow[d, "\mu_{X^{m-1}}"'] & E^{n+1}(X^m, X^{m-1}) \arrow[d, "\mu_{(X^m,X^{m-1})}"'] \\
F^{n-1}(X^{m-1}) \arrow[r, "\delta"] & F^n(X^m, X^{m-1}) \arrow[r, "j^*"] & F^n(X^m) \arrow[r, "i^*"] & F^n(X^{m-1}) \arrow[r, "\delta"] & F^{n+1}(X^m, X^{m-1})
\end{tikzcd}$$

and the five lemma it suffices to prove that μ is an isomorphism for $(X, A) = (X^m, X^{m-1})$.

Let $\Phi \colon \coprod_\alpha (D^m, \partial D^m) \to (X^m, X^{m-1})$ be the characteristic maps of the m-cells in X. By excision and the sum axiom, the horizontal arrows in the following diagram are isomorphisms.

$$\begin{tikzcd}
E^n(X^m, X^{m-1}) \arrow[r, "\Phi^*"] \arrow[d, "\mu"'] & E^n(\coprod_\alpha (D^m, \partial D^m)) \arrow[r] \arrow[d, "\mu"'] & \prod_\alpha E^n(D^m, \partial D^m) \arrow[d, "\prod_\alpha \mu"] \\
F^n(X^m, X^{m-1}) \arrow[r, "\Phi^*"] & F^n(\coprod_\alpha (D^m, \partial D^m)) \arrow[r] & \prod_\alpha F^n(D^m, \partial D^m)
\end{tikzcd}$$

Hence it suffices to verify that $\mu_{(D^m,\partial D^m)}$ is an isomorphism. In the first diagram above, μ_A is an isomorphism by the inductive hypothesis for $A = \partial D^m$, and μ_X is an isomorphism by homotopy invariance and the case of a point for $X = D^m$. The result then follows by the five lemma.

In the case $E^* = h^*$ and $F^* = k^*$, this suffices to complete the proof, since $h^n(X) \cong h^n(X^m)$ and $k^n(X) \cong k^n(X^m)$ for $m \gg n$.

In general, we use the Milnor lim-lim^1 sequence of Theorem 3F.8. This is a natural short exact sequence

$$0 \to \lim_m{}^1 E^{n-1}(X^m) \longrightarrow E^n(X) \longrightarrow \lim_m E^n(X^m) \to 0$$

for any cohomology theory E^*. Since $\mu_{X^m} \colon h^n(X^m) \to k^n(X^m)$ is an isomorphism for all n and finite m, it follows that $\lim_m{}^1 h^{n-1}(X^m) \to \lim_m{}^1 k^{n-1}(X^m)$ and $\lim_m h^n(X^m) \to \lim_m k^n(X^m)$ are isomorphisms. Hence $\mu_X \colon h^n(X) \to h^n(X)$ is also an isomorphism. $\square$

There are variants of this Künneth theorem for relative cohomology, and for reduced cohomology. When $Y = S^k$ is a sphere, $\tilde{H}^*(S^k)$ is $\mathbb{Z}$ concentrated in degree k, and

$$\wedge \colon \tilde{H}^p(X) \otimes \tilde{H}^k(S^k) \xrightarrow{\cong} \tilde{H}^{p+k}(X \wedge S^k)$$

is an isomorphism. This agrees with the k-fold (right) suspension isomorphism

$$\sigma^k \colon \tilde{H}^*(X) \cong \tilde{H}^{*+1}(X \wedge S^1) \cong \ldots \cong \tilde{H}^{*+k}(X \wedge S^k)\,.$$

11. September 28th lecture

11.1. **Cohomology of projective spaces.** The following calculation forms the basis for many other cohomology computations.

Theorem 11.1.

(1) $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]$ *and* $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$*, where* $|x| = 1$.
(2) $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y]$ *and* $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[y]/(y^{n+1})$*, where* $|y| = 2$.
(3) $H^*(\mathbb{H}P^\infty) \cong \mathbb{Z}[z]$ *and* $H^*(\mathbb{H}P^n) \cong \mathbb{Z}[z]/(z^{n+1})$*, where* $|z| = 4$.
(4) $H^*(\mathbb{O}P^n) \cong \mathbb{Z}[w]/(w^{n+1})$*, where* $n \in \{0, 1, 2\}$ *and* $|w| = 8$.

Proof. We discuss the complex case. Recall the CW structure

$$\mathbb{C}P^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$$

where the $2n$-cell is attached by the canonical map $\phi\colon S^{2n-1} \to \mathbb{C}P^{n-1}$ taking a unit vector $z = (z_0, \dots, z_{n-1})$ in $S^{2n-1} \subset \mathbb{C}^n$ to the complex line $L = \mathbb{C}z \subset \mathbb{C}^n$, viewed as a point in $\mathbb{C}P^{n-1}$.

$$\begin{tikzcd} S^{2n-1} \arrow[r] \arrow[d, "\phi"'] & D^{2n} \arrow[d, "\Phi"] \\ \mathbb{C}P^{n-1} \arrow[r] & \mathbb{C}P^n \end{tikzcd}$$

The long exact sequence of the pair $(\mathbb{C}P^n, \mathbb{C}P^{n-1})$ simplifies to isomorphisms

$$i^*\colon H^k(\mathbb{C}P^n) \xrightarrow{\cong} H^k(\mathbb{C}P^{n-1})$$

for $k \neq 2n$ and an isomorphism

$$j^*\colon \mathbb{Z} \cong H^{2n}(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \xrightarrow{\cong} H^{2n}(\mathbb{C}P^n)$$

in the remaining degree.

The result is clear for $n \in \{0, 1\}$. Let $n \geq 2$ and suppose by induction that $H^*(\mathbb{C}P^{n-1}) \cong \mathbb{Z}[y]/(y^n)$ with $|y| = 2$. Choose $y \in H^2(\mathbb{C}P^n)$ restricting by i^* to the generator $y \in H^2(\mathbb{C}P^{n-1})$. Then

$$H^*(\mathbb{C}P^n) = \mathbb{Z}\{1, y, \dots, y^{n-1}, y^n\}$$

except possibly in degree y^n. To finish the proof, we need only show that y^n generates $H^{2n}(\mathbb{C}P^n)$. For this, it certainly suffices to show that the cup product $y^i \cup y^j$ generates $H^{2n}(\mathbb{C}P^n)$, where $0 < i < n$ and $i + j = n$.

Roughly speaking, this will be established by expressing this cup product in cohomology as an intersection product in homology. The submanifold $\mathbb{C}P^i \subset \mathbb{C}P^n$ is dual to the cohomology class y^j. Similarly, an "opposite" submanifold $\mathbb{C}P^j$ is dual to y^i. The intersection $\mathbb{C}P^i \cap \mathbb{C}P^j$ is a single point, dual to y^n. Since the class of a point generates $H_0(\mathbb{C}P^n)$, it follows that y^n generates $H^{2n}(\mathbb{C}P^n)$. As stated, this relies on the Poincaré duality between homology and cohomology, and the fact that this duality takes the intersection product to the cup product. We instead follow (Dold and) Hatcher and give a direct proof.

We view $\mathbb{C}P^n$ as the space of complex lines in

$$\mathbb{C}^{n+1} \cong \mathbb{C}^i \oplus \mathbb{C} \oplus \mathbb{C}^j\,.$$

The lines in $\mathbb{C}^i \oplus \mathbb{C} \oplus 0 = \mathbb{C}^{i+1}$ then define a subspace $\mathbb{C}P^i \subset \mathbb{C}^{n+1}$, and the lines in $0 \oplus \mathbb{C} \oplus \mathbb{C}^j \cong \mathbb{C}^{j+1}$ define a subspace $\mathbb{C}P^j \subset \mathbb{C}^{n+1}$. The latter embedding is not the one from the skeleton filtration: The intersection $\mathbb{C}P^i \cap \mathbb{C}P^j = \{p\}$ is the point given by the line $0 \oplus \mathbb{C} \oplus 0 \subset \mathbb{C}^{n+1}$. See the figure on page 220 of Hatcher.

Note that $\mathbb{C}P^{i-1} \cap \mathbb{C}P^j = \emptyset$. The inclusion

$$\mathbb{C}P^{i-1} \xrightarrow{\simeq} \mathbb{C}P^n - \mathbb{C}P^j$$

is a homotopy equivalence. To see this, construct a retraction sending

$$[z_0 : \dots : z_n] = \mathbb{C}(z_0, \dots, z_n)$$

with $(z_0, \dots, z_{i-1}) \neq 0$ to

$$[z_0 : \dots : z_{i-1} : 0 : \dots : 0] = \mathbb{C}(z_0, \dots, z_{i-1}, 0, \dots, 0)\,.$$

This is a deformation retraction, in view of the homotopy given for $t \in [0, 1]$ by sending $[z_0 : \dots : z_n]$ to

$$[z_0 : \dots : z_{i-1} : tz_i : \dots : tz_n]\,.$$

When $i = n$ and $j = 0$ this simplifies to the deformation retraction $\mathbb{C}P^n - \Phi(0) \to \mathbb{C}P^{n-1}$ obtained by radially deforming $D^{2n} - 0$ to ∂D^{2n}. It follows that

$$H^*(\mathbb{C}P^n, \mathbb{C}P^n - \mathbb{C}P^j) \xrightarrow{\cong} H^*(\mathbb{C}P^n, \mathbb{C}P^{i-1})$$

is an isomorphism. In degree $2i$, cellular cohomology tells us that

$$H^{2i}(\mathbb{C}P^n, \mathbb{C}P^{i-1}) \xrightarrow{\cong} H^{2i}(\mathbb{C}P^n)$$

is an isomorphism. Hence the upper vertical maps in the commutative diagram

$$\begin{tikzcd}
H^{2i}(\mathbb{C}P^n) \otimes H^{2j}(\mathbb{C}P^n) \arrow[r, "\cup"] & H^{2n}(\mathbb{C}P^n) \\
H^{2i}(\mathbb{C}P^n, \mathbb{C}P^n - \mathbb{C}P^j) \otimes H^{2j}(\mathbb{C}P^n, \mathbb{C}P^n - \mathbb{C}P^i) \arrow[r, "\cup"] \arrow[u] \arrow[d] & H^{2n}(\mathbb{C}P^n, \mathbb{C}P^n - \{p\}) \arrow[u] \arrow[d] \\
H^{2i}(\mathbb{C}^n, (\mathbb{C}^i - 0) \times \mathbb{C}^j) \otimes H^{2j}(\mathbb{C}^n, \mathbb{C}^i \times (\mathbb{C}^j - 0)) \arrow[r, "\cup"] & H^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\}) \\
H^{2i}(\mathbb{C}^i, \mathbb{C}^i - 0) \otimes H^{2j}(\mathbb{C}^j, \mathbb{C}^j - 0) \arrow[u, "\pi_1^* \otimes \pi_2^*"] \arrow[ur, "\times"'] &
\end{tikzcd}$$

are isomorphisms. Restriction over $\mathbb{C}^n = \mathbb{C}P^n - \mathbb{C}P^{n-1} \subset \mathbb{C}P^n$ gives the middle vertical arrows. The lower triangle is the definition of the relative cross product, which we know is an isomorphism by the Künneth theorem for $(X, A) = (\mathbb{C}^i, \mathbb{C}^i - 0)$ and $(Y, B) = (\mathbb{C}^j, \mathbb{C}^j - 0)$. The maps π_1^* and π_2^* are isomorphisms, because $\mathbb{C}^j$ and $\mathbb{C}^i$ are contractible. It remains to prove that the middle vertical maps are isomorphisms. At the right hand side, this follows by the excision theorem for $Z = \mathbb{C}P^{n-1} \subset A = \mathbb{C}P^n - \{p\} \subset X = \mathbb{C}P^n$. At the left hand side, it follows from the commutative diagram

$$\begin{tikzcd}
H^{2i}(\mathbb{C}P^n, \mathbb{C}P^{i-1}) \arrow[r, "\cong"] & H^{2i}(\mathbb{C}P^i, \mathbb{C}P^{i-1}) \\
H^{2i}(\mathbb{C}P^n, \mathbb{C}P^n - \mathbb{C}P^j) \arrow[r] \arrow[u, "\cong"] \arrow[d] & H^{2i}(\mathbb{C}P^i, \mathbb{C}P^i - \{p\}) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
H^{2i}(\mathbb{C}^n, (\mathbb{C}^i - 0) \times \mathbb{C}^j) \arrow[r, "\cong"] & H^{2i}(\mathbb{C}^i, \mathbb{C}^i - 0)
\end{tikzcd}$$

It follows that $\cup\colon H^{2i}(\mathbb{C}P^n) \otimes H^{2j}(\mathbb{C}P^n) \to H^{2n}(\mathbb{C}P^n)$ is an isomorphism, so $y^i \cup y^j$ generates $H^{2n}(\mathbb{C}P^n)$.

The case of $H^*(\mathbb{C}P^\infty)$ follows, since $H^k(\mathbb{C}P^\infty) \to H^k(\mathbb{C}P^n)$ is an isomorphism for $k < 2n + 2$. □

Definition 11.2. A real division algebra is a real vector space V with a bilinear pairing $\cdot\colon V \times V \to V$ such that $u \mapsto u \cdot v$ is an isomorphism for each $v \neq 0$, and $v \mapsto u \cdot v$ is an isomorphism for each $u \neq 0$. If V is finite-dimensional, then this is equivalent to the condition that $u \cdot v = 0$ only if $u = 0$ or $v = 0$.

Example 11.3. The real numbers $\mathbb{R}$, complex numbers $\mathbb{C} = \mathbb{R}\{1, i\}$, quaternions $\mathbb{H} = \mathbb{R}\{1, i, j, k\}$ and octonions $\mathbb{O} \cong \mathbb{R}^8$ are examples of real division algebras. Here $i^2 = -1$ in $\mathbb{C}$ and

$$i^2 = j^2 = k^2 = -1, ij = k = -ji, jk = i = -kj, ki = j = -ik$$

in $\mathbb{H}$.

Theorem 11.4 (Heinz Hopf, 1940)**.** *If $\mathbb{R}^n$ admits the structure of a real division algebra, then $n = 2^i$ for some $i \geq 0$.*

Proof. Let $\cdot\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a bilinear pairing, with $u \cdot v \neq 0$ for $u \neq 0$ and $v \neq 0$. The restricted map

$$\cdot\colon (\mathbb{R}^n - \{0\}) \times (\mathbb{R}^n - \{0\}) \longrightarrow \mathbb{R}^n - \{0\}$$

then induces a map

$$h\colon \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^{n-1} .$$

Its restriction $\mathbb{R}P^{n-1} \times \{q\} \to \mathbb{R}P^{n-1}$ is a homeomorphism for each $q \in \mathbb{R}P^{n-1}$. Likewise, its restriction $\{p\} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$ is a homeomorphism for each $p \in \mathbb{R}P^{n-1}$.

$$\begin{tikzcd}
\mathbb{R}P^{n-1} \times \{q\} \arrow[r] \arrow[dr, "\cong"'] & \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \arrow[d, "h"] & \{p\} \times \mathbb{R}P^{n-1} \arrow[l] \arrow[dl, "\cong"] \\
& \mathbb{R}P^{n-1} &
\end{tikzcd}$$

Using the Künneth theorem, we get a graded $\mathbb{Z}/2$-algebra homomorphism

$$\begin{aligned} h^*\colon \mathbb{Z}/2[x](x^n) = H^*(\mathbb{R}P^{n-1};\mathbb{Z}/2) &\longrightarrow H^*(\mathbb{R}P^{n-1}\times\mathbb{R}P^{n-1};\mathbb{Z}/2) \\ &\cong H^*(\mathbb{R}P^{n-1};\mathbb{Z}/2)\otimes H^*(\mathbb{R}P^{n-1};\mathbb{Z}/2) = \mathbb{Z}/2[x](x^n)\otimes\mathbb{Z}/2[x](x^n) = \mathbb{Z}/2[x_1,x_2]/(x_1^n,x_2^n) \end{aligned}$$

where $x_1 = x\otimes 1$ and $x_2 = 1\otimes x$. We can write $h^*(x) = c_1x_1+c_2x_2$ for some $c_i \in \mathbb{Z}/2$. The inclusion $\mathbb{R}P^{n-1}\times\{q\} \to \mathbb{R}P^{n-1}\times\mathbb{R}P^{n-1}$ induces $x_1 \mapsto x$ and $x_2 \mapsto 0$, hence sends $h^*(x)$ to c_1x. Since a homeomorphism induces an isomorphism in cohomology, it follows that c_1x generates $H^1(\mathbb{R}P^{n-1};\mathbb{Z}/2) = \mathbb{Z}/2\{x\}$, so $c_1 = 1$. Similarly, $c_2 = 1$. Hence

$$h^*(x) = x_1 + x_2\,.$$

From $x^n = 0$ it follows that

$$0 = h^*(x^n) = h^*(x)^n = (x_1+x_2)^n = \sum_{k=0}^{n}\binom{n}{k}x_1^kx_2^{n-k}$$

in $\mathbb{Z}/2[x_1,x_2]/(x_1^n,x_2^n)$. Hence $\binom{n}{k} \equiv 0 \mod 2$ for each $0<k<n$. This holds if and only if n is a power of 2. $\square$

Lemma 11.5 (Édouard Lucas, 1878)**.** *For any prime p, we have*

$$\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i} \mod p$$

where $n = \sum_i n_ip^i$ and $k = \sum_i k_ip^i$ are the p-adic expansions of n and k. Here $0 \le n_i, k_i < p$ for all i.

Proof. In $\mathbb{Z}/p[x]$ we have $(1+x)^{p^i} = 1 + x^{p^i}$ for all $i \ge 0$. Hence

$$\begin{aligned} (1+x)^n &= (1+x)^{n_0}(1+x^p)^{n_1}(1+x^{p^2})^{n_2}\cdot\ldots \\ &= [\sum_{k_0=0}^{n_0}\binom{n_0}{k_0}x^{k_0}]\cdot[\sum_{k_1=0}^{n_1}\binom{n_1}{k_1}x^{k_1p}]\cdot[\sum_{k_2=0}^{n_2}\binom{n_2}{k_2}x^{k_2p^2}]\cdot\ldots \\ &= \sum_k[\binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdot\ldots]x^k \end{aligned}$$

where $k = k_0 + k_1p + k_2p^2 + \ldots$, since there can be no cancellation of terms. $\square$

12. September 30th lecture

12.1. Poincaré duality.

Definition 12.1. An n-dimensional manifold is a (second countable) Hausdorff space M that is locally homeomorphic to $\mathbb{R}^n$. A closed n-manifold is an n-dimensional manifold that is compact.

Example 12.2. S^n and $\mathbb{R}P^n$ are n-manifolds, $\mathbb{C}P^n$ is a $2n$-manifold, $\mathbb{H}P^n$ is a $4n$-manifold, and $\mathbb{O}P^2$ is a 16-manifold. The orthogonal group $O(n)$ is an $n(n-1)/2$-manifold, and the unitary group $U(n)$ is an n^2-manifold. These examples are all closed.

The connected sum $M\#N$ of two n-manifolds is an n-manifold. The Cartesian product $M\times N$ of an n-manifold and an m-manifold is an $(n+m)$-manifold. If M and N are closed, then so are $M\#N$ (when defined) and $M\times N$. Any open subset of an n-manifold is an n-manifold, which is usually not closed.

These are 'manifolds without boundary'. We may consider manifolds with boundary later. Each finite simplicial complex embedded in $\mathbb{R}^n$ is a deformation retract of a suitable closed neighborhood that is a compact n-manifold with boundary. Hence these realize a large variety of homotopy types.

Many closed n-manifolds M admit "dual" CW structures X and Y, such that the cellular cochain complex $W^*(X)$ is isomorphic to the chain complex $W_{n-*}(Y)$:

$$\begin{tikzcd} \ldots \arrow[r] & W^k(X) \arrow[r, "\delta"] \arrow[d, "\cong"'] & W^{k+1}(X) \arrow[r] \arrow[d, "\cong"'] & \ldots \\ \ldots \arrow[r] & W_{n-k}(Y) \arrow[r, "\partial"] & W_{n-k-1}(Y) \arrow[r] & \ldots \end{tikzcd}$$

Hence $H^k(M) \cong H^k(X)$ is isomorphic to $H_{n-k}(Y) \cong H_{n-k}(M)$. This is the Poincaré duality isomorphism:

$$H^k(M) \cong H_{n-k}(M)\,.$$

Hence these closed manifolds only realize some rather special homotopy types. To clarify for which closed manifolds M this isomorphism holds, we must discuss orientability of manifolds.

12.2. **Orientations.** The isomorphisms

$$\begin{array}{ccc} H_n(D^n, S^{n-1}; G) & \xrightarrow{\partial} & \tilde{H}_{n-1}(S^{n-1}; G) \\ \downarrow & & \downarrow \\ H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; G) & \xrightarrow{\partial} & \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}; G) \end{array}$$

show that each of these groups is isomorphic to G. For $G = \mathbb{Z}$ we have a choice of two generators for $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. If

$$\sigma = [v_0, v_1, \dots, v_n]$$

is an affine n-simplex in $\mathbb{R}^n$, with $0 \in \mathrm{int}(\sigma)$, then $\sigma \in C_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ is an n-cycle. If the ordered basis

$$v_1 - v_0, \dots, v_n - v_0$$

for $\mathbb{R}^n$ is positively oriented, then $[\sigma] = g_n$ is one of the generators for $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$, while if the basis is negatively oriented, then $[\sigma] = -g_n$ is the other generator. Here 'positively oriented' can be interpreted to say that the matrix with the standard coordinates of the given vectors as columns has positive determinant. We can therefore view a choice of generator for $H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ as a choice of orientation of $\mathbb{R}^n$ (at 0). The homology cross product

$$H_p(\mathbb{R}^p, \mathbb{R}^p - \{0\}) \times H_q(\mathbb{R}^p, \mathbb{R}^p - \{0\}) \xrightarrow{\times} H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$$

for $n = p + q$ takes the positive orientations g_p and g_q to the positive orientation $g_p \times g_q = g_n$.

For any n-ball $B \subset \mathbb{R}^n$ (open or closed) and points $x, y \in B$, the inclusions induce isomorphisms

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \xleftarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n - B) \xrightarrow{\cong} H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})\,.$$

Hence an orientation of $\mathbb{R}^n$ at x, i.e., a choice of generator $\mu_x \in H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$, uniquely determines an orientation $\mu_y \in H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$ of $\mathbb{R}^n$ at y, by the condition that μ_x and μ_y come from the same class in $H_n(\mathbb{R}^n, \mathbb{R}^n - B)$.

Each point $x \in M$ in an n-manifold has an open neighborhood U that is homeomorphic to $\mathbb{R}^n$, sending x to 0. By the excision theorem, we have isomorphisms

$$H_n(M, M - \{x\}) \xleftarrow{\cong} H_n(U, U - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\})\,.$$

Definition 12.3. Let M be an n-manifold. A local orientation of M at x is a choice of generator $\mu_x \in H_n(M, M - \{x\}) \cong \mathbb{Z}$.

An orientation of M is a function $x \mapsto \mu_x$ assigning to each $x \in M$ a local orientation μ_x of M at x. The assignment is to be locally consistent, in the sense that each $x \in M$ has neighborhoods $x \in B \subset U \cong \mathbb{R}^n$ such that B corresponds to an open ball of finite radius in $\mathbb{R}^n$, and there is a class $\mu_B \in H_n(M, M - B)$ mapping to μ_y for each $y \in B$ under the natural homomorphisms

$$H_n(M, M - B) \longrightarrow H_n(M, M - \{y\})\,.$$

Following Hatcher, we introduce the notation

$$H_p(X|A) = H_p(X, X - A)$$

for the local homology of X at any subspace A. By the excision theorem, $H_p(U|A) \cong H_p(X|A)$ for any $U \subset X$ containing the closure of A in its interior, so the local homology at A only depends on how A sits in a neighborhood in X. If $A \subset B \subset X$ we refer to the homomorphism

$$H_p(X|B) \longrightarrow H_p(X|A)$$

induced by the inclusion $(X, X - B) \subset (X, X - A)$ as the 'restriction' from the local homology at B to the local homology at A. For $B = X$, this is a homomorphism $H_p(X) \to H_p(X|A)$.

An orientation of M is then a choice of generator $\mu_x \in H_n(M|x)$ for each $x \in X$, such that μ_y corresponds to μ_x under the restriction isomorphisms

$$H_n(M|x) \xleftarrow{\cong} H_n(M|B) \xrightarrow{\cong} H_n(M|y)$$

for all $y \in B$, where $x \in B \subset U$ corresponds to a point in an open ball in $\mathbb{R}^n$ under a homeomorphism $U \cong \mathbb{R}^n$.

12.3. **The orientation cover.**

Lemma 12.4. *Each n-manifold M has an orientable double covering space $\tilde{M} \to M$.*

Proof. Let

$$\tilde{M} = \{\mu_x \mid x \in M,\, \mu_x \text{ generates } H_n(M|x)\}\,.$$

The rule $\mu_x \mapsto x$ defines a projection $p\colon \tilde{M} \to M$. Consider any open ball B in a coordinate chart $U \subset M$, and let $\mu_B \in H_n(M|B) \cong \mathbb{Z}$ be one of the two possible generators. Let

$$U(\mu_B) = \{\mu_x \mid \mu_B \text{ restricts to } \mu_x\} \subset \tilde{M}\,.$$

These form a basis for a topology on $\tilde{M}$ such that $p^{-1}(B) = U(\mu_B) \sqcup U(-\mu_B)$, making p a covering projection. We get a canonical orientation of $\tilde{M}$ by assigning to $\mu_x \in \tilde{M}$ the generator $\tilde{\mu}_x$ mapping to μ_x under the isomorphism

$$p_*\colon H_n(\tilde{M}|\mu_x) \xrightarrow{\cong} H_n(M|x)\,.$$

□

Proposition 12.5. *Let M be connected. Then M is orientable if and only if $\tilde{M}$ has two components. If M is simply-connected, then M is orientable.*

Proof. If M is oriented by μ, then the rule $x \mapsto \mu_x$ defines a section $\mu\colon M \to \tilde{M}$ with $p\mu = 1$. The opposite orientation defines another section $-\mu\colon M \to \tilde{M}$. Their images give the two connected components of $\tilde{M}$.

Conversely, if $\tilde{M} = \tilde{M}_1 \sqcup \tilde{M}_2$ has two components, then $p\colon \tilde{M}_1 \to M$ is a homeomorphism, and $\tilde{M}_1$ is (canonically) oriented, so M is orientable.

If M is simply-connected, then any double cover has two components. □

12.4. **The local homology sheaf.** Let

$$M_{\mathbb{Z}} = \coprod_{x \in M} H_n(M|x)$$

be the set of classes $\alpha \in H_n(M|x) \cong \mathbb{Z}$, with $x \in M$. For any open ball B in a coordinate chart $U \subset M$ and class $\alpha_B \in H_n(M|B) \cong \mathbb{Z}$ let

$$U(\alpha_B) = \{\alpha_x \mid \alpha_B \text{ restricts to } \alpha_x\} \subset M_{\mathbb{Z}}\,.$$

These form a basis for a topology on $M_{\mathbb{Z}}$, making $p\colon M_{\mathbb{Z}} \to M$ a covering space with fibers

$$p^{-1}(x) = H_n(M|x) \cong \mathbb{Z}\,.$$

In other words, this is a sheaf of abelian groups over M. For $k \geq 0$ let $M_k \subset M_{\mathbb{Z}}$ be the subspace of classes α_k corresponding to $\pm k$ times any generator of $H_n(M|x)$. Then $M_0 \cong M$, $M_1 = \tilde{M}$, and multiplication by k defines a homeomorphism $\tilde{M} \cong M_k$ for each $k \geq 1$. Hence

$$\tilde{M} = \coprod_{k \geq 0} M_k \cong M \sqcup \coprod_{k \geq 1} \tilde{M}$$

A map $\alpha\colon M \to M_{\mathbb{Z}}$ with $p\alpha = 1$ is called a section of this covering space (or sheaf).

12.5. **R-orientations.** Non-orientable manifolds are nonetheless orientable in a weaker sense, namely with $\mathbb{Z}/2$-coefficients. We therefore generalize the notion of an orientation to an R-orientation, where R is any ring. The main examples will be $R = \mathbb{Z}$ (with the two generators 1 and -1) and $R = \mathbb{Z}/2$ (with the unique generator 1).

Let

$$M_R = \coprod_{x \in M} H_n(M|x; R)$$

be the set of classes $\alpha \in H_n(M|x; R) \cong R$, with $x \in M$. As before, there is a topology on M_R making $p\colon M_R \to M$ a covering space with fibers

$$p^{-1}(x) = H_n(M|x; R) \cong R\,.$$

In other words, this is a sheaf of R-modules over M. Let

$$\Gamma_R(M) = \{x \mapsto \alpha_x \in p^{-1}(x)\}$$

be the R-module of sections to $p\colon M_R \to M$. An R-orientation of M is a section $\mu\colon M \to M_R$ such that μ_x generates $H_n(M|x; R)$ as an R-module for each $x \in M$. For any isomorphism $H_n(M|x; R) \cong R$, this is equivalent to asking that μ_x corresponds to an invertible element, i.e., a unit, in R.

If $R = \mathbb{Z}/2$, then the isomorphism $H_n(M|x; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is uniquely defined, so there is a well-defined decomposition

$$M_{\mathbb{Z}/2} = M_0 \sqcup M_1$$

and a unique $\mathbb{Z}/2$-orientation $\mu\colon M \to M_1 \subset M_{\mathbb{Z}/2}$ taking each $x \in M$ to the nonzero class $\mu_x \in H_n(M|x; \mathbb{Z}/2) \cong \mathbb{Z}/2$. This is the (canonical and unique) $\mathbb{Z}/2$-orientation of M.

12.6. **The fundamental class.** Let R be any ring.

Theorem 12.6. *Let M be a closed, connected n-manifold. Then $H_i(M; R) = 0$ for all $i > n$. If M is R-orientable, then the restriction map*

$$H_n(M; R) \longrightarrow H_n(M|x; R) \cong R$$

is an isomorphism for each $x \in M$. If $x \mapsto \mu_x$ is an R-orientation of M, then there is a unique class $[M] \in H_n(M; R)$ that restricts to μ_x for each $x \in M$, and $[M]$ generates $H_n(M; R) \cong R$ as an R-module.

We call $[M]$ the fundamental class, or orientation class, of the closed, oriented n-manifold M.

To prove the theorem, it is better to work with a relative statement for pairs (M, A), to give some flexibility in varying A.

Proposition 12.7. *Let M be an n-manifold and let $A \subset M$ be compact.*

(1) *$H_i(M|A; R) = 0$ for all $i > n$.*
(2) *A class in $H_n(M|A; R)$ is zero if (and only if) its restriction in $H_n(M|x; R)$ is zero for each $x \in A$.*
(3) *Let $x \mapsto \alpha_x$ be a section to $p\colon M_R \to M$. Then there is a (unique) class $\alpha_A \in H_n(M|A; R)$ that restricts to α_x for each $x \in A$.*

Proof of theorem. We apply the proposition with $M = A$. The vanishing for $i > n$ is then clear. Recall the R-module $\Gamma_R(M)$ of sections to $p\colon M_R \to M$. We have an isomorphism

$$H_n(M; R) \xrightarrow{\cong} \Gamma_R(M)$$

sending α to the section $x \mapsto \alpha_x$, where α_x is the restriction of α under

$$H_n(M; R) \longrightarrow H_n(M|x; R)\,.$$

If $x \mapsto \mu_x$ is an R-orientation of M, then $M_R \cong M \times R$ with $r \cdot \mu_x$ corresponding to (x, r), for $x \in M$ and $r \in R$. Hence sections in $M_R \to M$ correspond to maps $M \to R$. Since M is connected and R is discrete, any such map is constant. It follows that evaluation at any one point $x \in R$ defines an isomorphism

$$\Gamma_R(M) \xrightarrow{\cong} R\,.$$

This is the isomorphism claimed in the theorem. □

13. October 5th lecture

To prove Proposition 12.7, we use the following lemma.

Lemma 13.1. *If the proposition holds for compact subsets A, B and $A \cap B \subset M$, then it holds for $A \cup B$.*

Proof. Let $U = M - A$ and $V = M - B$. We work with coefficients in R, but omit them from the notation. We have a long exact Mayer–Vietoris sequence in relative homology

$$\begin{aligned} \dots \to H_{i+1}(M, U \cup V) & \\ \xrightarrow{\partial} H_i(M, U \cap V) \xrightarrow{\Phi} H_i(M, U) \oplus H_i(M, V) \xrightarrow{\Psi} H_i(M, U \cup V) & \\ \xrightarrow{\partial} H_{i-1}(M, U \cap V) \to \dots . & \end{aligned}$$

This uses that $\{U, V\}$ is an excisive pair, so that the inclusion $C_*(U) + C_*(V) = C_*(U + V) \to C_*(U \cup V)$ is a quasi-isomorphism. Rewritten in terms of local homology, we obtain

$$\begin{aligned} \dots \to H_{i+1}(M|A \cap B) & \\ \xrightarrow{\partial} H_i(M|A \cup B) \xrightarrow{\Phi} H_i(M|A) \oplus H_i(M|B) \xrightarrow{\Psi} H_i(M|A \cap B) & \\ \xrightarrow{\partial} H_{i-1}(M|A \cup B) \to \dots . & \end{aligned}$$

(1) For $i > n$, the assumption that $H_i(M|A) = 0$, $H_i(M|B) = 0$ and $H_{i+1}(M|A \cap B) = 0$ implies that $H_i(M|A \cup B) = 0$ by exactness.

(2) For $i = n$, we have an exact sequence

$$0 \to H_n(M|A \cup B) \longrightarrow H_n(M|A) \oplus H_n(M|B) \longrightarrow H_n(M|A \cap B) .$$

If $\alpha \in H_n(M|A \cup B)$ restricts to zero for each $x \in A \cup B$, then the restriction $\alpha_A \in H_n(M|A)$ restricts to zero for each $x \in A$, and the restriction $\alpha_B \in H_n(M|B)$ restricts to zero for each $x \in B$. By the assumptions for A and B it follows that $\alpha_A = 0$ and $\alpha_B = 0$, so $\Phi(\alpha) = 0$, which implies $\alpha = 0$ since Φ is injective.

(3) Let $x \mapsto \alpha_x$ be a section. By assumption there are classes $\alpha_A \in H_n(M|A)$, $\alpha_B \in H_n(M|B)$ and $\alpha_{A\cap B} \in H_n(M|A \cap B)$ that restrict to α_x for all $x \in A$, $x \in B$ and $x \in A \cap B$, respectively. This uniquely determines these classes, by (2). It follows that α_A and α_B restrict to $\alpha_{A\cap B}$, so Ψ maps α_A and α_B to $\alpha_{A\cap B} - \alpha_{A\cap B} = 0$. By exactness, there is therefore a class $\alpha_{A\cup B}$ that restricts to α_A and α_B. It then restricts to α_x for each $x \in A$ and each $x \in B$, hence for each $x \in A \cup B$. □

Sketch proof of Proposition 12.7. We first check that the result holds for $M = \mathbb{R}^n$ and $A = [v_0, \dots, v_p]$ an affine p-simplex. In this case $M - A \subset M - \{x\}$ is a homotopy equivalence for each $x \in A$, so each restriction $H_i(M|A) \to H_i(M|x)$ is an isomorphism. The claims (1), (2) and (3) then readily follow.

Next, we use the lemma and induction to confirm the result for $M = \mathbb{R}^n$ and A a finite simplicial complex, with linearly embedded simplices.

Next, we deduce the result for $M = \mathbb{R}^n$ and A any compact subset. For injectivity/uniqueness of α_A this requires a chain level argument to reduce to the case of a suitable finite simplicial complex K containing A. See case (4) on page 237 of Hatcher.

Finally, we deduce the result for arbitrary M, since any compact A can be written as a finite union of compact subsets, each contained in some coordinate chart that is homeomorphic to $\mathbb{R}^n$. □

13.1. The cap product.

Let $\phi \in C^p(X; R)$ and $\psi \in C^q(X; R)$ be cochains, and let $\sigma \colon \Delta^{p+q} \to X$ be a singular simplex. Recall that the cup product $\phi \cup \psi$ is defined by

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_p]) \cdot \psi(\sigma|[v_p, \dots, v_{p+q}]) .$$

This expression can be calculated in two steps, as

$$(\phi \cup \psi)(\sigma) = \phi(\psi \cap \sigma)$$

where the 'cap product'

$$\begin{aligned} \psi \cap \sigma &= \sigma|[v_0, \dots, v_p] \cdot \psi(\sigma|[v_p, \dots, v_{p+q}]) \\ &= \psi(\sigma|[v_p, \dots, v_{p+q}]) \cdot \sigma|[v_0, \dots, v_p] \end{aligned}$$

lies in $C_p(X;R)$. We extend this R-linearly to R-chains $\alpha = \sum_i r_i\sigma_i \in C_{p+q}(X;R)$, to obtain a homomorphism

$$\psi \cap -\colon C_{p+q}(X;R) \longrightarrow C_p(X;R)\,.$$

Written in terms of the evaluation pairing

$$\begin{aligned} C^p(X;R) \otimes_R C_p(X;R) &\xrightarrow{\langle -,-\rangle} R \\ \phi \otimes \alpha &\longmapsto \phi(\alpha) \end{aligned}$$

the cup product $- \cup \psi$ is adjoint to the cap product $\psi \cap -$:

$$\langle \phi \cup \psi, \alpha\rangle = \langle \phi, \psi \cap \alpha\rangle\,.$$

We get a left action of $C^*(X;R)$ on $C_*(X;R)$.

Lemma 13.2. *For $\phi \in C^p(X;R)$, $\psi \in C^q(X;R)$ and $\alpha \in C_n(X;R)$ we have*

$$\phi \cap (\psi \cap \alpha) = (\phi \cup \psi) \cap \alpha$$

in $C_{n-p-q}(X;R)$.

Proof. For all $\theta \in C^{n-p-q}(X;R)$ we have $\langle \theta, \phi \cap (\psi \cap \alpha)\rangle = \langle \theta \cup \phi, \psi \cap \alpha\rangle = \langle \theta \cup \phi \cup \psi, \alpha\rangle = \langle \theta, (\phi \cup \psi) \cap \alpha\rangle$. □

The Leibniz rule takes the following form.

Lemma 13.3. *For $\psi \in C^q(X;R)$ and $\alpha \in C_{p+q}(X;R)$ we have*

$$\partial(\psi \cap \alpha) = \psi \cap \partial\alpha + (-1)^p \delta\psi \cap \alpha$$

Proof. For all $\phi \in C^{p-1}(X;R)$,

$$\begin{aligned} \langle \delta\phi \cup \psi, \alpha\rangle &= \langle \delta\psi, \psi \cap \alpha\rangle = \langle \phi, \partial(\psi \cap \alpha)\rangle \\ \langle \delta(\phi \cup \psi), \alpha\rangle &= \langle \phi \cup \psi, \partial\alpha\rangle = \langle \phi, \psi \cap \partial\alpha\rangle \\ \langle \phi \cup \delta\psi, \alpha\rangle &= \langle \phi, \delta\psi \cap \alpha\rangle\,. \end{aligned}$$

Hence $\delta\phi \cup \psi = \delta(\phi \cup \psi) + (-1)^p \phi \cup \delta\psi$ implies the lemma. □

We can therefore make the following definition.

Definition 13.4. The cap product

$$H^q(X;R) \otimes_R H_{p+q}(X;R) \xrightarrow{\cap} H_p(X;R)$$

is given by

$$[\psi] \cap [\alpha] = [\psi \cap \alpha]$$

for all q-cocycles ψ and $(p+q)$-cycles α.

Lemma 13.5. *For $x \in H^p(X;R)$, $y \in H^q(X;R)$, $\xi \in H_{p+q}(X;R)$ and $\eta \in H_n(X;R)$ we have*

$$\begin{aligned} \langle x \cup y, \xi\rangle &= \langle x, y \cap \xi\rangle \\ x \cap (y \cap \eta) &= (x \cup y) \cap \eta \end{aligned}$$

Proof. The evaluation pairing $\langle -,-\rangle\colon H^p(X;R) \otimes_R H_p(X;R) \to R$ is given by $\langle [\phi], [\alpha]\rangle = \langle \phi, \alpha\rangle = \phi(\alpha)$, so these formulas follow from the chain level statements. □

Example 13.6. We can explicitly calculate cap products in simplicial homology for Δ-complexes. Alternatively, we can often extract the same information from earlier cup product calculations. When $X = T^2$ with $H_1(X) = \mathbb{Z}\{\alpha, \beta\}$, $H^1(X) = \mathbb{Z}\{a, b\}$ with the ordered basis (a, b) being dual to the ordered basis (α, β), and $H^2(X) = \mathbb{Z}\{a \cup b\}$, we can choose a generator $[X] \in H_2(X)$ with $\langle a \cup b, [X]\rangle = 1$. Then $b \cap [X] \in H_1(X)$ satisfies $\langle a, b \cap [X]\rangle = \langle a \cup b, [X]\rangle = 1$ and $\langle b, b \cap [X]\rangle = \langle b \cup b, [X]\rangle = 0$, so $b \cap [X] = \alpha$. Similarly, $a \cap [X] \in H_1(X)$ satisfies $\langle a, a \cap [X]\rangle = \langle a \cup a, [X]\rangle = 0$ and $\langle b, a \cap [X]\rangle = \langle b \cup a, [X]\rangle = -1$, so $a \cap [X] = -\beta$. Hence $y \mapsto y \cap [X]$ is the isomorphism

$$\begin{aligned} H^1(X) &\xrightarrow{\cong} H_1(X) \\ a &\longmapsto -\beta \\ b &\longmapsto \alpha \end{aligned}$$

This example is typical for closed oriented manifolds.

Theorem 13.7 (Closed Poincaré duality). *Let M be a closed R-oriented n-manifold, with fundamental class $[M] \in H_n(M;R)$. The homomorphism*

$$D = D_M \colon H^k(M;R) \xrightarrow{\cong} H_{n-k}(M;R)$$

defined by $D(y) = y \cap [M]$ is an isomorphism for all k.

Remark 13.8. Hatcher's definition of the cap product is a little different. If ours is a 'left cap product', then his is a 'right cap product'

$$C_{p+q}(X;R) \otimes_R C^p(X;R) \longrightarrow C_q(X;R)$$

given by

$$\sigma \cap \phi = \phi(\sigma|[v_0, \dots, v_p]) \cdot \sigma|[v_p, \dots, v_{p+q}] .$$

Using the chain map ρ from the proof of graded commutativity, the definitions are related by

$$\rho(\psi \cap \sigma) = (-1)^{pq} \rho(\sigma) \cap \rho(\psi)$$

at the chain level, so that

$$y \cap \xi = (-1)^{pq} \xi \cap y$$

in cohomology, where $y \in H^q(X;R)$ and $\xi \in H_{p+q}(X;R)$. We made this choice to emphasize the adjoint role of $- \cup \psi$ and $\psi \cap -$, but a reader wishing to keep track of signs in Hatcher's proof of Poincaré duality may find it easier to adhere to his conventions.

14. October 7th lecture

14.1. **Relative cap products, and naturality.** The proof of Poincaré duality will rely on an inductive argument involving a relative statement for non-compact R-oriented manifolds. To make sense of this, we will need a relative cap product.

Consider $A, B \subset X$. If $\psi \in C^q(X, B; R)$ vanishes on chains in B, and $\alpha \in C_{p+q}(A + B; R) \subset C_{p+q}(X;R)$ is a sum of chains in A and in B, then $\psi \cap \alpha \in C_p(A;R) \subset C_p(X;R)$. Hence, if $\alpha \in C_{p+q}(X, A+B; R)$ then $\psi \cap \alpha \in C_p(X, A; R)$ and we get relative cap products

$$\cap \colon C^q(X, B; R) \otimes_R C_{p+q}(X, A+B; R) \longrightarrow C_p(X, A; R)$$

and

$$\cap \colon H^q(X, B; R) \otimes_R H_{p+q}(X, A+B; R) \longrightarrow H_p(X, A; R) .$$

When $\{A, B\}$ is an excisive pair, so that $H_*(X, A+B; R) \cong H_*(X, A \cup B; R)$, then we can write this as

$$\cap \colon H^q(X, B; R) \otimes_R H_{p+q}(X, A \cup B; R) \longrightarrow H_p(X, A; R) .$$

The excision hypothesis is satisfied if A and B are open in X, or are subcomplex of a CW complex X, or in the following two cases.

$$\cap \colon H^q(X, B; R) \otimes_R H_{p+q}(X, B; R) \longrightarrow H_p(X; R)$$
$$\cap \colon H^q(X; R) \otimes_R H_{p+q}(X, A; R) \longrightarrow H_p(X, A; R) .$$

The following naturality statement is sometimes known as Frobenius reciprocity, or the projection formula.

Lemma 14.1. *For any map $f \colon X \to Y$ and classes $y \in H^q(Y;R)$ and $\xi \in H_{p+q}(X;R)$ the relation*

$$y \cap f_*(\xi) = f_*(f^*(y) \cap \xi)$$

holds. In other words, $f_* \colon H_*(X;R) \to H_*(Y;R)$ is a left $H^*(Y;R)$-module homomorphism, where $H^*(Y;R)$ acts via $f^* \colon H^*(Y;R) \to H^*(X;R)$ and the cap product on $H_*(X)$, and via the cap product on $H_*(Y)$.

Proof. We prove this at the chain level. For $\phi \in C^p(Y;R)$, $\psi \in C^q(X;R)$ and $\alpha \in C_{p+q}(X;R)$ we have $f^\#(\phi \cup \psi) = f^\#(\phi) \cup f^\#(\psi)$, which implies that $\langle f^\#(\phi \cup \psi), \alpha \rangle = \langle \phi \cup \psi, f_\#(\alpha) \rangle = \langle \phi, \psi \cap f_\#(\alpha) \rangle$ equals $\langle f^\#(\phi) \cup f^\#(\psi), \alpha \rangle = \langle f^\#(\phi), f^\#(\psi) \cap \alpha \rangle = \langle \phi, f_\#(f^\#(\psi) \cap \alpha) \rangle$. Hence

$$\psi \cap f_\#(\alpha) = f_\#(f^\#(\psi) \cap \alpha) .$$

□

14.2. **Directed colimits.** For an n-manifold M with R-orientation $x \mapsto \mu_x$, and any compact subset $K \subset M$, we showed that there is a unique class $\mu_K \in H_n(M|K;R) = H_n(M, M-K;R)$ that restricts to μ_x for each $x \in K$. Cap product with μ_K defines homomorphisms

$$- \cap \mu_K \colon H^k(M, M-K;R) \longrightarrow H_{n-k}(M;R)$$
$$- \cap \mu_K \colon H^k(M;R) \longrightarrow H_{n-k}(M, M-K;R)$$

We shall concentrate on the first of these, in the colimit as K expands over all compact subsets of M. This leads to the notion

$$H^k_c(M;R) = \operatorname*{colim}_K H^k(M, M-K;R)$$

of cohomology with compact supports. (The second cap product leads instead to consideration of a limit

$$H^{lf}_{n-k}(M;R) = \lim_K H_{n-k}(M, M-K;R)$$

sometimes known as locally finite homology.)

Let X be any space. As A ranges over the partially ordered set of compact subsets of X, the homology groups $H_p(A;G)$ form a diagram of abelian groups, all mapping to $H_p(X;G)$. For each inclusion $A \subset B$ of compact subsets, the diagram contains the homomorphism $H_p(A;G) \to H_p(B;G)$ induced by this inclusion. We shall see that the algebraic colimit of this diagram maps isomorphically to $H_p(X;G)$, so that

$$\operatorname*{colim}_A H_p(A;G) \cong H_p(X;G)\,.$$

Note that for $A \subset B \subset X$ the inclusion $(X, X-B) \subset (X, X-A)$ induces a homomorphism

$$H^*(X, X-A;G) \longrightarrow H^*(X, X-B;G)$$

that we may refer to as 'extension' along $A \subset B$. If we write

$$H^*(X|A;G) = H^*(X, X-A;G)$$

for the local cohomology of X at A, then extension from A to B is a homomorphism $H^*(X|A;G) \to H^*(X|B;G)$. As A ranges over the compact subsets of X, the local cohomology groups $H^q(X|A;G)$ also form a diagram of G-modules (or abelian groups), all mapping to $H^q(X;G)$. The algebraic colimit

$$H^q_c(X;G) = \operatorname*{colim}_A H^q(X|A;G)$$

of this diagram maps to $H^q(X;G)$, but this map may not be an isomorphism.

Definition 14.2. A nonempty partially ordered set $(I, \le)$ is 'directed' if for each pair $\alpha, \beta \in I$ there is an element $\gamma \in I$ with $\alpha \le \gamma$ and $\beta \le \gamma$. Equivalently, I is directed if each finite subset of I has an upper bound.

An I-shaped diagram of abelian groups is a functor $G \colon I \to \mathrm{Ab}$. It associates to each $\alpha \in I$ an abelian group G_α, and for each pair (α, β) with $\alpha \le \beta$ it associates a homomorphism $f_{\alpha,\beta} \colon G_\alpha \to G_\beta$, such that $f_{\alpha,\alpha} = 1$ and $f_{\beta,\gamma} f_{\alpha,\beta} = f_{\alpha,\gamma}$ for all $\alpha \le \beta \le \gamma$ in I. The colimit

$$\operatorname*{colim}_I G = \operatorname*{colim}_{\alpha \in I} G_\alpha$$

is the initial abelian group H receiving compatible homomorphisms

$$\iota_\beta \colon G_\beta \longrightarrow H$$

for all $\beta \in I$. It can be constructed as the quotient group

$$\operatorname*{colim}_{\alpha \in I} G_\alpha = \bigoplus_{\alpha \in I} G_\alpha / (\sim)$$

where $\sim$ is the equivalence relation identifying $x \in G_\alpha \subset \bigoplus_{\alpha \in I} G_\alpha$ with $f_{\alpha,\beta}(x) \in G_\beta \subset \bigoplus_{\alpha \in I} G_\alpha$. When I is directed we call this a directed colimit.

Any element in $\operatorname{colim}_{\alpha\in I} G_\alpha$ is the equivalence class of a finite sum $x_1+\cdots+x_n$ with $x_i \in G_{\alpha_i}$. When I is directed, we can choose β with $\alpha_1,\dots,\alpha_n \le \beta$. Then $x_1+\cdots+x_n$ is identified with $f_{\alpha_1,\beta}(x_1)+\cdots+f_{\alpha_n,\beta}(x_n)$ in G_β. Hence each element in $\operatorname{colim}_{\alpha\in I} G_\alpha$ is in the image of some ι_β.

Conversely, if $x, y \in G_\beta$ have the same image in $\operatorname{colim}_{\alpha\in I} G_\alpha$, then $y - x \in \bigoplus_{\alpha\in I} G_\alpha$ is a finite sum of expressions $f_{\alpha_i,\beta_i}(x_i) - x_i$. When I is directed we can choose γ with $\alpha_i, \beta_i, \beta \le \gamma$, and then each expression $f_{\alpha_i,\beta_i}(x_i) - x_i$ maps to zero in G_γ. Hence $f_{\beta,\gamma}(y - x) = 0$, so x becomes equal to y under $G_\beta \to G_\gamma$ for some γ.

It follows that a directed colimit of abelian groups can be calculated at the level of underlying sets, since the latter colimit has a canonical abelian group structure.

Lemma 14.3. *Let $G\colon I \to \mathrm{Ab}$ be a directed diagram. If each homomorphism $f_{\alpha,\beta}\colon G_\alpha \to G_\beta$ is an isomorphism, then each structure map $\iota_\beta\colon G_\beta \to \operatorname{colim}_{\alpha\in I} G_\alpha$ is an isomorphism.*

Example 14.4. Let X be any space. The set I of compact subsets $A \subset X$ is partially ordered, with $A \le B$ meaning that $A \subset B$. It is directed, because (the empty set is compact and) for any two compact subsets $A, B \subset X$, their union $A \cup B$ is compact, and $A \subset A \cup B$ and $B \subset A \cup B$.

Proposition 14.5. *If a space $X = \bigcup_{\alpha\in I} X_\alpha$ is the union of a directed set of subspaces, with the property that each compact subset of X is contained in some X_α, then the canonical map*

$$\operatorname*{colim}_{\alpha\in I} H_p(X_\alpha; G) \xrightarrow{\cong} H_p(X; G)$$

is an isomorphism.

Proof. Each $\xi \in H_p(X;G)$ is the class of a p-cycle in $C_p(X;G)$, which is a finite linear combination of singular simplices. The union of the images of these simplices is compact, hence is contained in some X_α. Thus ξ is in the image of $H_p(X_\alpha;G)$, and therefore also in the image of the canonical map displayed above.

If $\eta \in H_p(X_\alpha;G)$ maps to zero in $H_p(X;G)$ then η is the class of some cycle in $C_p(X_\alpha;G)$, whose image in $C_p(X;G)$ is the boundary of some $(p+1)$-chain. This chain is a finite linear combination of singular simplices, and the union of their images is compact, hence is contained in some X_β. Choosing γ greater than α and β, it follows that η maps to zero in $H_p(X_\gamma;G)$, so that η represents zero in $\operatorname{colim}_{\alpha\in I} H_p(X_\alpha;G)$. □

Remark 14.6. Many authors sloppily omit the condition that each compact subset of X is contained in some X_α, saying that homology commutes with directed (or filtered) colimits of spaces. This leads to incorrect statements. For instance, if (X, d) is a metric space, and X_α ranges over the closed, countable subsets of X, then X has the topology of the union $\bigcup_\alpha X_\alpha$, but no uncountable subset of X is contained in one of the X_α. The case $X = S^1$ is representative. Here $H_1(X_\alpha) = 0$ for each α, so that $\operatorname{colim}_\alpha H_1(X_\alpha) = 0$, but $H_1(X) \cong \mathbb{Z} \neq 0$.

14.3. **Cohomology with compact supports.**

Definition 14.7. The q-th compactly supported cohomology group of a space X is given by the directed colimit

$$H^q_c(X;G) = \operatorname*{colim}_{A} H^q(X|A;G)$$

where A ranges over the partially ordered set of compact subsets of X. The diagram consists of the extension homomorphisms

$$H^q(X|A;G) \longrightarrow H^q(X|B;G)$$

for compact $A \subset B$ in X. The compatible homomorphisms $j\colon H^q(X|A;G) \to H^q(X;G)$ induce a canonical homomorphism $H^q_c(X;G) \to H_q(X;G)$.

Definition 14.8. A subset J of a directed set $(I, \le)$ is 'cofinal' if for each $\alpha \in I$ there exists a $\beta \in J$ with $\alpha \le \beta$. (Mac Lane simply says that J is 'final' in I.)

Example 14.9. If I has a greatest element β, then $J = \{\beta\}$ is cofinal in I.

Lemma 14.10. *If $J \subset I$ is cofinal, then the canonical homomorphism*

$$\operatorname*{colim}_{\beta\in J} G_\beta \xrightarrow{\cong} \operatorname*{colim}_{\alpha\in I} G_\alpha$$

is an isomorphism for any I-shaped diagram G (such that the left hand colimit exists). If I has a greatest element β, then $\iota_\beta \colon G_\beta \to \operatorname{colim}_{\alpha\in I} G_\alpha$ is an isomorphism.

Example 14.11. $H^q_c(X;G) \to H^q(X;G)$ is an isomorphism if X is compact.

Lemma 14.12.

$$H^q_c(\mathbb{R}^n;G) \cong \begin{cases} G & \text{for } q=n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The set J of closed balls $B_k = B(0,k)$ of integer radius $k \ge 1$ centered at the origin is a cofinal subset of the set I of compact subsets of $\mathbb{R}^n$. Hence

$$\operatorname*{colim}_k H^q(\mathbb{R}^n|B_k;G) \xrightarrow{\cong} H^q_c(\mathbb{R}^n;G)$$

is an isomorphism. Here $H^q(\mathbb{R}^n|B_k;G) = 0$ for $q \ne n$ and $H^n(\mathbb{R}^n|B_k;G) \cong G$ for each k. Each inclusion $B_k \subset B_{k+1}$ induces an isomorphism $H^n(\mathbb{R}^n|B_k;G) \to H^n(\mathbb{R}^n|B_{k+1};G)$, so $\operatorname{colim}_k H^n(\mathbb{R}^n|B_k;G) \cong G$. □

Remark 14.13. We note that this implies that $H^q_c(X;G)$ is a homotopy invariant of X if X is compact, but not for Euclidean spaces. A map $f \colon X \to Y$ does not in general induce a homomorphism $H^q_c(Y;G) \to H^q_c(X;G)$, but this holds if f is proper, i.e., if $f^{-1}(L) \subset X$ is compact for each compact $L \subset Y$. It follows that $H^q_c(X;G)$ is a proper homotopy invariant, and that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not proper homotopy equivalent for $n \ne m$.

14.4. **Poincaré duality for non-compact manifolds.** Let M be an R-oriented n-manifold, possibly non-compact. For each compact $K \subset M$ we have an orientation class $\mu_K \in H_n(M|K;R)$ that restricts to the local orientation $\mu_x \in H_n(M|x;R)$ for each $x \in K$. The cap product with μ_K then defines the homomorphism

$$- \cap \mu_k \colon H^k(M|K;R) \longrightarrow H_{n-k}(M;R)\,.$$

We claim that these are compatible as K varies through the directed set of compact subsets, partially ordered by inclusions.

Lemma 14.14. *Let $K \subset L$ be an inclusion of compact subsets of M. Then the square*

$$\begin{tikzcd} H^k(M|K;R) \arrow[r, "-\cap\mu_K"] \arrow[d] & H_{n-k}(M;R) \arrow[d, equal] \\ H^k(M|L;R) \arrow[r, "-\cap\mu_L"] & H_{n-k}(M;R) \end{tikzcd}$$

commutes.

Proof. The projection formula for the map $i \colon (M, M-L) \to (M, M-K)$ gives the relation

$$y \cap i_*(\mu_L) = i_*(i^*(y) \cap \mu_L)$$

for all $y \in H^k(M|K;R)$. Here $i^*(y) \cap \mu_L$ is the image of y under the extension map to $H^k(M|L;R)$ and the cap product to $H_{n-k}(K;R)$. Since $i \colon M \to M$ is the identity, the right hand i_* is also the identity. Furthermore, $i_*(\mu_L)$ is the restriction of the orientation class μ_L to $H_n(M|K;R)$. Since its restriction to $H_n(M|x;R)$ is μ_x for each $x \in K$, it must be equal to μ_K. Hence we can rewrite the projection formula in this special case as

$$y \cap \mu_K = i^*(y) \cap \mu_L\,.$$

□

Passing to directed colimits, as K varies of the partially ordered set of compact subsets, we obtain a homomorphism

$$D_M \colon H^k_c(M;R) = \operatorname*{colim}_K H^k(M|K;R) \xrightarrow{\operatorname{colim}_K -\cap\mu_K} \operatorname*{colim}_K H_{n-k}(M;R) \cong H_{n-k}(M;R)\,.$$

This generalizes the duality isomorphism for compact M, since the left hand colimit is realized by $H^k(M;R) = H^k(M|M;R)$ if M is compact.

Theorem 14.15 (Poincaré duality)**.** *Let M be an R-oriented n-manifold. The homomorphism*

$$D = D_M \colon H^k_c(M;R) \xrightarrow{\cong} H_{n-k}(M;R)$$

defined by $D(y) = y \cap \mu_K$ for $y \in H^k(M|K;R) \to H^k_c(M;R)$ is an isomorphism for all k.

15. October 12th lecture

15.1. **Directed colimits are exact.** Let $(I, \le)$ be a directed set. Consider maps i and j of I-shaped diagrams of abelian groups

$$\begin{tikzcd}
\dots \arrow[r] & A_\alpha \arrow[r, "f_{\alpha,\beta}"] \arrow[d, "i_\alpha"] & A_\beta \arrow[r] \arrow[d, "i_\beta"] & \dots \\
\dots \arrow[r] & B_\alpha \arrow[r, "g_{\alpha,\beta}"] \arrow[d, "j_\alpha"] & B_\beta \arrow[r] \arrow[d, "j_\beta"] & \dots \\
\dots \arrow[r] & C_\alpha \arrow[r, "h_{\alpha,\beta}"] & C_\beta \arrow[r] & \dots
\end{tikzcd}$$

Lemma 15.1. *Suppose that that* $\mathrm{im}(i_\alpha) = \ker(j_\alpha)$ *for each $\alpha \in I$, so that $A_\alpha \to B_\alpha \to C_\alpha$ is exact for each α. Then the induced homomorphisms*

$$\operatorname*{colim}_\alpha A_\alpha \xrightarrow{i} \operatorname*{colim}_\alpha B_\alpha \xrightarrow{j} \operatorname*{colim}_\alpha C_\alpha$$

satisfy $\mathrm{im}(i) = \ker(j)$*. Hence passage to directed colimits preserve exactness.*

Proof. If $a \in \mathrm{colim}_\alpha A_\alpha$ then $a = \iota_\alpha(a_\alpha)$ for some $\alpha \in I$ and $a_\alpha \in A_\alpha$. Then $ji(a) = \iota_\alpha(j_\alpha i_\alpha(a)) = 0$, since $j_\alpha i_\alpha = 0$. Hence $ji = 0$.

Conversely, if $b \in \mathrm{colim}_\alpha B_\alpha$ then $b = \iota_\alpha(b_\alpha)$ for some $\alpha \in I$ and $b_\alpha \in B_\alpha$. If $j(b) = 0$ then $j_\alpha(b_\alpha)$ maps to 0 in $\mathrm{colim}_\alpha C_\alpha$, so for some $\beta \ge \alpha$ we have $h_{\alpha,\beta} j_\alpha(b_\alpha) = 0$ in C_β. This equals $j_\beta(b_\beta)$, where $b_\beta = g_{\alpha,\beta}(b_\alpha)$ in B_β. By exactness at B_β it follows that $b_\beta = i_\beta(a_\beta)$ for some $a_\beta \in A_\beta$. Then $b = \iota_\beta(b_\beta) = \iota_\beta i_\beta(a_\beta) = i\iota_\beta(a_\beta) = i(a)$, with $a = \iota_\beta(a_\beta)$. Hence $\ker(j) = \mathrm{im}(i)$. □

Corollary 15.2. *Suppose that* $\mathrm{im}(i_\alpha) \subset \ker(j_\alpha)$*. Then*

$$\operatorname*{colim}_\alpha \frac{\ker(j_\alpha)}{\mathrm{im}(i_\alpha)} \cong \frac{\ker(j)}{\mathrm{im}(i)} .$$

Hence passage to directed colimits preserves homology.

Proof. The exact sequences

$$0 \to \ker(i_\alpha) \longrightarrow A_\alpha \xrightarrow{i_\alpha} B_\alpha$$

$$0 \to \ker(j_\alpha) \longrightarrow B_\alpha \xrightarrow{j_\alpha} C_\alpha$$

for $\alpha \in I$ give exact sequences

$$0 \to \operatorname*{colim}_\alpha \ker(i_\alpha) \longrightarrow A \xrightarrow{i} B$$

$$0 \to \operatorname*{colim}_\alpha \ker(j_\alpha) \longrightarrow B \xrightarrow{j} C ,$$

showing that $\mathrm{colim}_\alpha \ker(i_\alpha) \cong \ker(i)$ and $\mathrm{colim}_\alpha \ker(j_\alpha) \cong \ker(j)$. The exact sequences

$$A_\alpha \longrightarrow \mathrm{im}(i_\alpha) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{im}(i_\alpha) \longrightarrow B_\alpha$$

then give exact sequences

$$A \longrightarrow \operatorname*{colim}_\alpha \mathrm{im}(i_\alpha) \longrightarrow 0$$

$$0 \longrightarrow \operatorname*{colim}_\alpha \mathrm{im}(i_\alpha) \longrightarrow B$$

showing that $\operatorname{colim}_\alpha \operatorname{im}(i_\alpha) \cong \operatorname{im}(i)$. Finally, the exact sequences

$$\operatorname{im}(i_\alpha) \longrightarrow \ker(j_\alpha) \longrightarrow \frac{\ker(j_\alpha)}{\operatorname{im}(i_\alpha)} \longrightarrow 0$$

give an exact sequence

$$\operatorname{im}(i) \longrightarrow \ker(j) \longrightarrow \operatorname*{colim}_\alpha \frac{\ker(j_\alpha)}{\operatorname{im}(i_\alpha)} \longrightarrow 0\,,$$

showing that

$$\operatorname*{colim}_\alpha \frac{\ker(j_\alpha)}{\operatorname{im}(i_\alpha)} \cong \frac{\ker(j)}{\operatorname{im}(i)}\,.$$

□

15.2. **A gluing lemma.** Let M be an R-oriented n-manifold. We omit R from the notation.

Lemma 15.3. *If $M = U \cup V$ with U and V open, then there is a diagram of long exact Mayer–Vietoris sequences, commutative up to signs:*

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H_c^{k-1}(M) \arrow[r, "D_M"] \arrow[d, "\delta"'] & H_{n-k+1}(M) \arrow[d, "\partial"'] \\
H_c^k(U \cap V) \arrow[r, "D_{U\cap V}"] \arrow[d] & H_{n-k}(U \cap V) \arrow[d] \\
H_c^k(U) \oplus H_c^k(V) \arrow[r, "D_U \oplus D_V"] \arrow[d] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[d] \\
H_c^k(M) \arrow[r, "D_M"] \arrow[d, "\delta"'] & H_{n-k}(M) \arrow[d, "\partial"'] \\
H_c^{k+1}(U \cap V) \arrow[r, "D_{U\cap V}"] \arrow[d] & H_{n-k-1}(U \cap V) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Sketch proof. For compact $K \subset U$ and $L \subset V$ we have a diagram of vertical Mayer–Vietoris sequences and horizontal excision isomorphisms and cap products.

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
H^{k-1}(M|K \cup L) \arrow[rr, "-\cap\mu_{K\cup L}"] \arrow[d, "\delta"'] & & H_{n-k+1}(M) \arrow[d, "\partial"'] \\
H^k(M|K \cap L) \arrow[r, "\cong"] \arrow[d] & H^k(U \cap V|K \cap L) \arrow[r, "-\cap\mu_{K\cap L}"] & H_{n-k}(U \cap V) \arrow[d] \\
H^k(M|K) \oplus H^k(M|L) \arrow[r, "\cong"] \arrow[d] & H^k(U|K) \oplus H^k(V|L) \arrow[r, "-\cap\mu_K \oplus -\cap\mu_L"] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[d] \\
H^k(M|K \cup L) \arrow[rr, "-\cap\mu_{K\cup L}"] \arrow[d, "\delta"'] & & H_{n-k}(M) \arrow[d, "\partial"'] \\
H^{k+1}(M|K \cap L) \arrow[r, "\cong"] \arrow[d] & H^{k+1}(U \cap V|K \cap L) \arrow[r, "-\cap\mu_{K\cap L}"] & H_{n-k-1}(U \cap V) \arrow[d] \\
\vdots & & \vdots
\end{tikzcd}$$

An elaborate check of definitions confirms that each square commutes, up to signs. See Hatcher, pages 246–247. Taking this for granted, we can pass to directed colimits over K and L. As we saw in Lemma 15.1, the directed colimit of an exact sequence is still exact. By definition, $\operatorname{colim}_K H^k(U|K) = H^k_c(U)$ and $\operatorname{colim}_L H^k(V|L) = H^k_c(V)$. By cofinality, $\operatorname{colim}_{K,L} H^k(M|K \cup L) = H^k_c(M)$ and $\operatorname{colim}_{K,L} H^k(M|K \cap L) = H^k_c(U \cap V)$. This leads to the diagram of the lemma. $\square$

15.3. **An exhaustion lemma.**

Lemma 15.4. *If $M = \bigcup_i U_i$ is the union of an increasing sequence*

$$U_1 \subset U_2 \subset \dots$$

of open subsets, and each $D_{U_i} \colon H^k_c(U_i) \to H_{n-k}(U_i)$ is an isomorphism, then so is $D_M \colon H^k_c(M) \to H_{n-k}(M)$.

Proof. We have commutative diagrams

$$\begin{tikzcd}
H^k_c(U_i) \arrow[r, "D_{U_i}", "\cong"'] \arrow[d] & H_{n-k}(U_i) \arrow[d] \\
H^k_c(U_{i+1}) \arrow[r, "D_{U_{i+1}}", "\cong"'] & H_{n-k}(U_{i+1})
\end{tikzcd}$$

since each compact $K \subset U_i$ is also a compact $K \subset U_{i+1}$. Passing to sequential (hence directed) colimits over i, we get an isomorphism

$$\operatorname*{colim}_i H^k_c(U_i) \xrightarrow{\cong} \operatorname*{colim}_i H_{n-k}(U_i)\,.$$

Here the left hand side is $H^k_c(M)$, and the right hand side is $H_{n-k}(M)$, because each compact $K \subset M$ is contained in some U_i. The composite isomorphism from $H^k_c(M)$ to $H_{n-k}(M)$ equals D_M. $\square$

15.4. **Sketch proof of Theorem 14.15.**

Proof. We first check that the theorem holds for $M = \mathbb{R}^n$. For any closed ball $B \subset \mathbb{R}^n$ the structure map

$$H^k(\mathbb{R}^n|B) \xrightarrow{\cong} H^k_c(\mathbb{R}^n)$$

is an isomorphism, so it suffices to check that

$$- \cap \mu_B \colon H^k(\mathbb{R}^n|B) \longrightarrow H_{n-k}(\mathbb{R}^n)$$

is an isomorphism. This is trivial if $k \neq n$, and easy for $k = n$. We may even take $B = \{0\}$.

Next, the theorem holds for any open subset $M \subset \mathbb{R}^n$. We can write such an M as the union of a countable collection of bounded, convex, open subsets V_i, each of which is homeomorphic to $\mathbb{R}^n$. Let $U_i = V_1 \cup \dots \cup V_i$. Then the theorem holds for each V_i, hence for each U_i by induction and the gluing result, and then for M by the exhaustion result.

If M is second countable, it is the union of countably many open subsets, each homeomorphic to $\mathbb{R}^n$. The conclusion then follows, first for finite unions by induction and the gluing result, and finally for countable unions by the exhaustion result. $\square$

15.5. **Perfect pairings.**

Definition 15.5. Let R be commutative. An R-bilinear pairing $\cdot \colon A \times B \to R$ (or $A \otimes_R B \to R$) is perfect if the adjoint homomorphisms

$$\begin{aligned} A &\longrightarrow \operatorname{Hom}_R(B, R) \\ a &\longmapsto (b \mapsto a \cdot b) \end{aligned}$$

and

$$\begin{aligned} B &\longrightarrow \operatorname{Hom}_R(A, R) \\ b &\longmapsto (a \mapsto a \cdot b) \end{aligned}$$

are isomorphisms. If $R = F$ is a field and $A = F\{a_1, \dots, a_m\}$ and $B = F\{b_1, \dots, b_n\}$ are F-vector spaces, this means that the $m \times n$ coefficient matrix

$$C = (c_{i,j} = a_i \cdot b_j)_{i,j}$$

is invertible, so that $m = n$.

Suppose that M is a closed, R-oriented n-manifold, with fundamental class $[M] \in H_n(M; R)$. The cup product and evaluation define a pairing

$$\begin{aligned} H^k(M;R) \times H^{n-k}(M;R) &\longrightarrow R \\ (a,b) &\longmapsto \langle a \cup b, [M] \rangle \end{aligned}$$

In the case $R = \mathbb{Z}$, let $H^k_{\text{free}}(M)$ denote the torsion-free quotient of $H^k(M;\mathbb{Z})$ by its torsion subgroup.

Proposition 15.6. *When $R = F$ is a field, the cup pairing*

$$H^k(M;F) \times H^{n-k}(M;F) \longrightarrow F$$

is perfect. When $R = \mathbb{Z}$, the induced pairing

$$H^k_{\text{free}}(M) \times H^{n-k}_{\text{free}}(M) \longrightarrow \mathbb{Z}$$

is perfect.

Proof. The composition

$$H^k(M;R) \xrightarrow{h} \mathrm{Hom}_R(H_k(M;R), R) \xrightarrow{\mathrm{Hom}(D,1)} \mathrm{Hom}_R(H^{n-k}(M;R), R)$$

sends x to the homomorphism $y \mapsto \langle x, D(y) \rangle = \langle x, y \cap [M] \rangle = \langle x \cup y, [M] \rangle$, hence is the adjoint to the cup pairing. Here D is an isomorphism by Poincaré duality. The homomorphism h is an isomorphism when R is a field. When $R = \mathbb{Z}$ it is surjective, with kernel the torsion subgroup of $H^k(M;\mathbb{Z})$, so that

$$\begin{aligned} H^k_{\text{free}}(M) &\xrightarrow{h} \mathrm{Hom}(H_k(M), \mathbb{Z}) \\ \mathrm{Hom}(H^{n-k}_{\text{free}}(M), \mathbb{Z}) &\longrightarrow \mathrm{Hom}(H^{n-k}(M;\mathbb{Z}), \mathbb{Z}) \end{aligned}$$

are isomorphisms. This proves the first condition for the cup pairing to be perfect. The second condition follows by graded commutativity of the cup product. □

16. October 14th lecture

16.1. **Projective spaces.**

Proposition 16.1. $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$ *with* $|x| = 1$.

Proof. We know that $H^k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $0 \le k \le n$. Let $x \in H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ be the generator, i.e., the nonzero element. By induction, we may assume that x^k generates $H^k(\mathbb{R}P^n; \mathbb{Z}/2) \cong H^k(\mathbb{R}P^{n-1}; \mathbb{Z}/2)$ for $0 \le k < n$. It remains to prove that x^n generates $H^n(\mathbb{R}P^n; \mathbb{Z}/2)$. By Poincaré duality, the cup pairing

$$H^1(\mathbb{R}P^n; \mathbb{Z}/2) \times H^{n-1}(\mathbb{R}P^n; \mathbb{Z}/2) \longrightarrow H^n(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

is perfect, hence not zero. It follows that $x \cup x^{n-1} = x^n$ cannot be zero, and must generate $H^n(\mathbb{R}P^n; \mathbb{Z}/2)$. □

Similar arguments prove that $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[y]/(y^{n+1})$ with $|y| = 2$, $H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[z]/(z^{n+1})$ with $|z| = 4$ and $H^*(\mathbb{O}P^2; \mathbb{Z}) \cong \mathbb{Z}[w]/(w^3)$ with $|w| = 8$, using that these are all closed orientable manifolds.

16.2. **3-manifolds.** Let M be a closed, connected 3-manifold with fundamental group π. Then $H_0(M) \cong \mathbb{Z}$ and $H_1(M) = \pi^{ab} = \pi/[\pi, \pi]$. It follows by the universal coefficient theorem that $H^0(M) = \mathbb{Z}$ and $H^1(M) \cong \mathrm{Hom}(\pi, \mathbb{Z})$.

If M is orientable, then Poincaré duality implies that $H^3(M) \cong \mathbb{Z}$, $H^2(M) \cong \pi^{ab}$, $H_3(M) \cong \mathbb{Z}$ and $H_2(M) \cong \mathrm{Hom}(\pi, \mathbb{Z})$. Hence the fundamental group controls the homology and cohomology of M.

If M is not orientable, then $H_0(M; \mathbb{Z}/2) = \mathbb{Z}/2$ and $H_1(M; \mathbb{Z}/2) \cong \pi^{ab} \otimes \mathbb{Z}/2$. By the universal coefficient theorem, $H^0(M; \mathbb{Z}/2) = \mathbb{Z}/2$ and $H^1(M, \mathbb{Z}/2) \cong \mathrm{Hom}(\pi, \mathbb{Z}/2)$. By Poincaré duality, $H^3(M; \mathbb{Z}/2) = \mathbb{Z}/2$, $H^2(M; \mathbb{Z}/2) \cong \pi^{ab} \otimes \mathbb{Z}/2$, $H_3(M; \mathbb{Z}/2) = \mathbb{Z}/2$ and $H_2(M, \mathbb{Z}/2) \cong \mathrm{Hom}(\pi, \mathbb{Z}/2)$. Again, the fundamental group controls the mod 2 homology and cohomology of M.

The case of integral (co-)homology for non-orientable M is a little more complex, but can be deduced from that of the orientation cover $\tilde{M} \to M$. ((ETC: Work this out?))

16.3. **Simply-connected** 4**-manifolds.** Let M be a closed and simply-connected 4-manifold. Then $H_0(M) = \mathbb{Z}$ and $H_1(M) = 0$, so that $H^0(M) = \mathbb{Z}$, $H^1(M) = 0$ and $H^2(M) \cong \mathrm{Hom}(H_2(M), \mathbb{Z})$ is torsion-free, by the universal coefficient theorem. Any simply-connected manifold can be oriented, so Poincaré duality applies, giving $H^4(M) = \mathbb{Z}$, $H^3(M) = 0$, $H_4(M) \cong \mathbb{Z}$, $H_3(M) = 0$ and $H^2(M) \cong H_2(M)$. It follows that $H^2(M) \cong \mathbb{Z}\{x_1, \dots, x_r\}$ for some $r \geq 0$, and the cup pairing

$$H^2(M) \times H^2(M) \longrightarrow H^4(M) \cong \mathbb{Z}$$

is perfect. This is known as the intersection form of M, and is given by a non-singular symmetric $r \times r$ coefficient matrix with integer entries

$$C = (c_{i,j} = \langle x_i \cup x_j, [M] \rangle)_{i,j} .$$

There are finitely many such for each r, up to the equivalence relation given by a change of basis in $H^2(M) \cong \mathbb{Z}^r$.

Proposition 16.2 (J.H.C. Whitehead (1949))**.** *Two closed, simply-connected* 4*-manifold are homotopy equivalent if and only if their intersection forms are equivalent.*

It is a striking fact that if M is smooth, then the intersection forms that can arise are severely restricted.

Theorem 16.3 (Simon Donaldson (1982)))**.** *Let M be a closed, simply-connected smooth* 4*-manifold. If the intersection form is positive definite, then it is diagonalizable.*

Donaldson's proof uses 'instanton gauge theory'. The smooth structure lets one study a space of covariant derivations, or connections, in a vector bundle over M, modulo the symmetries given by the self-isomorphisms, or gauge automorphisms, of that vector bundle. In physical terms, the connections correspond to potentials for so-called instantons on M. In Donaldson's case, this moduli space can be compactified by adding a copy of M at the boundary, using work of Karen Uhlenbeck. Approaching a point on the boundary corresponds to the situation where the curvature of the connection, or the energy distribution of the instanton, becomes concentrated near a single point, and then 'bubbles off' from that point. The result W is almost a 5-manifold, except for some singular points with neighborhoods of the form $C(\mathbb{C}P^2)$, i.e., cones on $\mathbb{C}P^2$. Removing the open cones one obtains a cobordism from M to a disjoint union of copies of $\mathbb{C}P^2$. It follows that the intersection form of M is the direct sum of those for the copies of $\mathbb{C}P^2$, i.e., a diagonal matrix with entries $+1$ or -1 (depending on orientations).

Part 3. Homotopy Theory

16.4. **The Steenrod problem.** Let X be a space and G an abelian group. The singular homology $H_n(X; G)$ of X with coefficients in G is constructed by considering maps $\sigma \colon \Delta^n \to X$ from standard simplices, forming linear combinations $\alpha = \sum_i g_i \sigma_i$ with coefficients in G, and passing to n-cycles modulo n-boundaries.

Poincaré and Thom considered a variation of this idea, where the n-cycles α in X are replaced by closed smooth n-manifolds

$$f \colon M \longrightarrow X$$

mapping to X, and n-boundaries are replaced by compact smooth $(n+1)$-manifolds

$$F \colon W \longrightarrow X$$

mapping to X. The sum of $f\colon M \to X$ and $f'\colon M' \to X$ is given by the disjoint union $f \sqcup f'\colon M \sqcup M' \to X$. If $\partial W = M \sqcup M'$, $F|M = f$ and $F|M' = f'$ then we then set $F|\partial W = f + f'$ equal to zero. This defines the unoriented bordism group $\mathscr{N}_n(X) = MO_n(X)$. There is a natural homomorphism

$$MO_n(X) \longrightarrow H_n(X;\mathbb{Z}/2)$$

mapping $f\colon M \to X$ to the image $f_*[M] \in H_n(X;\mathbb{Z}/2)$ of the unoriented fundamental class $[M] \in H_n(M;\mathbb{Z}/2)$, and Thom (1954) proved that this is always surjective. More precisely, the homology theory $X \mapsto MO_*(X)$ is represented by a spectrum MO, in the sense of algebraic topology, and MO is a wedge sum of suspensions of the spectrum $H\mathbb{Z}/2$ representing $X \mapsto H_*(X;\mathbb{Z}/2)$.

To map to $H_n(X;\mathbb{Z})$ we consider oriented bordism, considering closed oriented smooth n-manifolds

$$f\colon M \longrightarrow X$$

mapping to X, and compact oriented smooth $(n+1)$-manifolds

$$F\colon W \longrightarrow X$$

mapping to X. If $\partial W = (-M) \sqcup M'$, where $-M$ denotes M with the opposite orientation, then we set $F|\partial W = -f + f'$ equal to zero. Equivalently, we set $f = f'$. This defines the oriented bordism group $\Omega_n(X) = MSO_n(X)$. There is a natural homomorphism

$$MSO_n(X) \longrightarrow H_n(X;\mathbb{Z})$$

mapping $f\colon M \to X$ to the image $f_*[M] \in H_n(X;\mathbb{Z})$ of the oriented fundamental class $[M] \in H_n(M;\mathbb{Z})$. Thom (1954) proved that this is surjective for $n \le 6$, but not for $n = 7$. The Steenrod problem asks which homology classes can be realized by oriented bordism classes. This amounts to a study of the spectrum MSO representing $X \mapsto MSO_*(X)$. It was proved by C.T.C. Wall that, localized at 2, $MSO_{(2)}$ is a wedge sum of suspensions of spectra $H\mathbb{Z}/2$ and $H\mathbb{Z}_{(2)}$, where the latter represents $X \mapsto H_*(X;\mathbb{Z}_{(2)})$. Hence, for each class x in $H_n(X;\mathbb{Z})$ there is some odd multiple $(2k+1)x$ that is realized by an oriented n-manifold.

Variants of this for manifolds with singularities were then developed by Thom (1960s) and by Dennis Sullivan (1971), see also the work by Nils Baas.

16.5. **Homotopy groups.** If we concentrate on the simplest closed n-manifold, namely the n-sphere $M = S^n$, and on the simplest cobordisms, namely the cylinders $W = I \times S^n$, where $I = [0,1]$ is an interval, then we are led to consider the maps

$$f\colon S^n \longrightarrow X$$

modulo the equivalence relation that $f \simeq f'$ if there is a map

$$F\colon I \times S^n \longrightarrow X$$

such that $F|\{0\} \times S^n = f$ and $F|\{1\} \times S^n = f'$. In other words, we consider homotopy classes of maps $S^n \to X$. To define a sum without referring to $S^n \sqcup S^n$, it is convenient to pass to the category of based spaces and base-point preserving (= based) maps. In a sense, this means that we obtain a reduced theory, whose value at the one-point space is trivial.

Definition 16.4 (Eduard Čech, 1932)**.** Let $S^n \subset \mathbb{R}^{n+1}$ be based at $s_0 = (1, 0, \dots, 0)$. Let

$$I \ltimes S^n = \frac{I \times S^n}{I \times \{s_0\}}$$

be the reduced cylinder. Two based maps $f, f'\colon S^n \to X$ are based homotopic, written $f \simeq f'$, if there is a based map

$$F\colon I \ltimes S^n \longrightarrow S^n$$

with $F(0,s) = f(s)$ and $F(1,s) = f'(s)$ for all $s \in S^n$. We let

$$\pi_n(X, x_0) = \{f\colon S^n \to X\}/{\simeq}$$

be the set of based homotopy classes of based maps $S^n \to X$. We write

$$[f] = \{f'\colon S^n \to X \mid f \simeq f'\}$$

for the equivalence class of f, i.e., its based homotopy class. Let $[c]$ be the class of the constant map to x_0.

More generally, for a space A and a based space (Y, y_0) we call

$$A \ltimes Y = \frac{A \times Y}{A \times \{y_0\}}$$

the half-smash product of A and Y. For based spaces (X, x_0) and (Y, y_0) the wedge sum is

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$$

and the smash product is

$$X \wedge Y = \frac{X \times Y}{X \vee Y}.$$

The half-smash product can be written as

$$A \ltimes Y \cong A_+ \wedge Y$$

where A_+ denotes A with an additional, disjoint, basepoint.

16.6. **Coordinates.** For explicit constructions, it can be more convenient to use a model for S^n with simpler coordinates than those of the unit sphere in $\mathbb{R}^{n+1}$. For this we let

$$I^n = I \times \cdots \times I$$

be the n-dimensional cube, with points $(s_1, \dots, s_n)$ where $0 \le s_i \le 1$ for each $1 \le i \le n$. Its boundary ∂I^n consists of those points where $s_i \in \{0, 1\}$ for some i. We fix a homeomorphism

$$I^n/\partial I^n \cong S^n$$

for each $n \ge 0$. A based map $f \colon S^n \to X$ then corresponds to a map $g \colon I^n \to X$ with $g(\partial I^n) \subset \{x_0\}$. In other words, g is a map of pairs $(I^n, \partial I^n) \to (X, x_0)$, where x_0 is short for $\{x_0\}$.

A based homotopy $F \colon I \ltimes S^n \to X$ then corresponds to a map $G \colon I \times I^n \to X$ with $G(I \times \partial I^n) \subset \{x_0\}$. This is then a homotopy $G \colon g \simeq g'$ from $g = G|\{0\} \times I^n$ to $g' = G|\{1\} \times I^n$, which are maps $g, g' \colon (I^n, \partial I^n) \to (X, x_0)$. Here we identify $\{0\} \times I^n$ and $\{1\} \times I^n$ with I^n in the evident way.

Lemma 16.5.

$$\pi_n(X, x_0) \cong \{g \colon (I^n, \partial I^n) \to (X, x_0)\}/\simeq$$

with $[c]$ the class of the constant map to x_0.

17. October 19th lecture

17.1. **Group structure.** When $n = 0$, $\pi_n(X, x_0)$ is the set of path components of X, with the component of x_0 as a preferred base element $[c]$. When $n = 1$, $\pi_1(X, x_0)$ is the fundamental group of X, with unit element $[c]$. The definition of the group structure generalizes to all $n \ge 1$.

Definition 17.1. Let $n \ge 1$. For maps $f, g \colon (I^n, \partial I^n) \to (X, x_0)$ let $f + g \colon (I^n, \partial I^n) \to (X, x_0)$ be the map

$$(f+g)(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & \text{for } 0 \le s_1 \le 1/2, \\ g(2s_1 - 1, s_2, \dots, s_n) & \text{for } 1/2 \le s_1 \le 1. \end{cases}$$

In particular, $(f + g)(s) = x_0$ if $s = (s_1, \dots, s_n) \in \partial I^n$, or if $s_1 = 1/2$.

Lemma 17.2. *If $F \colon f \simeq f'$ and $G \colon g \simeq g'$ then there is a based homotopy $F + G \colon f + g \simeq f' + g'$. Hence there is a well-defined pairing*

$$+ \colon \pi_n(X, x_0) \times \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$$

given by

$$[f] + [g] = [f + g].$$

When $n = 1$ this is the pairing we usually denote $[f] * [g]$. For $n \ge 2$ it is commutative, which justifies the notation $[f] + [g]$. We also write 0 for $[c]$.

Lemma 17.3. $[c] + [f] = [f] = [f] + [c]$.

Lemma 17.4. *If $n \ge 2$, then $[f] + [g] = [g] + [f]$ in $\pi_n(X, x_0)$.*

Proof. For the duration of this proof, define a second pairing

$$\circ\colon \pi_n(X,x_0)\times\pi_n(X,x_0)\longrightarrow\pi_n(X,x_0)$$

by $[f]\circ[g]=[f\circ g]$ where

$$(f\circ g)(s_1,s_2,\dots,s_n)=\begin{cases} f(s_1,2s_2,\dots,s_n) & \text{for } 0\le s_2\le 1/2,\\ g(s_1,2s_2-1,\dots,s_n) & \text{for } 1/2\le s_2\le 1.\end{cases}$$

Then check that

$$f+g\simeq(f\circ c)+(c\circ g)=(f+c)\circ(c+g)\simeq(c+f)\circ(g+c)=(c\circ g)+(f\circ c)\simeq g+f\,.$$

□

Definition 17.5. Let $n\ge 1$. For $f\colon(I^n,\partial I^n)\to(X,x_0)$ let $-f\colon(I^n,\partial I^n)\to(X,x_0)$ be the map

$$(-f)(s_1,s_2,\dots,s_n)=(1-s_1,s_2,\dots,s_n)\,.$$

Lemma 17.6. *If $F\colon f\simeq f'$ then $-F\colon -f\simeq -f'$. Hence*

$$-\colon\pi_n(X,x_0)\longrightarrow\pi_n(X,x_0)$$

can be defined by $-[f]=[-f]$.

When $n=1$, we usually write $\bar f$ for $-f$.

Lemma 17.7. $-[f]+[f]=[c]=[f]-[f]$.

Proposition 17.8. *$\pi_n(X,x_0)$ is a group for $n\ge 1$, which is abelian for $n\ge 2$.*

17.2. **Dependence on base point.** If $g\colon I\to X$ is a path from x_0 to x_1, then there is a preferred isomorphism

$$\beta_g\colon\pi_n(X,x_1)\xrightarrow{\cong}\pi_n(X,x_0)$$

for each $n\ge 0$. Hence $\pi_n(X,x_0)$ only depends, up to non-canonical isomorphism, on the class of x_0 in $\pi_0(X)$. If X is path connected, this is no choice. If X is also simply connected, then any two paths g and g' from x_0 to x_1 are homotopic, so $\beta_g=\beta_{g'}$ and $\pi_n(X,x_0)$ is independent of x_0 up to canonical isomorphism.

Definition 17.9. For $n\ge 0$ we say that X is n-connected if $\pi_i(X,x_0)=0$ for all $0\le i\le n$. Hence 0-connected means path connected, and 1-connected means (path and) simply connected.

17.3. **Functoriality.**

Definition 17.10. For any based map $\phi\colon(X,x_0)\to(Y,y_0)$ let

$$\phi_*\colon\pi_n(X,x_0)\longrightarrow\pi_n(Y,y)$$

be given by $\phi_*[f]=[\phi f]$.

$$(I^n,\partial I^n)\xrightarrow{f}(X,x_0)\xrightarrow{\phi}(Y,y_0)$$

Lemma 17.11. *If $F\colon f\simeq f'$ then $\phi F\colon\phi f\simeq\phi f'$, so ϕ_* is well-defined.*

Lemma 17.12. *For $n=0$, $n=1$ and $n\ge 2$ the rules $(X,x_0)\mapsto\pi_n(X,x_0)$ and $\phi\mapsto\phi_*$ define functors from the category $\mathscr{T}$ of based spaces to the categories of based sets, groups and abelian groups, respectively.*

Proof. If $\psi\colon(Y,y_0)\to(Z,z_0)$, then $\psi(\phi f)=(\psi\phi)f$ implies that $\psi_*\phi_*=(\psi\phi)_*$. □

Lemma 17.13. *If $\Phi\colon\phi\simeq\phi'$ is a based homotopy of maps $(X,x_0)\to(Y,y_0)$, then $\phi_*=\phi'_*$ are equal as functions $\pi_n(X,x_0)\to\pi_n(Y,y_0)$.*

Proof. Φf defines a based homotopy from ϕf to $\phi' f$. □

Lemma 17.14. *If $\phi\colon X\to Y$ is a based homotopy equivalence, then*

$$\phi_*\colon\pi_n(X,x_0)\longrightarrow\pi_n(Y,y_0)$$

is a bijection for $n=0$, and an isomorphism of groups for $n\ge 1$.

Proof. A based homotopy inverse $\psi\colon Y \to X$ induces an inverse $\psi_*\colon \pi_n(Y, y_0) \to \pi_n(X, x_0)$ to ϕ_*, for each $n \geq 0$. $\square$

Definition 17.15. A map $\phi\colon X \to Y$ of path connected spaces is called a weak homotopy equivalence if it induces isomorphisms $\pi_n(X, x_0) \cong \pi_n(Y, y_0)$ for all $n \geq 1$.

For $n \geq 0$ we say that a map $\phi\colon X \to Y$ of path connected spaces is n-connected if $\phi_*\colon \pi_i(X, x_0) \to \pi_i(Y, y_0)$ is an isomorphism for $0 \leq i < n$ and is surjective for $i = n$.

17.4. **Covering spaces and products.**

Proposition 17.16. *If $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering space projection, then*

$$p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_n(X, x_0)$$

is injective for $n = 1$ and an isomorphism for $n \geq 2$.

Proof. Let $n \geq 1$. If $g, g'\colon S^n \to \tilde{X}$ are such that there exists a homotopy $F\colon pg \simeq pg'\colon S^n \to X$, then by the covering homotopy property there is a homotopy $\tilde{F}\colon g \simeq g_1$ covering F. We must have $g_1 = g'$ by uniqueness of lifts, since S^n is path connected. Hence p_* is injective.

Let $n \geq 2$. Then $\pi_1(S^n) = 0$, so for any $f\colon S^n \to X$ the image of $f_*\colon \pi_1(S^n) \to \pi_1(X)$ is the trivial subgroup. This is contained in the image of $p_*\colon \pi_1(\tilde{X}_0) \to \pi_1(X)$. Since S^n is connected and locally path connected, there exists a lift $\tilde{f}\colon S^n \to \tilde{X}$ of f. Hence $p_*[\tilde{f}] = [f]$, and p_* is surjective. $\square$

Let M_g and N_h be the orientable and non-orientable surfaces obtained by taking the connected sum of g copies of $T^2 = S^1 \times S^1$ and h copies of $\mathbb{R}P^2$, respectively.

Corollary 17.17.

$$\pi_n(S^1) = \begin{cases} \mathbb{Z} & \textit{for } n = 1, \\ 0 & \textit{for } n \geq 2; \end{cases}$$

$$\pi_n(M_g) = \begin{cases} \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = e \rangle & \textit{for } n = 1, \\ 0 & \textit{for } g \geq 1 \textit{ and } n \geq 2; \end{cases}$$

$$\pi_n(N_h) = \begin{cases} \langle a_1, \dots, a_h \mid a_1^2 \cdots a_h^2 = e \rangle & \textit{for } n = 1, \\ 0 & \textit{for } h \geq 2 \textit{ and } n \geq 2. \end{cases}$$

Proof. We take the statements for $n = 1$ as known. There are universal covering projections $\mathbb{R} \to S^1$, $\mathbb{R}^2 \to M_g$ and $\mathbb{R}^2 \to N_h$, with contractible total spaces, except for $M_0 = S^2 = N_0$ and $N_1 = \mathbb{R}P^2$. Clearly $\pi_n(\mathbb{R}^1) = 0$ and $\pi_n(\mathbb{R}^2) = 0$ for $n \geq 2$. $\square$

Proposition 17.18. *For any product $\prod_\alpha X_\alpha$ the projection maps induce isomorphisms*

$$\prod_\alpha pr_{\alpha *}\colon \pi_n(\prod_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha \pi_n(X_\alpha)$$

for all $n \geq 0$. In particular

$$\pi_n(X \times Y) \xrightarrow{\cong} \pi_n(X) \times \pi_n(Y)\,.$$

Proof. Maps $f\colon S^n \to \prod_\alpha X_\alpha$ correspond bijectively to collections of maps $(f_\alpha\colon S^n \to X_\alpha)_\alpha$, with $f_\alpha = pr_\alpha f$, and homotopies $F\colon I \ltimes S^n \to \prod_\alpha X_\alpha$ correspond bijectively to collections of homotopies $(F_\alpha\colon I \ltimes S^n \to X_\alpha)_\alpha$, with $F_\alpha = pr_\alpha F$. $\square$

Let $T^k = S^1 \times \cdots \times S^1$ be the k-dimensional torus.

Corollary 17.19.

$$\pi_n(T^k) = \begin{cases} \mathbb{Z}^k & \textit{for } n = 1, \\ 0 & \textit{for } n \geq 2. \end{cases}$$

In other words, $S^1 \simeq K(\mathbb{Z}, 1)$ and $T^k \simeq K(\mathbb{Z}^k, 1)$ are Eilenberg–Mac Lane spaces of type $(\mathbb{Z}, 1)$ and $(\mathbb{Z}^k, 1)$, respectively.

18. October 21st lecture

18.1. **The Hurewicz homomorphism.** Consider $n \geq 1$, and let $[S^n] \in H_n(S^n)$ denote the fundamental class. To each map $f\colon S^n \to X$ we can associate the class

$$f_*([S^n]) \in H_n(X)\,.$$

If $f \simeq f'$ then $f_* = f'_*$, so this element in $H_n(X)$ only depends on the homotopy class of f.

Proposition 18.1. *$[f] \mapsto f_*([S^n])$ defines a group homomorphism*

$$h_n\colon \pi_n(X, x_0) \longrightarrow H_n(X)\,,$$

called the Hurewicz homomorphism.

For a proof, see Hatcher's Proposition 4.36.

Lemma 18.2. *When $X = S^n$ the Hurewicz homomorphism*

$$h_n\colon \pi_n(S^n) \longrightarrow H_n(S^n) \cong \mathbb{Z}$$

is surjective. Hence $\pi_n(S^n)$ has infinite order.

Proof. For $f = 1\colon S^n \to S^n$ the identity map, h_n takes $[f]$ to the generator $[S^n]$ of $H_n(S^n)$. More generally, if f has degree d then $h_n([f]) = d[S^n]$. □

18.2. **Cell attachment.** Consider $n \geq 1$. Let $X \cup_f e^n = X \cup_f D^n$ be obtained by attaching a based n-cell to X along a based map $f\colon S^{n-1} = \partial D^n \to X$. This is the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "f"'] & D^n \arrow[d] \\ X \arrow[r] & X \cup_f D^n\,. \end{tikzcd}$$

If $F\colon f \simeq f'$ then we can construct a homotopy equivalence

$$\hat{F}\colon X \cup_f D^n \xrightarrow{\simeq} X \cup_{f'} D^n\,.$$

This implies that the homotopy type of $X \cup_f D^n$ only depends on the homotopy class $[f] \in \pi_{n-1}(X, x_0)$. Here $\hat{F}$ is the identity on X, traces out the homotopy F on a reduced cylinder $I \ltimes S^{n-1} \subset D^n$, and identifies the remainder of D^n in the source with the copy of D^n in the target. The homotopy inverse is $\hat{G}$, likewise associated to $G = \bar{F}\colon f' \simeq f$. See page 13 and Proposition 0.18 in Hatcher's book. It follows that $\pi_{n-1}(X, x_0)$ accounts for all the possible ways of building a new homotopy type by attaching an n-cell to X.

If $\phi\colon X \to Y$ is a based homotopy equivalence, and $f\colon S^{n-1} \to X$ is as above, then there is a homotopy equivalence

$$\phi \cup 1\colon X \cup_f D^n \xrightarrow{\simeq} Y \cup_{\phi f} D^n\,.$$

Hence the question of which homotopy types can be obtained by attaching a cell to X only depends on the homotopy type of X. In particular, this interpretation of $\pi_{n-1}(X, x_0)$ is preserved by the isomorphism $\phi_*\colon \pi_{n-1}(X, x_0) \cong \pi_{n-1}(Y, y_0)$.

18.3. **The Hopf invariant.** We now make use of the cup product in cohomology to detect some nontrivial higher homotopy groups that are not seen by the Hurewicz homomorphism to homology. See Hatcher's Section 4.B. Consider the diagram

$$S^{n-1} \xrightarrow{f} X \xrightarrow{i} X \cup_f D^n \xrightarrow{q} S^n$$

where i is the structural inclusion and q is the quotient map to

$$(X \cup_f D^n)/X \cong D^n/S^{n-1} \cong S^n\,.$$

We obtain an exact sequence in cohomology

$$H^{*-1}(X) \xrightarrow{f^*} \tilde{H}^*(S^n) \xrightarrow{q^*} H^*(X \cup_f D^n) \xrightarrow{i^*} H^*(X) \xrightarrow{f^*} \tilde{H}^{*+1}(S^n)\,.$$

If the cohomology homomorphism $f^*\colon H^*(X) \to \tilde{H}^*(S^{n-1}) \cong \tilde{H}^{*+1}(S^n)$ is zero, then this is an extension

$$\tilde{H}^*(S^n) \xrightarrow{q^*} H^*(X \cup_f D^n) \xrightarrow{i^*} H^*(X)$$

of graded commutative rings, with respect to the cup product. In other words, i^* is a ring surjection with kernel isomorphic, via q^*, to $\mathbb{Z}$ concentrated in degree $* = n$.

Definition 18.3. The Hopf invariant $H(f)$ of $f\colon S^{n-1} \to X$ is the isomorphism class of the (graded commutative) ring extension above.

This is an invariant of the homotopy class of f.

Lemma 18.4. *If $f \simeq f'$ then $H(f) = H(f')$.*

Proof. The homotopy equivalence $\hat{F}$ induces a degree 1 map $S^n \to S^n$ and an isomorphism of algebra extensions

$$\begin{tikzcd}
\tilde{H}^*(S^n) \arrow[r, "q^*"] \arrow[d, "\cong"] & H^*(X \cup_{f'} D^n) \arrow[r, "i^*"] \arrow[d, "\hat{F}^*"] & H^*(X) \arrow[d, "="] \\
\tilde{H}^*(S^n) \arrow[r, "q^*"] & H^*(X \cup_f D^n) \arrow[r, "i^*"] & H^*(X)
\end{tikzcd}$$

□

When $X = S^m$ is itself a sphere, the algebra extension can only be nontrivial for $n = 2m$. We are considering

$$S^{2m-1} \xrightarrow{f} S^m \xrightarrow{i} S^m \cup_f D^{2m} \xrightarrow{q} S^{2m}\,.$$

For $m \geq 2$, $f^*\colon H^*(S^m) \to \tilde{H}^*(S^{2m-1})$ is zero, so we get an extension

$$\tilde{H}^*(S^{2m}) \xrightarrow{q^*} H^*(S^m \cup_f D^{2m}) \xrightarrow{i^*} H^*(S^m)\,.$$

The iterated suspension isomorphisms

$$\tilde{H}^0(S^0) \cong \tilde{H}^m(S^m) \cong \tilde{H}^{2m}(S^{2m})$$

specify generators $\sigma^m \in H^m(S^m)$ and $\sigma^{2m} \in H^{2m}(S^{2m})$. We let

$$\alpha \in H^m(S^m \cup_f D^{2m})$$
$$\beta \in H^{2m}(S^m \cup_f D^{2m})$$

be the unique classes with $i^*(\alpha) = \sigma^m$ and $\beta = q^*(\sigma^{2m})$. With this notation, the extension above has the form

$$\mathbb{Z}\{\beta\} \xrightarrow{q^*} H^*(S^m \cup_f D^{2m}) \xrightarrow{i^*} \mathbb{Z}[\alpha]/(\alpha^2)\,.$$

Since the cup product in $H^*(S^m \cup_f D^{2m})$ is unital, it is completely specified by the bilinear pairing

$$\cup\colon H^m(S^m \cup_f D^{2m}) \times H^m(S^m \cup_f D^{2m}) \longrightarrow H^{2m}(S^m \cup_f D^{2m})\,,$$

which in turn is given by the coefficient $H(f) \in \mathbb{Z}$ in the formula $\alpha \cup \alpha = H(f)\beta$.

Definition 18.5. The Hopf invariant $H(f) \in \mathbb{Z}$ of $f\colon S^{2m-1} \to S^m$ is the integer such that

$$\alpha^2 = H(f)\beta$$

in $H^*(S^m \cup_f D^{2m})$.

Proposition 18.6. $[f] \mapsto H(f)$ *defines a group homomorphism*

$$H\colon \pi_{2m-1}(S^m) \longrightarrow \mathbb{Z}\,.$$

18.4. **The Hopf fibrations.**

Definition 18.7. The complex Hopf fibration $\eta\colon S^3 \to S^2$ is the composite

$$S^3 \xrightarrow{p} \mathbb{C}P^1 \cong S^2$$

where p maps a unit vector $v \in \mathbb{C}^2$ to the class of the complex line $\mathbb{C}v$. It is the attaching map of the 4-cell in $\mathbb{C}P^2 = \mathbb{C}P^1 \cup_\eta D^4$.

The quaternionic Hopf fibration $\nu\colon S^7 \to S^4$ is the composite

$$S^7 \xrightarrow{p} \mathbb{H}P^1 \cong S^4$$

where p maps a unit vector $v \in \mathbb{H}^2$ to the class of the quaternionic line $\mathbb{H}v$. It is the attaching map of the 8-cell in $\mathbb{H}P^2 = \mathbb{H}P^1 \cup_\nu D^8$.

The octonionic Hopf fibration $\sigma\colon S^{15} \to S^8$ is a composite

$$S^{15} \xrightarrow{p} \mathbb{O}P^1 \cong S^8\,.$$

It is the attaching map of the 16-cell in $\mathbb{O}P^2 = \mathbb{O}P^1 \cup_\sigma D^{16}$.

We use the same notation for the homotopy classes of these maps.

Proposition 18.8. *$H(\eta) = 1$, $H(\nu) = 1$, $H(\sigma) = 1$, so $\eta \in \pi_3(S^2)$, $\nu \in \pi_7(S^4)$ and $\sigma \in \pi_{15}(S^8)$ are nontrivial classes of infinite order.*

Proof. We do the complex case, with $S^2 \cup_\eta D^4 \cong \mathbb{C}P^2$. The extension

$$\tilde{H}^4(S^4) \xrightarrow{q^*} H^*(\mathbb{C}P^2) \xrightarrow{i^*} H^*(\mathbb{C}P^1)$$

has the form

$$\mathbb{Z}\{\beta\} \xrightarrow{q^*} \mathbb{Z}\{1, \alpha, \alpha^2\} \xrightarrow{i^*} \mathbb{Z}\{1, \alpha\}\,.$$

Here $\beta = \pm\alpha^2$ by our cup product calculation, or Poincaré duality, so $H(\eta) = \pm 1$. (To get the correct sign, a comparison of orientations and suspension isomorphisms is needed.) □

18.5. **The Whitehead product.** The product $S^m \times S^m$ admits a CW structure with m-skeleton $S^m \vee S^m$, and a single $2m$-cell attached by a map

$$w\colon S^{2m-1} \longrightarrow S^m \vee S^m\,.$$

For example, when $m = 1$ this is the commutator $[a, b] = a * b * \bar{a} * \bar{b}$. (This can be generalized by considering the attaching map $S^{m+n-1} \to S^m \vee S^n$ of the top cell in $S^m \times S^n$.) Composition with w defines a pairing

$$[-,-]\colon \pi_m(X) \times \pi_m(X) \longrightarrow \pi_{2m-1}(X)$$

called the Whitehead product, taking the homotopy classes of $a\colon S^m \to X$ and $b\colon S^m \to X$ to the homotopy class of the composite

$$[a, b]\colon S^{2m-1} \xrightarrow{w} S^m \vee S^m \xrightarrow{a \vee b} X\,.$$

In the special case $a = b = 1\colon S^m \to S^m$, the map $1 \vee 1\colon S^m \vee S^m \to S^m$ is the fold map, also denoted ∇. (It is in a sense dual to the diagonal map Δ.)

Writing $\iota_m\colon S^m \to S^m$ for the identity map, we thus have $[\iota_m, \iota_m] = \nabla w\colon S^{2m-1} \to S^m$, and map of cell attachment diagrams

$$\begin{tikzcd}
S^{2m-1} \arrow[r, "w"] \arrow[d, "="] & S^m \vee S^m \arrow[r, "i"] \arrow[d, "\nabla"] & S^m \times S^m \arrow[r, "q"] \arrow[d, "\phi"] & S^{2m} \arrow[d, "="] \\
S^{2m-1} \arrow[r, "\nabla w"] & S^m \arrow[r, "i"] & S^m \cup_{\nabla w} D^{2m} \arrow[r, "q"] & S^{2m}\,.
\end{tikzcd}$$

Here ϕ identifies the two m-cells in $S^m \times S^m$ to one.

Proposition 18.9. *The Whitehead product $[\iota_m, \iota_m]\colon S^{2m-1} \to S^m$ has Hopf invariant*

$$H([\iota_m, \iota_m]) = \begin{cases} 2 & \text{for } m \ge 2 \text{ even,} \\ 0 & \text{for } m \ge 1 \text{ odd.} \end{cases}$$

It follows that the homotopy groups $\pi_{2m-1}(S^m)$ have infinite order for all even $m \geq 2$. Jean–Pierre Serre (1951) proved that $\pi_n(S^n)$ and $\pi_{4k-1}(S^{2k})$ are the only homotopy groups of spheres that are of infinite order. In fact, each of these has rank 1.

Proof. Consider $m \geq 2$. Passing to cohomology we obtain a map of ring extensions

$$\begin{tikzcd} \tilde{H}^{2m}(S^{2m}) \arrow[r, "q^*"] \arrow[d, "="] & H^*(S^m \cup_{\nabla w} D^{2m}) \arrow[r, "i^*"] \arrow[d, "\phi^*"] & H^*(S^m) \arrow[d, "\nabla^*"] \\ \tilde{H}^{2m}(S^{2m}) \arrow[r, "q^*"] & H^*(S^m \times S^m) \arrow[r, "i^*"] & H^*(S^m \vee S^m)\,. \end{tikzcd}$$

Recall the generators $\alpha, \beta \in H^*(S^m \cup_{\nabla c} D^{2m})$, with $i^*(\alpha) = \sigma^m$ and $\beta = q^*(\sigma^{2m})$. We need to express α^2 as a multiple of β in $H^*(S^m \cup_{\nabla w} D^{2m})$. This can be done by comparison with the cup product structure in $H^*(S^m \times S^m)$.

By the Künneth theorem, $H^*(S^m \times S^m) \cong \mathbb{Z}\{1, a, b, ab\}$ where $|a| = |b| = m$, $a^2 = 0$, $b^2 = 0$ and $ba = (-1)^{m^2} ab = (-1)^m ab$. The restriction i^* sends ab to 0, so q^* maps σ^{2m} to $\pm ab$. We fix orientation so that $q^*(\sigma^{2m}) = ab$. We then have the following map of ring extensions:

$$\begin{tikzcd} \mathbb{Z}\{\beta\} \arrow[r, "q^*"] \arrow[d, "="] & \mathbb{Z}\{1, \alpha, \beta\} \arrow[r, "i^*"] \arrow[d, "\phi^*"] & \mathbb{Z}\{1, \alpha\} \arrow[d, "\nabla^*"] \\ \mathbb{Z}\{\beta\} \arrow[r, "q^*"] & \mathbb{Z}\{1, a, b, ab\} \arrow[r, "i^*"] & \mathbb{Z}\{1, a, b\}\,. \end{tikzcd}$$

The map ∇^* takes σ^m to $a+b$ in $H^m(S^m \vee S^m)$, since $\nabla in_1 = 1 = \nabla in_2$ implies $in_1^* \nabla^* = 1 = in_2^* \nabla^*$. Hence $\phi^*(\alpha) = a + b$ and $\phi^*(\beta) = ab$ in $H^*(S^m \times S^m)$.

Since ϕ^* is a ring homomorphism, we can calculate

$$\phi^*(\alpha^2) = (a+b)^2 = a^2 + ab + ba + b^2 = \begin{cases} 2ab & \text{for } m \text{ even,} \\ 0 & \text{for } m \text{ odd.} \end{cases}$$

Since ϕ^* is injective, it follows that

$$\alpha^2 = \begin{cases} 2\beta & \text{for } m \text{ even,} \\ 0 & \text{for } m \text{ odd.} \end{cases}$$

This proves that $H(\nabla w) = 2$ for m even and 0 for m odd. □

This leads to the question, for which $m \geq 2$ does there exist a map $f\colon S^{2m-1} \to S^m$ of Hopf invariant one? Frank Adams (1960) proved that this only happens in the dimensions associated to the projective planes over real division algebras, i.e., for $m = 2$, $m = 4$ and $m = 8$. Using Steenrod operations and the Adem relations it is easy to see that only powers of 2 are possible, but to eliminate all possibilities $m = 2^i$ for $i \geq 4$, further machinery such as the Adams spectral sequence is required. It follows that S^0, S^1, S^3 and S^7 are the only spheres that are H-spaces, i.e., which admit a pairing

$$\mu\colon S^{m-1} \times S^{m-1} \longrightarrow S^{m-1}$$

that is unital and associative up to homotopy. Such spaces were studied by Hopf as generalizations of Lie groups, hence the 'H'.

18.6. **Cylinders, cones and suspensions.** We work in based spaces. Recall that $I = [0,1]$, with 0 as the base point, and that the reduced cylinder on (X, x_0) is $I_+ \wedge X$. For $t \in I$, let $i_t\colon X \to I_+ \wedge X$ be the inclusion $x \mapsto t \wedge x$. A map $F\colon I_+ \wedge X \to Y$ is a based homotopy from $f_0 = Fi_0\colon X \to Y$ to $f_1 = Fi_1\colon X \to Y$. In the case $f_0 = c$ is the constant map, we call F a null-homotopy (of or) to f_1. Let

$$CX = I \wedge X \cong \frac{I_+ \wedge X}{\{0\}_+ \wedge X}$$

be the quotient space, with the standard inclusion $i\colon X \to CX$ induced by i_1. A map $F\colon CX \to Y$ is essentially the same as a null-homotopy to $f = Fi\colon X \to Y$.

The map $i_1 \colon X \to I_+ \wedge X$ is a (Hurewicz) cofibration, i.e., has the homotopy extension property. These are "good" inclusions in a homotopical sense. It follows that $i \colon X \to CX$ is a cofibration, since the homotopy extension property essentially only depends on a neighborhood of X in CX, which is homeomorphic to the neighborhood in $I_+ \wedge X$. Hence it is homotopically meaningful to consider the quotient space of CX given by collapsing $X \cong i(X)$ to a point.

Recall that we have fixed an identification $I/\partial I \cong S^1$. Let

$$\Sigma X = S^1 \wedge X \cong \frac{I \wedge X}{\partial I \wedge X} = CX/X$$

be the reduced suspension of X. Let us write $[t, x]$ for the image of $(t, x) \in I \times X$ in CX or ΣX. We write $\Sigma^m X = \Sigma(\Sigma^{m-1} X)$ for the m-fold iterated suspension. The homeomorphisms

$$I^m/\partial I^m \wedge I^n/\partial I^n \cong I^{m+n}/\partial I^{m+n}$$

correspond to homeomorphisms

$$S^m \wedge S^n \cong S^{m+n} .$$

In particular, $\Sigma S^n \cong S^{1+n}$ and $\Sigma^m S^n \cong S^{m+n}$. Alternatively, if we identify $S^n \cong \mathbb{R}^n \cup \{\infty\}$ as a one-point compactification, then $S^m \wedge S^n \cong S^{m+n}$ extends the linear homeomorphism $\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$.

19. October 28th lecture

See Hatcher's Section 4.3, especially the proof of Theorem 4.58, for a discussion with similar aims as the following.

19.1. **Mapping cylinders.** Any map $f \colon X \to Y$ can be factored as a cofibration $c \colon X \to Mf$ followed by a homotopy equivalence $d \colon Mf \simeq Y$. Let the reduced mapping cylinder

$$Mf = Y \cup_X (I_+ \wedge X)$$

of f be defined by the left hand pushout square in the following diagram.

$$\begin{tikzcd}
& X \arrow[d, "i_0"'] \arrow[dr, "1"] & \\
X \arrow[r, "i_1"] \arrow[d, "f"'] & I_+ \wedge X \arrow[r, "p"] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r, "\simeq"'] & Mf \arrow[r, "d", "\simeq"'] & Y
\end{tikzcd}$$

Note the structure maps $I_+ \wedge X \to Mf$ and $Y \to Mf$. The inclusion $i_0 \colon X \to I_+ \wedge X$ is a cofibration, and so is the composite

$$c \colon X \xrightarrow{i_0} I_+ \wedge X \longrightarrow Mf ,$$

since a neighborhood of X in Mf is homeomorphic to a suitable neighborhood in $I_+ \wedge X$.

The cylinder projection $p \colon I_+ \wedge X \to X$ is a deformation retraction relative to $X \cong \{1\}_+ \wedge X$, hence induces a deformation retraction

$$d \colon Mf \longrightarrow Y \cup_X X \cong Y$$

relative to Y. In particular, d is a homotopy equivalence. Then $f = dc \colon X \to Y$ gives the claimed factorization.

19.2. **Mapping cones.** Let the reduced mapping cone

$$Cf = Y \cup_X CX = Y \cup_f CX$$

of $f \colon X \to Y$ be the pushout in the following diagram.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & CX \arrow[d] \\
Y \arrow[r, "i"] & Cf
\end{tikzcd}$$

The homeomorphism $CX \cong I_+ \wedge X/X$, where X is included by $i_0 \colon X \to CX$, induces a homeomorphism

$$Cf \cong Mf/X\,,$$

where X is included by the cofibration $c\colon X \to Mf$. We also call Cf the homotopy cofiber of f. The diagram

$$X \xrightarrow{f} Y \xrightarrow{i} Cf$$

is called the canonical homotopy cofiber sequence of f.

Example 19.1. We can identify the cone on S^{n-1} with D^n. Then the mapping cone of a map $f\colon S^{n-1} \to X$ is the same as the space obtained by attaching an n-cell to X along f:

$$Cf \cong X \cup_f D^n\,.$$

The construction of homotopy cofibers thus generalizes cell attachment.

Example 19.2. Let $*$ be a one-point space. The mapping cone of $X \to *$ is the suspension ΣX.

The mapping cone is a homotopy invariant construction.

Proposition 19.3. *Given a commutative square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

where ϕ and ψ are homotopy equivalences, the induced map

$$\theta = \psi \cup C\phi \colon Cf \longrightarrow Cf'$$

is a homotopy equivalence.

19.3. **Homotopy cofiber sequences.** Consider maps $f\colon X \to Y$ and $g\colon Y \to Z$. To extend g over Cf (to a map $\bar{g}\colon Cf \to Z$ with $\bar{g}i = g$) is equivalent to extending $gf\colon X \to Z$ over CX, i.e., to give a null-homotopy F to gf.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & CX \arrow[d] \arrow[ddr, bend left, "F"] & \\ Y \arrow[r, "i"] \arrow[drr, bend right, "g"'] & Cf \arrow[dr, "\bar{g}"] & \\ & & Z \end{tikzcd}$$

A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

together with a choice of null-homotopy F to $gf\colon X \to Z$ is called a homotopy cofiber sequence if the induced map $Cf \to Z$ is a homotopy equivalence. In this case the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "="'] & Y \arrow[r, "i"] \arrow[d, "="'] & Cf \arrow[d, "\simeq"', "\bar{g}"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

exhibits a homotopy equivalence from the canonical homotopy cofiber sequence for f to the one given by f, g and F. The canonical homotopy cofiber sequence is a homotopy cofiber sequence, in this sense, by way of the null-homotopy to $if\colon X \to Cf$ given by the structure map $CX \to Cf$.

Recall Hatcher's Proposition 0.17:

Proposition 19.4. *If $i\colon A \to X$ is a cofibration (satisfies the homotopy extension property) and A is contractible, then the quotient map $q\colon X \to X/A$ is a homotopy equivalence.*

This implies that for cofibrations the homotopy cofiber and the cofiber are homotopy equivalent.

Corollary 19.5. *If $f: X \to Y$ is a cofibration, then the quotient map*

$$Cf = Y \cup_X CX \xrightarrow{\simeq} Y/X$$

from the homotopy cofiber of f to the cofiber of f is a homotopy equivalence. Hence

$$X \xrightarrow{f} Y \xrightarrow{g} Y/X$$

is a homotopy cofiber sequence (with respect to the constant nullhomotopy to $c = gf$).

Proof. If $f: X \to Y$ is a cofibration, then so is $\bar{f}: CX \to Cf$. This follows formally from the pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & CX \arrow[d, "\bar{f}"] \\ Y \arrow[r, "i"] & Cf \end{tikzcd}$$

Since CX is contractible, the quotient map $q: Cf \to Cf/CX$ is an equivalence, by the previous proposition. Here $Cf/CX \cong Y/X$ is the pushout in the following square.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & * \arrow[d] \\ Y \arrow[r, "g"] & Y/X \end{tikzcd}$$

□

19.4. **Exact and coexact sequences.** For general based spaces Y and Z let $\pi(Y, Z)$ denote the set of homotopy classes $[g]$ of maps $g: Y \to Z$. This has a preferred base point, given by the homotopy class $[c]$ of the constant map. (We shall reserve the notation $[Y, Z]$ for $\pi(Y^c, Z)$ where Y^c is a CW complex that is weakly homotopy equivalent to Y. When Y has the homotopy type of a CW complex, this is the same as $\pi(Y, Z)$.)

For maps $f: X \to Y$ and $h: Z \to W$ we get natural functions

$$f^*: \pi(Y, Z) \longrightarrow \pi(X, Z)$$
$$h_*: \pi(Y, Z) \longrightarrow \pi(Y, W)$$

given by $f^*([g]) = [gf]$ and $h_*([g]) = [hg]$.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

These only depend on the homotopy classes of f and h, respectively. If f is a homotopy equivalence, then f^* is a bijection. Likewise, if h is a homotopy equivalence, then h_* is a bijection.

We say that a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact if the induced diagram

$$\pi(T, X) \xrightarrow{f_*} \pi(T, Y) \xrightarrow{g_*} \pi(T, Z)$$

is exact for each based space T, in the sense that the image of f_* is equal to the kernel of g_*, meaning the preimage for g_* of the base point. Dually, we say that the diagram is coexact if the induced diagram

$$\pi(Z, T) \xrightarrow{g^*} \pi(Y, T) \xrightarrow{f^*} \pi(X, T)$$

is exact for each based space T, now meaning that the image of g^* is equal to the kernel of f^*, i.e., the preimage for f^* of the base point.

Lemma 19.6 (Puppe (1958))**.** *Each homotopy cofiber sequence is coexact.*

Proof. It suffices to prove that

$$\pi(Cf,T) \xrightarrow{i^*} \pi(Y,T) \xrightarrow{f^*} \pi(X,T)$$

is exact. A class $[g] \in \pi(Y,T)$ is in the kernel of f^* if and only if $f^*[g] = [c]$, meaning that there is a null-homotopy $F\colon c \simeq gf$. This is equivalent to the existence of an extension $\bar{g}\colon Cf \to T$ of g over i, i.e., to saying that $[g]$ is in the image of i^*. □

Remark 19.7. Later we shall consider the dual problem, constructing homotopy fiber sequences

$$Fg \xrightarrow{p} Y \xrightarrow{g} Z$$

that are exact in the sense that

$$\pi(T,Fg) \xrightarrow{p_*} \pi(T,Y) \xrightarrow{g_*} \pi(T,Z)$$

is exact for each T. Taking $T = S^n$, this asserts that

$$\pi_n(Fg) \xrightarrow{p_*} \pi_n(Y) \xrightarrow{g_*} \pi_n(Z)$$

is exact for each $n \ge 0$. When (Z,Y) is a pair, $\pi_n(Fg)$ will thus play the role of a relative homotopy group $\pi_{n+1}(Z,Y)$, with p_* the connecting homomorphism.

19.5. **Co-H-spaces and co-H-groups.** The loop sum

$$in_1 * in_2 \colon S^1 \longrightarrow S^1 \vee S^1$$

pinches two points to one. It corepresents the group operation in each fundamental group $\pi_1(Z)$, and induces a pinch map

$$\psi\colon \Sigma X = S^1 \wedge X \longrightarrow (S^1 \vee S^1) \wedge X \cong \Sigma X \vee \Sigma X$$

for any based space X. More precisely, $\psi\colon \Sigma X \to \Sigma X \vee \Sigma X$ and $\epsilon\colon \Sigma X \to *$ satisfy associativity and unitality properties, up to homotopy, which make ΣX a co-H-space. There is also a map $\Sigma X \to \Sigma X$ giving an inverse, up to homotopy, making ΣX a co-H-group.

The pinch map corepresents a pairing

$$+\colon \pi(\Sigma X, Z) \times \pi(\Sigma X, Z) \longrightarrow \pi(\Sigma X, Z)$$

that gives $\pi(Y,Z)$ a group structure whenever $Y = \Sigma X$ is a suspension. This is natural in X and Z. For each homotopy cofiber sequence the suspended diagram

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma Z$$

(with the suspended null-homotopy) is a again a homotopy cofiber sequence, and

$$\pi(\Sigma Z,T) \xrightarrow{\Sigma g^*} \pi(\Sigma Y,T) \xrightarrow{\Sigma f^*} \pi(\Sigma X,T)$$

is now an exact sequence of groups.

There are also pinch maps $CX \to \Sigma X \vee CX$ and $Cf \to \Sigma X \vee Cf$, with the latter inducing an action

$$\pi(\Sigma X,T) \times \pi(Cf,T) \longrightarrow \pi(Cf,T)$$

by the group $\pi(\Sigma X,T)$ on the set $\pi(Cf,T)$, for any T.

The proof that $\pi_n(X)$ is commutative for $n \ge 2$ generalizes to show that $\pi(\Sigma^2 X,T)$ is commutative for any X and T.

19.6. **Iterated homotopy cofibers.** Consider a based map $f\colon X \to Y$, with mapping cone Cf. The inclusion $i\colon Y \to Cf$ is a cofibration, with cofiber

$$q\colon Cf \longrightarrow Cf/Y = (Y \cup_X CX)/Y \cong CX/X \cong \Sigma X\,,$$

and we can form its mapping cone

$$Ci = Cf \cup_Y CY = Cf \cup_i CY\,.$$

The inclusion $j\colon Cf \to Ci$ is again a cofibration, with cofiber

$$Ci/Cf = (Cf \cup_Y CY)/Cf \cong CY/Y \cong \Sigma Y\,,$$

and we can form its mapping cone

$$Cj = Ci \cup_{Cf} CCf = Ci \cup_j CCf\,.$$

Let $k\colon Ci \to Cj$ denote the inclusion.

$$\begin{tikzcd}
 & & & \Sigma X \arrow[d, "\simeq"] & \\
X \arrow[r, "f"] & Y \arrow[r, "i"] & Cf \arrow[r, "j"] \arrow[dr, "q"'] & Ci \arrow[r, "k"] \arrow[d, "\simeq"] & Cj \arrow[d, "\simeq"] \\
 & & & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

Lemma 19.8. *The collapse map*

$$Ci \cong CX \cup_X CY \xrightarrow{\simeq} CX/X \cong \Sigma X$$

is a homotopy equivalence, with a homotopy inverse

$$\Sigma X \xrightarrow{\simeq} Ci$$

given by

$$[t, x] \longmapsto \begin{cases} [2t, x] & \textit{for } 0 \le t \le 1/2, \\ [2(1-t), f(x)] & \textit{for } 1/2 \le t \le 1. \end{cases}$$

Proof. Since $i\colon Y \to Cf$ is a cofibration, so is the inclusion $CY \to Ci$, and since CY is contractible, the collapse map $Ci \to Ci/CY$ is a homotopy equivalence. The displayed map sends $[t, x] \in \Sigma X$ to $[2t, x]$ in $CX \subset Ci$ for $t \le 1/2$, and to $[2(1-t), f(x)]$ in $CY \subset Ci$. Its composite with the collapse map is

$$[t, x] \longmapsto \begin{cases} [2t, x] & \text{for } 0 \le t \le 1/2, \\ [1, x] & \text{for } 1/2 \le t \le 1, \end{cases}$$

where $[1, x]$ equals the base point in ΣX, and this is homotopic to the identity map. □

Lemma 19.9. *The collapse map*

$$Cj \cong CY \cup_Y C(Cf) \xrightarrow{\simeq} CY/Y \cong \Sigma Y$$

is a homotopy equivalence. The composite

$$\Sigma X \xrightarrow{\simeq} Ci \xrightarrow{k} Cj \xrightarrow{\simeq} \Sigma Y$$

is homotopic to $-\Sigma f\colon \Sigma X \to \Sigma Y$, given by

$$[t, x] \longmapsto [1-t, f(x)]\,.$$

Proof. Since $j\colon Cf \to Ci$ is a cofibration, so is the inclusion $C(Cf) \to Cj$, and since $C(Cf)$ is contractible, the collapse map $Cj \to Cj/C(Cf)$ is a homotopy equivalence. The displayed composite is

$$[t, x] \longmapsto \begin{cases} [1, f(x)] & \text{for } 0 \le t \le 1/2, \\ [2(1-t), f(x)] & \text{for } 1/2 \le t \le 1, \end{cases}$$

where $[1, f(x)]$ equals the base point in Y, and this is homotopic to $-\Sigma f$. □

19.7. **The coexact Puppe sequence.** Repeating the process, we obtain the diagram

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{q} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma Cf \xrightarrow{-\Sigma q} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \dots$$

known as the (coexact) Puppe (homotopy cofiber) sequence.

Proposition 19.10. *Each pair of composable maps in the Puppe sequence is a homotopy cofiber sequence. Hence there is an exact sequence*

$$\cdots \to \pi(\Sigma^2 Y, T) \xrightarrow{\Sigma^2 f^*} \pi(\Sigma^2 X, T) \xrightarrow{-\Sigma q^*} \pi(\Sigma Cf, T) \xrightarrow{-\Sigma i^*} \pi(\Sigma Y, T) \xrightarrow{-\Sigma f^*} \pi(\Sigma X, T)$$
$$\xrightarrow{q^*} \pi(Cf, T) \xrightarrow{i^*} \pi(Y, T) \xrightarrow{f^*} \pi(X, T)$$

for each based space T, where the upper row consists of groups and group homomorphisms. Two elements in $\pi(Cf,T)$ have the same image in $\pi(Y,T)$ if and only if they are in the same orbit for the action by $\pi(\Sigma X,T)$ on $\pi(Cf,T)$,

Proof. The equivalence $Ci \to \Sigma X$ exhibits

$$Y \xrightarrow{i} Cf \xrightarrow{q} \Sigma X$$

as a homotopy cofiber sequence. The equivalence $Cj \to \Sigma Y$ exhibits

$$Cf \xrightarrow{j} Ci \longrightarrow \Sigma Y$$

as a homotopy cofiber sequence, which in turn is equivalent to

$$Cf \xrightarrow{q} \Sigma X \xrightarrow{-\Sigma f} \Sigma y\,.$$

This pattern continues. □

Remark 19.11. When $f\colon X \to Y$ is the inclusion in a pair (Y,X), this sequence should be reminiscent of the long exact sequence in reduced cohomology

$$\dots \to \tilde{H}^{n-1}(X) \xrightarrow{\delta} H^n(Y,X) \longrightarrow \tilde{H}^n(Y) \longrightarrow \tilde{H}^n(X) \to \dots\,.$$

Indeed, excision gives an isomorphism $H^*(Y,X) \cong \tilde{H}^*(Y \cup_X CX) = \tilde{H}^*(Cf)$ such that $H^n(Y,X) \to \tilde{H}^n(Y)$ corresponds to i^*, and $\delta\colon \tilde{H}^{n-1}(X) \to H^n(Y,X)$ corresponds to $\pm\Sigma f^*$ via the suspension isomorphism $\sigma\colon \tilde{H}^{n-1}(X) \cong \tilde{H}^n(\Sigma X)$. We will see later that the Puppe sequence for the Eilenberg–Mac Lane space $T = K(\mathbb{Z},n)$ recovers the cohomology long exact sequence, up to degree n.

19.8. **Whitehead's theorem.** Here is a version of Hatcher's Theorem 4.5.

Theorem 19.12. *Let $\phi\colon X \to Y$ be a weak homotopy equivalence (of based, path connected spaces). Then*

$$\phi_*\colon \pi(W,X) \xrightarrow{\cong} \pi(W,Y)$$

is a bijection for each based CW complex W.

Proof. Let W^n denote the n-skeleton of W. We may assume W^0 is a point. We prove that

$$\phi_*\colon \pi(\Sigma^k W^n,X) \xrightarrow{\cong} \pi(\Sigma^k W^n,Y)$$

is a bijection for each $k \ge 0$, by induction on n. Let

$$f\colon V^n = \bigvee_\alpha S^n \longrightarrow W^n$$

be the attaching map for the $(n+1)$-cells in W, so that $Cf = W^{n+1}$. Mapping the Puppe sequence for f to X and to Y gives us a map of exact sequences

$$\begin{array}{ccccccccc}
\pi(\Sigma W^n,X) & \xrightarrow{-\Sigma f^*} & \pi(\Sigma V^n,X) & \xrightarrow{q^*} & \pi(W^{n+1},X) & \xrightarrow{i^*} & \pi(W^n,X) & \xrightarrow{f^*} & \pi(V^n,X) \\
\phi_* \downarrow \cong & & \phi_* \downarrow \cong & & \phi_* \downarrow & & \phi_* \downarrow \cong & & \phi_* \downarrow \cong \\
\pi(\Sigma W^n,Y) & \xrightarrow{-\Sigma f^*} & \pi(\Sigma V^n,Y) & \xrightarrow{q^*} & \pi(W^{n+1},Y) & \xrightarrow{i^*} & \pi(W^n,Y) & \xrightarrow{f^*} & \pi(V^n,Y)
\end{array}$$

The maps in the second and fifth column are products over α of $\phi_*\colon \pi_{n+1}(X) \to \pi_{n+1}(Y)$ and $\phi_*\colon \pi_n(X) \to \pi_n(Y)$, respectively, which we assume are bijections. The maps in the first and fourth column are bijections by the inductive hypothesis. By (a variant of) the five lemma, the middle map is a bijection. This verifies the inductive hypothesis for $k = 0$. The general case follows in the same way from the Puppe sequence for $\Sigma^k f$. The extension from finite-dimensional to general W follows by a passage to limits, using the Milnor lim-Rlim short exact sequence

$$0 \to \operatorname*{Rlim}_n \pi(\Sigma W^n,X) \longrightarrow \pi(W,X) \longrightarrow \lim_n \pi(W^n,X) \to 0$$

and its variant for Y. □

20. November 2nd lecture

Corollary 20.1. *A weak homotopy equivalence of CW complexes is a homotopy equivalence.*

Proof. Let $\phi\colon X \to Y$ be a weak homotopy equivalence of CW complexes. Since

$$\phi_*\colon \pi(Y,X) \xrightarrow{\cong} \pi(Y,Y)$$

is surjective, there is a map $\psi\colon Y \to X$ such that $\phi\psi \simeq 1\colon Y \to Y$. Since

$$\phi_*\colon \pi(X,X) \xrightarrow{\cong} \pi(X,Y)$$

is injective, $\psi\phi \simeq 1\colon X \to X$ follows from $\phi\psi\phi \simeq \phi$. Hence ψ is a homotopy inverse to ϕ. □

Example 20.2. Let $S^\infty = \operatorname{colim}_k S^k$ be the unit sphere in $\mathbb{R}^\infty$, where the colimit is formed over the equatorial inclusion maps $i\colon S^k \to S^{k+1}$ By compactness, each map $f\colon S^n \to S^\infty$ factors through some S^k, hence also through the upper hemisphere $D^{k+1}_+ \subset S^{k+1}$, which is contractible. It follows that $if\colon S^n \to S^{k+1}$ is null-homotopic, so $[f] = 0$ in $\pi_n(S^\infty)$. Thus $S^\infty \to *$ is a weak homotopy equivalence between CW complexes, and is therefore also a homotopy equivalence. Hence S^∞ is contractible.

Corollary 20.3. *If X and Y are homotopy equivalent to CW complexes, and $\phi\colon X \to Y$ is a weak homotopy equivalence, then ϕ is a homotopy equivalence*

Proof. Let $X^c \to X$ and $Y^c \to Y$ be homotopy equivalences, where X^c and Y^c are CW complexes. Then

$$\psi\colon X^c \simeq X \xrightarrow{\phi} Y \simeq Y^c$$

is a composite of weak homotopy equivalences, hence is a weak homotopy equivalence between CW complexes. It follows that ψ is a homotopy equivalence, and that ϕ is homotopic to the composite of homotopy equivalences

$$X \simeq X^c \xrightarrow{\psi} Y^c \simeq Y\,,$$

hence is itself a homotopy equivalence. □

20.1. **Cellular approximation of maps.**

Definition 20.4. A map $f\colon X \to Y$ of CW complexes is cellular if it preserves the skeleton filtration, i.e., if $f(X^n) \subset Y^n$ for each $n \geq 0$.

Here is a version of Hatcher's Theorem 4.8.

Theorem 20.5 (Cellular approximation)**.** *Let X and Y be CW complexes. Every map $f\colon X \to Y$ is homotopic to a cellular map. If $A \subset X$ is a subcomplex and $f|A$ is cellular, then the homotopy may be taken to be stationary on A.*

Sketch proof. By induction we may assume that f is cellular on $X^{n-1} \cup A$, and we need to find a homotopy of f on $X^n \cup A$, relative to $X^{n-1} \cup A$, to a cellular map. Hence, for each n-cell with characteristic map $\Phi\colon I^n \to X^n$ we must deform $f\Phi\colon I^n \to Y$ relative to ∂I^n, so as to take values in Y^n. By induction we may assume that $f\Phi$ has been deformed into Y^k, for $k > n$, and we need to compress it further into Y^{k-1}. Here we may also assume that $Y^k = Y^{k-1} \cup I^k$.

$$\begin{tikzcd}
\partial I^n \arrow[r] \arrow[dd] & I^n \arrow[d, "\Phi"] \arrow[ddl, dashed] \\
 & X^n \arrow[d, "f"] \\
Y^{k-1} \arrow[r] & Y^{k-1} \cup I^k
\end{tikzcd}$$

An argument (Lemma 4.10 in Hatcher's book) using the linear structure in I^n and I^k shows that the continuous map $f\Phi$ can be deformed, relative to the preimage of Y^{k-1}, to a map f_1 that is piecewise linear in the preimage of a simplex. More precisely, there is a linear k-simplex $\Delta \subset I^k$ such that

$$f_1^{-1}(\Delta) = P_1 \cup \dots \cup P_N \subset I^n$$

is a finite union of convex polyhedra, and on each P_i the map f_1 equals the restriction of a linear map $L_i\colon \mathbb{R}^n \to \mathbb{R}^k$. Shrinking Δ we may assume that it avoids the image of those L_i that are not surjective, which means that we may also assume that each L_i is surjective.

When $k > n$ there are no such surjections, so $f_1^{-1}(\Delta)$ is empty. Taking a point $y \in \Delta$, we see that f_1 factors through $Y^{k-1} \cup (I^k - \{y\})$, which deformation retracts to Y^{k-1} by means of a deformation retraction of $I^k - \{y\}$ to ∂I^{k-1}. This shows that $f\Phi$ and f_1 can be deformed, relative to ∂I^n, to map into Y^{k-1}. □

Theorem 20.6. *$\pi_n(S^k) = 0$ for $n < k$.*

Proof. Let S^n and S^k have the minimal CW structures. Any cellular map $S^n \to S^k$ is constant, so each map $S^n \to S^k$ is homotopic to the constant map. □

Corollary 20.7. *$\pi_n(\mathbb{R}P^k) = 0$ for $1 < n < k$.*

Proof. The covering space $p\colon S^k \to \mathbb{R}P^k$ induces isomorphisms of all higher homotopy groups. □

Example 20.8. For $k = \infty$,

$$\pi_n(\mathbb{R}P^\infty) \cong \begin{cases} \mathbb{Z}/2 & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 1)$ is an Eilenberg–Mac Lane space of type $(\mathbb{Z}/2, 1)$.

21. November 4th lecture

21.1. **CW approximation of spaces.** The following is a special case of Hatcher's Proposition 4.13.

Proposition 21.1. *For any space X there exists a CW complex W and a weak equivalence $f\colon W \to X$.*

We call W a CW approximation to X.

Proof. We may assume that X is path connected. We construct the skeleta W^n of W by induction. Let W^0 be a point, and let $f\colon W^0 \to X$ map to the base point. For $n \geq 0$ we inductively assume that we have constructed $f\colon W^n \to X$ to be n-connected, in the sense that $f_* = \pi_k(f)\colon \pi_k(W^n) \to \pi_k(X)$ is a bijection for $0 \leq k < n$ and a surjection for $k = n$. We shall then first attach $(n+1)$-cells to W^n to make $\pi_n(f)$ an isomorphism, and then we attach more $(n+1)$-cells to make $\pi_{n+1}(f)$ a surjection.

Choose cellular maps $\phi_\alpha\colon S^n \to W^n$ representing generators of the kernel of $\pi_n(f)$. Attach $(n+1)$-cells to W^n along these maps, and call the resulting CW complex V^{n+1}. We can extend f over $W^n \subset V^{n+1}$, since $f\phi_\alpha\colon S^n \to X$ is null-homotopic for each index α.

$$\begin{tikzcd}
\bigvee_\alpha S^n \arrow[r] \arrow[d, "\phi"'] & \bigvee_\alpha D^{n+1} \arrow[d] \arrow[dr] & \\
W^n \arrow[r] \arrow[rr, bend right, "f"'] & V^{n+1} \arrow[r, "g"'] & X
\end{tikzcd}$$

The extended homomorphism $\pi_k(g)\colon \pi_k(V^{n+1}) \to \pi_k(X)$ is then injective for each $k \leq n$, because any class in its kernel is represented by a map $S^k \to V^{n+1}$. This is homotopic to a cellular map, by the cellular approximation theorem, which factors through the n-skeleton W^n of V^{n+1}, and represents a class in the kernel of $\pi_k(f)$. Since we attached cells representing generators of this kernel (for $k = n$), it follows that the map $S^k \to V^{n+1}$ is nullhomotopic. The extended homomorphism is also surjective for $k \leq n$, since $\pi_k(f)$ is surjective in this range. Hence $\pi_k(g)$ is a bijection for $0 \leq k \leq n$.

Next, choose cellular maps $\psi_\beta\colon S^{n+1} \to X$ representing generators for $\pi_{n+1}(X)$. Let

$$W^{n+1} = V^{n+1} \vee \bigvee_\beta S^{n+1}$$

and extend g over $V^{n+1} \subset W^{n+1}$ to a map $f: W^{n+1} \to X$ using ψ_β on S^{n+1}_β.

$$\begin{tikzcd}
& \bigvee_\beta S^{n+1}_\beta \arrow[d] & \\
V^{n+1} \arrow[r] \arrow[rr, bend right, "g"'] & W^{n+1} \arrow[r, "f"'] & X
\end{tikzcd}$$

The inclusion $S^{n+1}_\beta \to W^{n+1}$ then represents a class in $\pi_{n+1}(W^{n+1})$ that gets sent by $\pi_{n+1}(f)$ to the relevant generator of $\pi_{n+1}(X)$, which shows that $\pi_{n+1}(f)$ is surjective. Injectivity of $\pi_k(f): \pi_k(W^{n+1}) \to \pi_k(X)$ for $k \le n$ follows from the injectivity of $\pi_k(g)$ by cellular approximation, since any cellular map $S^k \to W^{n+1}$ factors through V^{n+1}. Surjectivity of $\pi_k(f)$ for $k \le n$ follows algebraically from that of $\pi_k(g)$. Hence $f: W^{n+1} \to X$ is $(n+1)$-connected.

Let $W = \bigcup_n W^n$. Then

$$\operatorname*{colim}_n \pi_k(W^n) \cong \pi_k(W) \xrightarrow{\cong} \pi_k(X)$$

is an isomorphism for all $k \ge 0$, since the colimit system consists of isomorphisms for all $n \ge k+1$. □

21.2. **Compactly generated weak Hausdorff spaces.** Let A, X and Y be sets, and let Y^A be the set of functions $A \to Y$. There is then a bijection

$$\{\text{functions } A \times X \longrightarrow Y\} \xrightarrow{\cong} \{\text{functions } X \longrightarrow Y^A\}$$

taking a function $f: A \times X \to Y$ to the function $g: X \to Y^A$ given by

$$f(a, x) = g(x)(a)\,.$$

This bijection is natural in X and Y, hence is called an adjunction between the functors $X \mapsto A \times X$ and $Y \mapsto Y^A$ from sets to sets.

Next, let A, X and Y be unbased spaces, and let Y^A denote the space of maps $A \to Y$, with the compact-open topology. If A is locally compact and Hausdorff, then the bijection above restricts to a bijection

$$\{\text{maps } A \times X \longrightarrow Y\} \xrightarrow{\cong} \{\text{maps } X \longrightarrow Y^A\}\,.$$

Again, this is natural in X and Y, and defines an adjunction between the functors $X \mapsto A \times X$ and $Y \mapsto Y^A$ from spaces to spaces. Hence we can express a homotopy $F: I \times X \to Y$ as a map $G: X \to Y^I$ to the space Y^I of maps $I \to Y$. We call Y^I the 'free path space' of Y, where 'free' refers to the absence of any base point condition.

This works fine, because $A = I$ is (locally) compact and Hausdorff, but to avoid pathology in more general cases we hereafter restrict attention to a class of topological spaces that is intermediate between the locally compact Hausdorff spaces and general sets with topologies.

A first 'convenient category' CGH, of compactly generated Hausdorff spaces, was introduced by J. L. Kelley (1955) and developed by Norman Steenrod (1967). This category was slightly extended to the class CGWH, of compactly generated weak Hausdorff spaces, by John Moore, see M. C. McCord (1969). Other references include an appendix to Gaunce Lewis' PhD thesis (1978), Chapter 5 of May's "A Concise Course in Algebraic Topology" (1999), notes by Neil Strickland (2009) and Appendix A of Stefan Schwede's book "Global Homotopy Theory" from 2018.

Definition 21.2. Let K range over all compact Hausdorff spaces. A space X is weak Hausdorff if for each map $g: K \to X$ the image $g(K)$ is closed in X. A subset $A \subset X$ is compactly closed if for each map $g: K \to X$ the preimage $g^{-1}(A)$ is closed in K. The space X is compactly generated if each compactly closed subset is closed. Following May we let $\mathscr{U}$ be the category of compactly generated weak Hausdorff spaces.

Example 21.3. All locally compact Hausdorff spaces, and all metric spaces, are compactly generated weak Hausdorff.

21.3. **The exponential law.** We can form products $A \times X$ and mapping spaces Y^A within the category $\mathscr{U}$, refining the usual topologies by setting the compactly closed subsets to be the 'new' closed subsets. Then the correspondence $f \leftrightarrow g$ defines a bijection

$$\{\text{maps } A \times X \longrightarrow Y\} \xrightarrow{\cong} \{\text{maps } X \longrightarrow Y^A\}$$

for all CGWH spaces A, X and Y. Moreover, this bijection is a homeomorphism

$$Y^{A \times X} \cong (Y^A)^X$$

when we give both sets of maps the mapping space topologies. This is sometimes called the exponential law. In other words, $\mathscr{U}$ is a closed symmetric monoidal category.

Also following May, let $\mathscr{T}$ be the category of based compactly generated weak Hausdorff spaces. When A, X and Y are based CGWH spaces, we can form the smash product

$$A \wedge X = \frac{A \times X}{A \vee X}$$

and the based mapping space

$$\mathrm{Map}(A, Y) = \{\text{based maps } A \to Y\} \subset Y^A\,.$$

The correspondence $f \leftrightarrow g$ then defines a natural bijection

$$\{\text{based maps } A \wedge X \longrightarrow Y\} \xrightarrow{\cong} \{\text{based maps } X \longrightarrow \mathrm{Map}(A, Y)\}$$

which refines to a natural homeomorphism

$$\mathrm{Map}(A \wedge X, Y) \cong \mathrm{Map}(X, \mathrm{Map}(A, Y))\,.$$

Hence $\mathscr{T}$ is also a closed symmetric monoidal category. This is the category in which point set topological constructions are made in modern algebraic topology.

21.4. **Free and based path fibrations, loop spaces.** See Hatcher's pages 375 and 405–409 for a discussion similar to the following.

When $A = I_+$, let

$$Y^I = \mathrm{Map}(I_+, Y)$$

be the free path space of Y, based at the constant path. The adjunction

$$\mathrm{Map}(I \ltimes X, Y) = \mathrm{Map}(I_+ \wedge X, Y) \cong \mathrm{Map}(X, \mathrm{Map}(I_+, Y)) = \mathrm{Map}(X, Y^I)$$

shows that homotopies $F\colon I \ltimes X \to Y$ can equally well be expressed as maps $G\colon X \to Y^I$. Here G maps $x \in X$ to the path

$$G(x)\colon t \mapsto G(x)(t) = F(t, x)$$

in Y. By evaluating a path at time $t \in I$, we get a projection

$$p_t\colon Y^I \longrightarrow Y$$

sending $\eta\colon I \to Y$ to $p_t(\eta) = \eta(t)$. In particular, p_0 sends a path to its starting point and p_1 sends a path to its end point.

When $A = I = [0, 1]$, based at 0, let

$$PY = \mathrm{Map}(I, Y)$$

be the path space of Y. The elements of PY are paths $\eta\colon I \to Y$ with $\eta(0) = y_0$, the base point in Y. The end point projection p_1 restricts to a map $p\colon PY \longrightarrow Y$ sending η to $\eta(1) \in Y$. We call this the path space fibration. The adjunction

$$\mathrm{Map}(CX, Y) = \mathrm{Map}(I \wedge X, Y) \cong \mathrm{Map}(X, \mathrm{Map}(I, Y)) = \mathrm{Map}(X, PY)$$

shows that a nullhomotopy $F\colon CX \to Y$ to a map $f = Fi\colon X \to Y$ is equivalent to a map $G\colon X \to PY$ with $f = pG$.

The maps $p_1\colon Y^I \to Y$ and $p\colon PY \to Y$ are (Hurewicz) fibrations, i.e., have the homotopy lifting property. These are "good" projections in a homotopical sense. Hence it is homotopically meaningful to consider the preimage in PY of the base point in Y. Let

$$\Omega Y = \mathrm{Map}(S^1, Y) \cong \mathrm{Map}(I/\partial I, Y) = p^{-1}(y_0) \subset \mathrm{Map}(I, Y) = PY$$

be the loop space of Y. It is the space of loops $\eta\colon I \to Y$ in Y, starting and ending at y_0.

Recall that we identify $A = S^1$ with $I/\partial I$. The adjunction

$$\mathrm{Map}(\Sigma X, Y) = \mathrm{Map}(S^1 \wedge X, Y) \cong \mathrm{Map}(X, \mathrm{Map}(S^1, Y)) = \mathrm{Map}(X, \Omega Y)$$

exhibits the suspension functor $X \mapsto \Sigma X$ as the left adjoint of the loop space functor $Y \mapsto \Omega Y$. We write $\Omega^m Y = \Omega(\Omega^{m-1} Y)$ for the m-fold iterated loop space.

Lemma 21.4. *There are natural bijections $\pi_{n+m}(Y) \cong \pi_n(\Omega^m Y)$ for all $n, m \geq 0$. In particular, $\pi_{n+1}(Y) \cong \pi_n(\Omega Y)$ and $\pi_n(Y) \cong \pi_0(\Omega^n Y)$.*

21.5. **H-spaces and H-groups.** Hopf considered the following generalization of a Lie group. Let $*$ denote the one-point space.

Definition 21.5. An H-space is a space X with a pairing $\phi\colon X \times X \to X$ and a unit map $\eta\colon * \to X$ such that

$$\begin{tikzcd}
X \times X \times X \arrow[r, "\phi \times 1"] \arrow[d, "1 \times \phi"'] & X \times X \arrow[d, "\phi"] \\
X \times X \arrow[r, "\phi"] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
* \times X \arrow[r, "\eta \times 1"] \arrow[dr, "\cong"'] & X \times X \arrow[d, "\phi"] & X \times * \arrow[l, "1 \times \eta"'] \arrow[dl, "\cong"] \\
& X &
\end{tikzcd}$$

commute up to homotopy. An H-group is an H-space with a map $\chi\colon X \to X$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "{(\chi,1)}"] \arrow[d] & X \times X \arrow[d, "\phi"] & X \arrow[l, "{(1,\chi)}"'] \arrow[d] \\
* \arrow[r, "\eta"] & X & * \arrow[l, "\eta"']
\end{tikzcd}$$

commutes up to homotopy.

Example 21.6. A topological group G, with the multiplication $\phi\colon G \to G \to G$, the inclusion of the unit element $\eta\colon \{e\} \subset G$, and the group inverse map $\chi\colon G \to G$, gives an example of an H-group.

Example 21.7. A loop space $X = \Omega Y$, with the loop multiplication $\phi\colon \Omega Y \times \Omega Y \to \Omega Y$ sending (f, g) to $f * g$, the inclusion of the constant loop $\eta\colon \{c\} \subset \Omega Y$, and the reverse loop map $\chi\colon \Omega Y \to \Omega Y$ sending f to $\bar{f}$, exhibit ΩY as an H-group.

H-groups represent group valued functors. The product map ϕ represents a pairing

$$+\colon \pi(T, X) \times \pi(T, X) \longrightarrow \pi(T, X)$$

that gives $\pi(T, X)$ a natural group structure, for any space T. In particular $\pi(T, \Omega Y)$ is naturally a group, for any loop space ΩY. The natural bijection

$$\pi(\Sigma X, Y) \cong \pi(X, \Omega Y)$$

takes the group structure on the left, obtained from the co-H-space structure $\psi\colon \Sigma X \to \Sigma X \vee \Sigma X$ on ΣX, to the group structure on the right, obtained from the H-space structure $\phi\colon \Omega Y \times \Omega Y \to \Omega Y$ on ΩY.

There are also pairings $*\colon \Omega Z \times PZ \to PZ$ and $\Omega Z \times Fg \to Fg$, with the latter inducing an action

$$\pi(T, \Omega Z) \times \pi(T, Fg) \longrightarrow \pi(T, Fg)$$

by the group $\pi(T, \Omega Z)$ on the set $\pi(T, Fg)$, for any T.

The proof that $\pi_n(X)$ is commutative for $n \geq 2$ generalizes to show that $\pi(\Sigma^2 T, Y) \cong \pi(\Sigma T, \Omega Y) \cong \pi(T, \Omega^2 Y)$ is commutative for any T and Y.

21.6. **Path fibrations.** Any map $g\colon Y \to Z$ can be factored as a homotopy equivalence $e\colon Y \simeq Eg$ followed by a fibration $q\colon Eg \to Z$. Let the path fiber space

$$Eg = Y \times_Z Z^I$$

of g be defined by the lower pullback square in the following diagram.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "e"', "\simeq"] & Z \arrow[d, "s"'] \arrow[dr, "1"] & \\ Eg \arrow[r] \arrow[d, "\simeq"] & Z^I \arrow[r, "p_0"] \arrow[d, "p_1"'] & Z \\ Y \arrow[r, "g"] & Z & \end{tikzcd}$$

A point in Eg is thus a pair (y, ζ), with $y \in Y$ and $\zeta\colon I \to Z$, such that $g(y) = \zeta(1)$. The structure maps $Eg \to Y$ and $Eg \to Z^I$ remember y and ζ, respectively. The composite

$$q\colon Eg \longrightarrow Z^I \xrightarrow{p_0} Z$$

sends (y, ζ) to $\zeta(0)$ and has the homotopy lifting property, hence is a (Hurewicz) fibration. The constant path section $s\colon Z \to Z^I$ takes $z \in Z$ to $s(z)$ with $s(z,t) = z$ for all $t \in I$. It lifts to a map $e\colon Y \to Eg$, with $e(y) = (y, s(g(y)))$. The projection $p_0\colon Z^I \to Z$ is a deformation retraction relative to $Z \cong s(Z)$, hence induces a deformation retraction $Eg \to Z$ relative to $Y \cong e(Y)$. In particular, e is a homotopy equivalence. Then $g = qe\colon Y \to Z$ gives the claimed factorization.

21.7. **Homotopy fibers.** Let the homotopy fiber

$$Fg = Y \times_Z PZ$$

of $g\colon Y \to Z$ be the pullback in the following diagram

$$\begin{tikzcd} Fg \arrow[r] \arrow[d, "p"] & PZ \arrow[d, "p"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

A point in Fg is thus a pair (y, ζ) where $y \in Y$ and $\zeta\colon I \to Z$ is a path from the base point $\zeta(0) = z_0$ to $\zeta(1) = g(y)$. The inclusion $PZ = p_0^{-1}(z_0) \subset Z^I$ pulls back to an inclusion

$$Fg = q^{-1}(z_0) \subset Eg\,.$$

The diagram

$$Fg \xrightarrow{p} Y \xrightarrow{g} Z$$

is called the canonical homotopy fiber sequence of g.

Example 21.8. The homotopy fiber of $* \to Z$ is the loop space ΩZ.

The homotopy fiber is a homotopy invariant construction.

Proposition 21.9. *Given a commutative square*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\phi"'] & Z \arrow[d, "\psi"] \\ Y' \arrow[r, "g'"] & Z' \end{tikzcd}$$

where ϕ and ψ are homotopy equivalences, the induced map

$$\theta = \phi \times P\psi\colon Fg \longrightarrow Fg'$$

is a homotopy equivalence.

21.8. **Homotopy fiber sequences.** Consider maps $f\colon X \to Y$ and $g\colon Y \to Z$. To lift f over Fg (to a map $\tilde{f}\colon X \to Fg$ with $p\tilde{f} = f$) is equivalent to lifting $gf\colon X \to Z$ over PZ, i.e., to give a nullhomotopy G to gf.

$$\begin{tikzcd}
X \arrow[dr, "\tilde{f}"] \arrow[drr, bend left, "G"] \arrow[ddr, bend right, "f"'] & & \\
& Fg \arrow[r] \arrow[d, "p"] & PZ \arrow[d, "p"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

together with a choice of nullhomotopy G to $gf\colon X \to Z$ is called a homotopy fiber sequence if the induced map $X \to Fg$ is a homotopy equivalence. In this case the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\tilde{f}", "\simeq"'] & Y \arrow[r, "g"] \arrow[d, "="'] & Z \arrow[d, "="'] \\
Fg \arrow[r, "p"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

exhibits a homotopy equivalence from the given homotopy fiber sequence to the canonical homotopy fiber sequence for g.

Lemma 21.10. *Each homotopy fiber sequence is exact.*

Proof. It suffices to check that

$$\pi(T, Fg) \xrightarrow{p_*} \pi(T, Y) \xrightarrow{g_*} \pi(T, Z)$$

is exact. A class $[f] \in \pi(T, Y)$ is in the kernel of g_* if and only if $g_*[f] = [c]$, meaning that there is a null-homotopy $F\colon c \simeq gf$. This is equivalent to the existence of a lift $\tilde{f}\colon T \to Fg$ of f over p, i.e., to saying that $[f]$ is in the image of p_*. □

22. November 9th lecture

Remark 22.1. At this point we have two equivalent interpretations of a nullhomotopy to gf. One is a lift $\tilde{f}$ of f over $p\colon Fg \to Y$. The other is an extension $\bar{g}$ over g over $i\colon Y \to Cf$.

$$\begin{tikzcd}
& Fg \arrow[d, "p"] & \\
X \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & Y \arrow[r, "g"] \arrow[d, "i"'] & Z \\
& Cf \arrow[ur, dashed, "\bar{g}"'] &
\end{tikzcd}$$

22.1. **Fibers and homotopy fibers.** The following is dual to Hatcher's Proposition 0.17. Let $b_0 \in B$ denote the base point.

Proposition 22.2. *If $p\colon E \to B$ is a fibration (satisfies the homotopy lifting property) and B is contractible, then the fiber inclusion $j\colon F = p^{-1}(b_0) \to E$ is a homotopy equivalence.*

Proof. Let $G\colon I \ltimes B \to B$ be a contraction of B, i.e., a homotopy from c to 1_B. Let $H\colon I \ltimes E \to E$ be a homotopy to 1_E lifting $G(1 \ltimes p)$.

$$\begin{tikzcd}
E \arrow[rr, "1_E"] \arrow[d, "i_1"'] & & E \arrow[d, "p"] \\
I \ltimes E \arrow[r, "1 \ltimes p"'] \arrow[urr, dashed, "H"] & I \ltimes B \arrow[r, "G"'] & B
\end{tikzcd}$$

We have $pH(t,e) = G(t,p(e)) = b_0$ for all $e \in p^{-1}(b_0)$, so H restricts to a homotopy $\tilde{H}\colon I \ltimes F \to F$ to 1_F.

$$\begin{tikzcd}
I \ltimes F \arrow[r, "\tilde{H}"] \arrow[d, "1 \ltimes j"'] & F \arrow[d, "j"] \\
I \ltimes E \arrow[r, "H"'] & E
\end{tikzcd}$$

We also have $pH(0,e) = G(0,p(e)) = b_0$ for all $e \in E$, so $Hi_0\colon E \to E$ factors as jk for a map $k\colon E \to F$.

$$\begin{tikzcd}
F \arrow[r, "\tilde{H}i_0"] \arrow[d, "j"'] & F \arrow[d, "j"] \\
E \arrow[r, "Hi_0"'] \arrow[ur, "k"'] & E
\end{tikzcd}$$

It follows that $kj = \tilde{H}i_0$, since $jkj = Hi_0 j = j\tilde{H}i_0$ and j is injective, so both triangles commute in the diagram above. It follows that j and k are homotopy inverses, since H gives a homotopy from jk to 1_E and $\tilde{H}$ gives a homotopy from kj to 1_F. □

Corollary 22.3. *If $g\colon Y \to Z$ is a fibration, then the inclusion*

$$F = g^{-1}(z_0) \xrightarrow{\simeq} Fg$$

of the fiber of g (over the base point of Z) to the homotopy fiber of g is a homotopy equivalence. Hence

$$F \xrightarrow{f} Y \xrightarrow{g} Z$$

is a homotopy fiber sequence (with respect to the constant nullhomotopy to $c = gf$).

Proof. If $g\colon Y \to Z$ is a fibration, then so is the pullback $\tilde{g}\colon Fg \to PZ$. Here PZ is contractible, so the fiber of $Fg \to PZ$ is homeomorphic to $F = g^{-1}(z_0)$ and is homotopy equivalent to Fg. □

This corollary is also part of Hatcher's Proposition 4.65.

22.2. **Iterated homotopy fibers.** Consider a based map $g\colon Y \to Z$, with homotopy fiber Fg. The projection $p\colon Fg \to Y$ is a fibration, with fiber

$$j\colon \Omega Z = p^{-1}(z_0) \cong p^{-1}(y_0) \longrightarrow Fg\,.$$

We can form the repeated homotopy fiber

$$Fp = Fg \times_Y PY\,.$$

A point in Fp is a triple (y, ζ, η), where $y \in Y$, $\zeta\colon I \to Z$ is a path from z_0 to $g(y)$, and $\eta\colon I \to Y$ is a path from y_0 to y.

The projection $q\colon Fp \to Fg$ is again a fibration, with fiber

$$q^{-1}(y_0, c) \cong \Omega Y$$

consisting of triples (y_0, c, η), where η is any loop in (Y, y_0). We can form its iterated homotopy fiber

$$Fq = Fp \times_{Fg} P(Fg)\,.$$

Let $r\colon Fq \to Fp$ denote the projection.

$$\begin{tikzcd}
\Omega Y \arrow[r, "-\Omega g"] \arrow[d, "\simeq"'] & \Omega Z \arrow[d, "\simeq"'] \arrow[dr, "j"] & & & \\
Fq \arrow[r, "r"] & Fp \arrow[r, "q"] \arrow[d, "\simeq"'] & Fg \arrow[r, "p"] & Y \arrow[r, "g"] & Z \\
& \Omega Z & & &
\end{tikzcd}$$

Lemma 22.4. *The fiber inclusion $\Omega Z \xrightarrow{\simeq} Fp$ is a homotopy equivalence, with homotopy inverse $Fp \to \Omega Z$ given by*

$$(y, \zeta, \eta) \longmapsto \zeta * \overline{g\eta}\,.$$

Proof. p is a fibration. □

Lemma 22.5. *The fiber inclusion $\Omega Y \xrightarrow{\simeq} Fq$ is a homotopy equivalence. The composite*

$$\Omega Y \xrightarrow{\simeq} Fq \xrightarrow{r} Fp \xrightarrow{\simeq} \Omega Z$$

is homotopic to $-\Omega g \colon \Omega Y \to \Omega Z$.

Proof. q is a fibration. The calculation of the composite is very similar to that in the proof of Lemma 19.9. □

23. November 11th lecture

23.1. The exact Puppe sequence.

Repeating the process, we obtain the diagram

$$\dots \longrightarrow \Omega^2 Y \xrightarrow{\Omega^2 g} \Omega^2 Z \xrightarrow{-\Omega j} \Omega Fg \xrightarrow{-\Omega p} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{j} Fg \xrightarrow{p} Y \xrightarrow{g} Z$$

known as the (exact) Puppe (homotopy fiber) sequence.

Proposition 23.1. *Each pair of composable maps in the Puppe sequence is a homotopy fiber sequence. Hence there is an exact sequence*

$$\dots \longrightarrow \pi(T, \Omega^2 Y) \xrightarrow{\Omega^2 g_*} \pi(T, \Omega^2 Z) \xrightarrow{-\Omega j_*} \pi(T, \Omega Fg) \xrightarrow{-\Omega p_*} \pi(T, \Omega Y) \xrightarrow{-\Omega g_*} \pi(T, \Omega Z)$$
$$\xrightarrow{j_*} \pi(T, Fg) \xrightarrow{p_*} \pi(T, Y) \xrightarrow{g_*} \pi(T, Z)$$

for each based space T, where the upper row consists of groups and homomorphisms. Two elements in $\pi(T, Fg)$ have the same image in $\pi(T, Y)$ if and only if they are in the same orbit for the action of $\pi(T, \Omega Z)$ on $\pi(T, Fg)$.

Proof. The equivalence $\Omega Z \to Fp$ exhibits

$$\Omega Z \xrightarrow{j} Fg \xrightarrow{p} Y$$

as a homotopy fiber sequence. The equivalence $\Omega Y \to Fq$ exhibits

$$\Omega Y \longrightarrow Fp \xrightarrow{q} Fg$$

as a homotopy fiber sequence, which in turn is equivalent to

$$\Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{j} Fg\,.$$

This pattern continues. □

Theorem 23.2. *For any map $g \colon Y \to Z$ of based spaces, there is a long exact sequence of homotopy groups and sets*

$$\dots \longrightarrow \pi_n(Fg) \xrightarrow{p_*} \pi_n(Y) \xrightarrow{g_*} \pi_n(Z) \xrightarrow{j_*} \pi_{n-1}(Fg) \longrightarrow \dots$$

ending with

$$\dots \longrightarrow \pi_1(Z) \xrightarrow{j_*} \pi_0(Fg) \xrightarrow{p_*} \pi_0(Y) \xrightarrow{g_*} \pi_0(Z)\,.$$

Two elements in $\pi_0(Fg)$ have the same image in $\pi_0(Y)$ if and only if they are in the same orbit for the action of $\pi_1(Z)$ on $\pi_0(Fg)$.

Proof. This is the long exact Puppe sequence for $g \colon Y \to Z$, in the case $T = S^0$, where we use the identifications

$$\pi(S^0, \Omega^n X) \cong \pi(S^n, X) = \pi_n(X)$$

for $n \geq 0$ and $X = Fg$, Y and Z. □

23.2. **Long exact sequences for a fibration.** When $g\colon Y \to Z$ is a fibration, we can replace its homotopy fiber by its fiber.

Theorem 23.3. *For any fibration $p\colon E \to B$, with fiber $F = p^{-1}(b_0)$, there is a long exact sequence of homotopy groups and sets*

$$\dots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \dots$$

ending with

$$\dots \longrightarrow \pi_1(B) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B)\,.$$

Here $i\colon F \to E$ denotes the fiber inclusion. Two elements in $\pi_0(F)$ have the same image in $\pi_0(E)$ if and only if they are in the same orbit for the action of $\pi_1(B)$.

Proof. This follows directly from the long exact homotopy sequence for $Fp \to E \to B$ and the homotopy equivalence $F \simeq Fp$. A direct proof is given on page 376 in Hatcher's book, as Theorem 4.41. $\square$

23.3. **Fiber bundles.**

Definition 23.4. A fiber bundle with base B, total space E and fiber F is a map $p\colon E \to B$ that is locally trivial, in the sense that B is covered by open subsets U such that there exist homeomorphisms h making the triangles

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h", "\cong"'] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr"] \\ & U & \end{tikzcd}$$

commute. We call h a local trivialization of p over U. Note that h induces homeomorphisms $p^{-1}(b) \cong \{b\} \times F \cong F$ for each $b \in U$.

Theorem 23.5 (Huebsch (1955), Hurewicz (1955))**.** *A fiber bundle over a paracompact base space is a (Hurewicz) fibration.*

Proposition 23.6 (Hatcher's Proposition 4.48)**.** *Any fiber bundle is a Serre fibration, i.e., has the homotopy lifting property with respect to maps from CW complexes.*

Theorem 23.7. *For any fiber bundle $p\colon E \to B$ with fiber F, there is a long exact sequence of homotopy groups and sets*

$$\dots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \dots$$

ending with

$$\dots \longrightarrow \pi_1(B) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B)\,.$$

Here $i\colon F \to E$ denotes the fiber inclusion. Two elements in $\pi_0(F)$ have the same image in $\pi_0(E)$ if and only if they are in the same orbit for the action of $\pi_1(B)$.

Example 23.8. The projection $q\colon S^{2k+1} \to \mathbb{C}P^k$ is a fiber bundle with fiber S^1. Since $\pi_n(S^1) = 0$ for $n \neq 1$, the long exact sequence in homotopy gives isomorphisms

$$q\colon \pi_n(S^{2k+1}) \xrightarrow{\cong} \pi_n(\mathbb{C}P^k)$$

for $n \geq 3$, and an exact sequence

$$0 \to \pi_2(S^{2k+1}) \longrightarrow \pi_2(\mathbb{C}P^k) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(S^{2k+1}) \longrightarrow \pi_1(\mathbb{C}P^k) \longrightarrow 0\,.$$

For $k \geq 1$ we have $\pi_1(S^{2k+1}) = \pi_2(S^{2k+1}) = 0$, so $\mathbb{C}P^k$ is 1-connected with $\pi_2(\mathbb{C}P^k) \cong \pi_1(S^1) \cong \mathbb{Z}$. Furthermore, $\pi_n(\mathbb{C}P^k) = 0$ for $2 < n \leq 2k$.

For $k = 1$, we get isomorphisms

$$q_*\colon \pi_n(S^3) \xrightarrow{\cong} \pi_n(S^2)$$

for all $n \geq 3$.

For $k = \infty$, the projection $q\colon S^\infty \to \mathbb{C}P^\infty$ has contractible total space. Hence

$$\pi_n(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z} & \text{for } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ is an Eilenberg–Mac Lane space of type $(\mathbb{Z}, 2)$.

Example 23.9. The projection $q\colon S^{4k+3} \to \mathbb{H}P^k$ is a fiber bundle with fiber $S^3 = Sp(1)$. Since $\pi_n(S^{4k+3}) = 0$ for $n \le 4k+2$, the long exact sequence in homotopy gives isomorphisms

$$\partial\colon \pi_n(\mathbb{H}P^k) \xrightarrow{\cong} \pi_{n-1}(S^3)$$

for $n \le 4k+2$.

For $k = 1$ we get isomorphisms

$$\partial\colon \pi_n(S^4) \xrightarrow{\cong} \pi_{n-1}(S^3)$$

for $n \le 6$.

For $k = \infty$ we get isomorphisms $\pi_n(\mathbb{H}P^\infty) \cong \pi_{n-1}(S^3)$ for all n, but these are generally nonzero for most n, so $\mathbb{H}P^\infty$ is far from being an Eilenberg–Mac Lane space.

23.4. **Relative homotopy groups.** When $g\colon Y \to Z$ is the inclusion of a subspace, we can rewrite the homotopy groups of Fg are relative homotopy groups for the pair (Z, Y). See pages 343–344 in Hatcher's book.

Let $J^n \subset \partial(I \times I^n) \subset I \times I^n$ be the subspace

$$J^n = (I \times \partial I^n) \cup (\{0\} \times I^n)\,.$$

Then

$$\frac{I \times I^n}{J^n} \cong C(I^n/\partial I^n) \cong CS^n$$

and

$$\frac{\partial(I \times I^n)}{J^n} \cong \{1\} \times I^n/\{1\} \times \partial I^n \cong \{1\} \times S^n\,.$$

The inclusion $\partial(I \times I^n) \subset I \times I^n$ corresponds to the standard inclusion $i\colon S^n \subset CS^n$ induced by i_1.

Definition 23.10. Let (X, A) be a pair of spaces, based at $x_0 \in A \subset X$. For $n \ge 0$ let

$$\pi_{n+1}(X, A, x_0) = \{f\colon (I \times I^n, \partial(I \times I^n), J^n) \to (X, A, x_0)\}/{\simeq}$$

be the set of homotopy classes of maps $f\colon I \times I^n \to X$ taking the boundary $\partial(I \times I^n)$ to A and the subspace J^n to the base point. The homotopies are required to preserve these conditions.

For $n \ge 1$ we can define a group structure on $\pi_{n+1}(X, A, x_0)$, taking $f, g\colon (I \times I^n, \partial(I \times I^n), J^n) \to (X, A, x_0)$ to $f + g$ defined by

$$(f+g)(s_0, s_1, \dots, s_n) = \begin{cases} f(s_0, 2s_1, s_2, \dots, s_n) & \text{for } 0 \le s_1 \le 1/2, \\ g(s_0, 2s_1 - 1, s_2, \dots, s_n) & \text{for } 1/2 \le s_1 \le 1. \end{cases}$$

Proposition 23.11. *Let $i\colon A \to X$ be an inclusion, with homotopy fiber Fi. There is a natural bijection*

$$\pi_{n+1}(X, A) \cong \pi_n(Fi)$$

which is compatible with the group structures for $n \ge 1$.

Proof. The bijection takes the homotopy class of

$$f\colon (I \times I^n, \partial(I \times I^n), J^n) \longrightarrow (X, A, x_0)$$

to the homotopy class of

$$g\colon (I^n, \partial I^n) \longrightarrow (Fi, (x_0, c))$$

where $g(s_1, \dots, s_n) = (a, \xi) \in Fi$ is given by $a = f(0, s_1, \dots, s_n)$ and $\xi(t) = f(t, s_1, \dots, s_n)$. Note that $a \in A$ and ξ is a path in X from a to x_0. If $(s_1, \dots, s_n) \in \partial I^n$ then $g(s_1, \dots, s_n) = (x_0, c)$ is the base point of Fi. Compatibility with the group structure follows directly from the definitions.

Equivalently, the bijection takes the homotopy class of

$$f\colon (CS^n, S^n) \longrightarrow (X, A)$$

to the homotopy class of

$$g\colon S^n \longrightarrow Fi$$

where $g(s) = (a, \xi) \in F_i$ is given by $a = f(0, s)$ and $\xi(t) = f(t, s)$ for $s \in S^n$ and $t \in I$. □

Definition 23.12. Let $j\colon \pi_{n+1}(X, x_0) \to \pi_{n+1}(X, A, x_0)$ take the homotopy class of a map

$$f\colon (I \times I^n, \partial(I \times I^n)) \longrightarrow (X, x_0)$$

to the homotopy class of the same map, viewed as a map

$$j(f)\colon (I \times I^n, \partial(I \times I^n), J^n) \longrightarrow (X, A, x_0)\,.$$

Let $\partial\colon \pi_{n+1}(X, A, x_0) \to \pi_n(A, x_0)$ take the homotopy class of a map

$$f\colon (I \times I^n, \partial(I \times I^n), J^n) \longrightarrow (X, A, x_0)$$

to the homotopy class of its restriction to $I^n \cong \{1\} \times I^n$, viewed as a map

$$\partial(f)\colon (I^n, \partial I^n) \longrightarrow (A, x_0)\,.$$

Here $(I^n, \partial I^n) \cong (\{1\} \times I^n, \{1\} \times \partial I^n) \subset (\partial(I \times I^n), J^n)$.

Lemma 23.13. *The diagram*

$$\begin{tikzcd}
\pi_{n+1}(X, x_0) \arrow[r, "j"] \arrow[d, "="] & \pi_{n+1}(X, A, x_0) \arrow[r, "\partial"] \arrow[d, "\cong"] & \pi_n(A, x_0) \arrow[d, "="] \\
\pi_{n+1}(X, x_0) \arrow[r, "j_*"] & \pi_n(Fi, (x_0, c)) \arrow[r, "p_*"] & \pi_n(A, x_0)
\end{tikzcd}$$

commutes.

Proof. This follows directly from the definitions of j and ∂, and of the maps $j\colon \Omega A \to Fi$ and $p\colon Fi \to X$. □

Theorem 23.14 (Long exact sequence in homotopy for a pair)**.** *For any pair (X, A) of based spaces, there is a long exact sequence of homotopy groups and sets*

$$\dots \longrightarrow \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i} \pi_n(X) \xrightarrow{j} \pi_n(X, A) \longrightarrow \dots$$

ending with

$$\dots \longrightarrow \pi_1(X) \xrightarrow{j} \pi_1(X, A) \xrightarrow{\partial} \pi_0(A) \xrightarrow{i} \pi_0(X)\,.$$

Two elements in $\pi_1(X, A)$ have the same image in $\pi_0(A)$ if and only if they are in the same orbit for the action of $\pi_1(X)$ on $\pi_1(X, A)$.

Proof. This is isomorphic to the long exact sequence for $i\colon A \to X$. A direct proof is given on pages 344–345 in Hatcher's book (as Theorem 4.3). □

23.5. **Suspension and stable stems.** Let X be a based space. In the pair (CX, X), the homotopy groups $\pi_n(CX) = 0$ are trivial, so the connecting homomorphism

$$\partial\colon \pi_{n+1}(CX, X) \xrightarrow{\cong} \pi_n(X)$$

is an isomorphism for all $n \geq 0$. The collapse map $q\colon (CX, X) \to (\Sigma X, *)$ induces a homomorphism

$$q_*\colon \pi_{n+1}(CX, X) \longrightarrow \pi_{n+1}(\Sigma X, *) = \pi_{n+1}(\Sigma X)\,.$$

Lemma 23.15. *The composite*

$$E = q_*\partial^{-1}\colon \pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X)$$

is the suspension homomorphism (= "Einhängung") taking the homotopy class of $f\colon S^n \to X$ to the homotopy class of

$$\Sigma f\colon S^{n+1} \cong \Sigma S^n \longrightarrow \Sigma X\,.$$

Proof. Let $Cf\colon CS^n \to CX = I \wedge X$ be given by $Cf(t,x) = (t, f(x))$ for $t \in I$ and $x \in X$. Then ∂ maps the homotopy class of $Cf\colon (CS^n, S^n) \to (CX, X)$ to $[f]$, and q_* maps it to $[\Sigma f]$. □

Unlike the case of homology, the collapse map q_* and suspension E are not generally isomorphisms, even for well-behaved spaces X. However, Hans Freudenthal proved that they are isomorphisms in a "stable range" that is about twice as large as the connectivity of X. If X is $(k-1)$-connected, then q_* and E are isomorphisms for $n < 2k-1$, and surjections for $n = 2k-1$. This is the Freudenthal stability theorem, which we shall prove later. As an example, this proves that

$$E\colon \pi_n(S^k) \longrightarrow \pi_{n+1}(S^{k+1})$$

is an isomorphism for $n < 2k-1$ and a surjection for $n = 2k-1$. This, in turn, implies that for each $i \in \mathbb{Z}$ the colimit system

$$\dots \xrightarrow{E} \pi_{i+k}(S^k) \xrightarrow{E} \pi_{i+k+1}(S^{k+1}) \xrightarrow{E} \dots$$

consists of isomorphisms for $k \geq i+2$. The stable value,

$$\pi_i^S = \pi_i(S) = \operatorname*{colim}_k \pi_{i+k}(S^k)$$

is called the i-th stable stem, or the i-th stable homotopy group of spheres. We have proved that $\pi_i(S) = 0$ for $i < 0$, and the Brouwer degree gave us a surjective homomorphism $\pi_0(S) \to \mathbb{Z}$, which we will later see is an isomorphism. Serre's theorem about the finiteness of $\pi_n(S^k)$ (except for $n = k$ and $n = 2k-1$, with k even in the latter case) shows that $\pi_i(S)$ is a finite abelian group for each $i > 0$.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\pi_i(S)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$

Serre's class theory, or the later localization theories of Sullivan, Bousfield and Kan, allow us to focus on the p-components (= p-Sylow subgroups) of these groups, for one prime p at a time.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\pi_i(S)_{(2)}$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$(\mathbb{Z}/2)^2$

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\pi_i(S)_{(3)}$	$\mathbb{Z}_{(3)}$	0	0	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/9$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	0	0

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\pi_i(S)_{(5)}$	$\mathbb{Z}_{(5)}$	0	0	0	0	0	0	$\mathbb{Z}/5$	0	0	0	0	0	0	0	$\mathbb{Z}/5$	0	0	0	0	0

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\pi_i(S)_{(7)}$	$\mathbb{Z}_{(7)}$	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}/7$	0	0	0	0	0	0	0	0	0

23.6. **The Hurewicz–Strøm homotopy category.**

Definition 23.16. A map $f\colon X \to Y$ is a homotopy equivalence if there exists a map $g\colon Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$. We then say that g is a homotopy inverse to f, and any two homotopy inverses are homotopic.

Definition 23.17. A map $i\colon A \to X$ is a Hurewicz cofibration (or h-cofibration) if it has the homotopy extension property (HEP) with respect to any space T, i.e., if given any commutative diagram with solid arrows, the dashed arrow can be filled in.

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & I_+ \wedge A \arrow[d, "1\wedge i"] \arrow[ddr, bend left] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right] & I_+ \wedge X \arrow[dr, dashed] & \\
& & T
\end{tikzcd}$$

Using the mapping cylinder $Mi = X \cup_A I_+ \wedge A$, the diagram can be redrawn as follows.

$$\begin{tikzcd} Mi \arrow[r] \arrow[d] & T \\ I_+ \wedge X \arrow[ur, dashed] & \end{tikzcd}$$

Hence $i\colon A \to X$ is a Hurewicz cofibration if and only if $Mi \to I_+ \wedge X$ admits a retraction. By the exponential isomorphism, it is equivalent to ask for a lift in the following diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & T^I \arrow[d, "p_0"] \\ X \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$

Definition 23.18. A map $p\colon E \to B$ is a Hurewicz fibration (or h-fibration) if it has the homotopy lifting property (HLP) with respect to any space T, i.e., if given any commutative diagram with solid arrows, the dashed arrow can be filled in.

$$\begin{tikzcd} T \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ I_+ \wedge T \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

By the exponential isomorphism, this is equivalent to filling in the dashed arrow in the following diagram.

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & E^I \arrow[r, "p_0"'] \arrow[d, "p^I"] & E \arrow[d, "p"] \\ & B^I \arrow[r, "p_0"'] & B \end{tikzcd}$$

Using the path fibration $Ep = E \times_B B^I$, the diagram can be redrawn as follows.

$$\begin{tikzcd} & E^I \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & Ep \end{tikzcd}$$

Hence $p\colon E \to B$ is a Hurewicz fibration if and only if $E^I \to Ep$ admits a section.

24. November 16th lecture

24.1. **Model categories.** We now seem to have the choice of working with homotopy equivalences or weak homotopy equivalences, CW pairs or (Hurewicz) cofibrations, fiber bundles, (Hurewicz) fibrations or Serre fibrations. Quillen's theory of (closed) model categories clarifies the relationship between these notions.

Definition 24.1. A model category is a category $\mathscr{C}$, with all limits and colimits, together with three subcategories whose morphisms are called weak equivalences, cofibrations and fibrations, respectively. A cofibration that is also a weak equivalence is called a trivial cofibration, and a fibration that is also a weak equivalence is called a trivial fibration. These are assumed to satisfy the following axioms.

1. (2 out of 3) If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms in $\mathscr{C}$, and two of f, g and gf are weak equivalences, then so is the third.
2. (Retracts) If $f\colon X \to Y$ is a retract of $f'\colon X' \to Y'$ and f' is a weak equivalence, cofibration or fibration, then f is a weak equivalence, cofibration or fibration, respectively.

(3) (Lifting) Given a commutative square of solid arrows

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed] & B, \end{tikzcd}$$

the dashed lift exists if (a) $i\colon A \to X$ is a trivial cofibration and $p\colon E \to B$ is a fibration, or (b) $i\colon A \to X$ is a cofibration and $p\colon E \to B$ is a trivial fibration.

(4) (Factorization) Any morphism $f\colon X \to Y$ can be factored (a) as $f = pj$ where j is a trivial cofibration and p is a fibration, and (b) as $f = qi$ where i is a cofibration and q is a trivial fibration.

In axiom (2), f is a retract of f' if there is commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

where the horizontal composites $X \to X' \to X$ and $Y \to Y' \to Y$ are both the identities. In particular, X and Y are then retracts of X' and Y', respectively.

See Dwyer–Spalinski (1995) for an introduction to model categories, and Hovey (1999) and Hirschhorn (2003) for more detailed studies. Some authors refer to acyclic (co-)fibrations in place of trivial (co-)fibrations.

Theorem 24.2 (Arne Strøm (1972))**.** *The category of topological spaces, with the subcategories of homotopy equivalences, Hurewicz cofibrations and Hurewicz fibrations, is a model category.*

24.2. **The homotopy category.** A model category provides a model for the homotopy category of $\mathscr{C}$, i.e., the localization $\mathscr{C}[\mathscr{W}^{-1}]$ of $\mathscr{C}$ where each weak equivalence has been inverted to become an isomorphism. See Section 1.2 of Hovey's book "Model Categories". The original reference is Quillen's "Homotopical Algebra".

Let 0 and 1 be the initial and terminal objects in $\mathscr{C}$. The factorization axiom applied to the initial map $0 \to X$ gives a factorization

$$0 \longrightarrow X^c \longrightarrow X$$

where $0 \to X^c$ is a cofibration (we then say that X^c is cofibrant) and $X^c \to X$ is (a fibration and) a weak equivalence. We then call X^c a cofibrant replacement of X. The factorization axiom applied to the terminal map $Y \to 1$ gives a factorization

$$Y \longrightarrow Y^f \longrightarrow 1$$

where $Y^f \to 1$ is a fibration (we then say that Y^f is fibrant) and $Y \to Y^f$ is (a cofibration and) a weak equivalence. We then call Y^f a fibrant replacement of Y. A cylinder object IX for X is a factorization

$$X \sqcup X \longrightarrow IX \longrightarrow X$$

of the fold map $\nabla\colon X \sqcup X \to X$, where $X \sqcup X \to IX$ is a cofibration and $IX \to X$ is (a fibration and) a weak equivalence. Two maps $f, g\colon X \to Y$ are (left) homotopic if there exists a map $IX \to Y$ extending $f \sqcup g\colon X \sqcup X \to Y$. We let

$$\pi(X, Y) = \{X \to Y\}/\!\sim$$

be the set of equivalence classes of morphisms $X \to Y$ with respect to (left) homotopy. If X is cofibrant and Y is fibrant, this defines an equivalence relation that is compatible with composition. We let

$$\mathrm{Ho}(\mathscr{C})(X, Y) = \pi(X^c, Y^f) = \{X^c \to Y^f\}/\!\sim$$

be the set of morphisms from X to Y in the homotopy category $\mathrm{Ho}(\mathscr{C})$. The objects of $\mathrm{Ho}(\mathscr{C})$ are the same as the objects of $\mathscr{C}$. Composition of maps in $\mathscr{C}$ induces a pairing

$$\circ\colon \mathrm{Ho}(\mathscr{C})(Y, Z) \times \mathrm{Ho}(\mathscr{C})(X, Y) \longrightarrow \mathrm{Ho}(\mathscr{C})(X, Z)$$

that makes $\mathrm{Ho}(\mathscr{C})$ a category. There is a canonical functor

$$\mathscr{C} \longrightarrow \mathrm{Ho}(\mathscr{C})$$

that exhibits $\mathrm{Ho}(\mathscr{C})$ as the localization of $\mathscr{C}$ away from the weak equivalences. This is initial among the functors $\mathscr{C} \to \mathscr{D}$ that send the weak equivalences to isomorphisms. Moreover, the maps in $\mathscr{C}$ that become isomorphisms in $\mathrm{Ho}(\mathscr{C})$ are precisely the weak equivalences.

Example 24.3. In the category $\mathscr{T}$ of based spaces, the one-point space $*$ is both initial and terminal. The inclusion $* \to X$ is a Hurewicz cofibration and the projection $X \to *$ is a Hurewicz fibration, so each space is both cofibrant and fibrant. The reduced cylinder $I \ltimes X$ is a cylinder object for X, so $\mathrm{Ho}_h(\mathscr{T})(X, Y) = \pi(X, Y)$ is the usual set of homotopy classes of maps $X \to Y$. Hence $\mathrm{Ho}_h(\mathscr{T})$ is the usual (Hurewicz) homotopy category.

24.3. **The Serre–Quillen homotopy category.** The subcategory of weak homotopy equivalences is also part of a model structure.

Definition 24.4. A map $i\colon E \to B$ is a Serre fibration (or q-fibration) if it has the homotopy lifting property with respect to D^n for each $n \geq 0$, i.e., given any commutative diagram with solid arrows, the dashed arrow can be filled in.

$$\begin{tikzcd}
D^n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
I_+ \wedge D^n \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

Each Hurewicz fibration is a Serre fibration. Curiously (Steinberger–West, Cauty), each Serre fibration between CW complexes is a Hurewicz fibration. We mentioned earlier that each fiber bundle is a Serre fibration.

A relative cell complex (X, A) is a (possibly transfinite) composition

$$A \to \cdots \to X_\beta \to X_{\beta+1} \to \cdots \to X$$

where for each β there is a pushout square

$$\begin{tikzcd}
\partial D^n \arrow[r] \arrow[d] & D^n \arrow[d] \\
X_\beta \arrow[r] & X_{\beta+1} \,.
\end{tikzcd}$$

In particular, each relative CW complex is a relative cell complex. Even more particularly, each pair (X, A) with X a CW complex and A a subcomplex is a relative cell complex.

Definition 24.5. A map $i\colon A \to X$ is a Serre cofibration (or q-cofibration) if it is a retract of a map $i'\colon A' \to X'$, where (X', A') is a relative cell complex.

Theorem 24.6. *Quillen (1967) The category of topological spaces, with the subcategories of weak homotopy equivalences, Serre cofibrations and Serre fibrations, is a model category.*

The tricky thing to prove is the lifting property when $i\colon A \to X$ is a cell complex and $p\colon E \to B$ is a Serre fibration and a weak homotopy equivalence.

The direct proof of the long exact sequence in homotopy for a Hurewicz fibration only uses the homotopy lifting property with respect to CW complexes, which holds for all Serre fibrations.

Theorem 24.7. *For any Serre fibration $p\colon E \to B$, with fiber $F = p^{-1}(b_0)$, there is a long exact sequence of homotopy groups and sets*

$$\dots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \dots$$

ending with

$$\dots \longrightarrow \pi_1(B) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B)\,.$$

Here $i\colon F \to E$ denotes the fiber inclusion. Two elements in $\pi_0(F)$ have the same image in $\pi_0(E)$ if and only if they are in the same orbit for the action of $\pi_1(B)$.

24.4. **The weak homotopy category.** In the Serre–Quillen model structure, retracts of cell complexes are cofibrant and all spaces are fibrant. In particular, CW complexes are cofibrant, so any CW approximation

$$* \to W = X^c \xrightarrow{\sim} X$$

is a cofibrant replacement of X, while no fibrant replacement is necessary, meaning that we can take $Y = Y^f$. The reduced cylinder $I \ltimes X^c$ is a cylinder object for X^c, so the morphisms in the weak homotopy category are the homotopy classes

$$\mathrm{Ho}_q(X, Y) = [X, Y] = \pi(X^c, Y)$$

of maps from the CW approximation X^c to Y. These are the morphisms from X to Y in the weak homotopy category $\mathrm{Ho}_q(\mathscr{T})$. Composition of maps induces a pairing

$$\circ \colon [Y, Z] \otimes [X, Y] \longrightarrow [X, Z]$$

that makes $\mathrm{Ho}_q(\mathscr{T})$ a category. This is initial among functors $\mathscr{T} \to \mathscr{D}$ that send the weak homotopy equivalences to isomorphisms. The maps in $\mathscr{T}$ that become isomorphisms in $\mathrm{Ho}_q(\mathscr{T})$ are precisely the weak homotopy equivalences.

$$\begin{tikzcd} & \mathscr{T} \arrow[dl] \arrow[dr] & \\ \mathrm{Ho}_h(\mathscr{T}) \arrow[rr] \arrow[dr] & & \mathrm{Ho}_q(\mathscr{T}) \arrow[dl, "\pi_n"] \\ & \mathrm{Ab} & \end{tikzcd}$$

The homotopy groups of spaces are thus most informative when viewed as a functor from the homotopy category formed with respect to the Serre–Quillen model structure.

24.5. **Chain complexes.** Let R be a ring, and let $\mathrm{Ch}(R)$ be the category of (unbounded = integer graded) chain complexes (M_*, ∂) of R-modules. The morphisms in $\mathrm{Ch}(R)$ are the degree-preserving chain maps.

The homotopy category of R-module chain complexes, denoted $K(R)$, has the same objects as $\mathrm{Ch}(R)$, but the morphisms from M_* to N_* are the chain homotopy classes of chain maps

$$K(R)(M_*, N_*) = \{\text{chain maps } f \colon M_* \to N_*\}/\simeq .$$

The isomorphisms in $K(R)$ are the chain homotopy equivalences. The canonical functor

$$\mathrm{Ch}(R) \longrightarrow K(R)$$

inverts precisely the chain homotopy equivalences. There is a Hurewicz model structure on $\mathrm{Ch}(R)$: following Klaus Heiner Kamps (1978), a chain map $f \colon M_* \to N_*$ is a Hurewicz cofibration if $f_k \colon M_k \to N_k$ is split injective for each k, and it is a Hurewicz fibration if $f_k \colon M_k \to N_k$ is split surjective for each k.

Theorem 24.8. *(Marek Golasinski og Grzegorz Gromadzki (1982)) The category* $\mathrm{Ch}(R)$ *of chain complexes, with the subcategories of chain homotopy equivalences, Hurewicz cofibrations and Hurewicz fibrations, is a model category.*

Each object is both cofibrant and fibrant, so the homotopy category of this model structure is the usual homotopy category $K(R)$.

24.6. **The derived category.** A chain map $f \colon M_* \to N_*$ is a quasi-isomorphism if it induces an isomorphism in homology. For the purpose of homological algebra, where a chain complex is principally a carrier of its homology groups, it is more natural to consider two complexes to be equivalent if they are quasi-isomorphic (or linked by a chain of quasi-isomorphisms) than to ask that they be chain homotopy equivalent. The localization

$$\mathrm{Ch}(R) \longrightarrow D(R)$$

of $\mathrm{Ch}(R)$ that inverts all quasi-isomorphisms is called the derived category of R. This can be realized as the canonical functor to the homotopy category of a model structure on $\mathrm{Ch}(R)$, with the quasi-isomorphisms as the weak equivalences. Mark Hovey specified one such model structure in Section 2.3 of his book "Model Categories" (1999).

The projective cofibrant objects are defined in terms of cell objects. This is not completely explicit, but each cofibrant chain complex P_* is projective in each degree. Conversely, each bounded below and degreewise projective chain complex is cofibrant. A chain map $f_k\colon M_k \to N_k$ is defined to be a projective cofibration if it is a Hurewicz cofibration (split injective in each degree) with cofibrant cokernel

$$0 \to M_* \xrightarrow{f} N_* \longrightarrow P_* \to 0\,.$$

A chain map $f\colon M_* \to N_*$ is a projective fibration if $f_k\colon M_k \to N_k$ is surjective for each k.

Theorem 24.9 (Hovey–Palmieri–Strickland (1997), Schwede-Shipley (1998), Mark Hovey (1999))**.** *The category* $\mathrm{Ch}(R)$ *of chain complexes of R-modules, with the subcategories of quasi-isomorphisms, projective cofibration and projective fibrations, is a model category.*

Each chain complex is projectively fibrant. The morphisms in the homotopy category $D(R) = \mathrm{Ho}(\mathrm{Ch}(R))$ from M_* to N_* are thus the chain homotopy classes of maps

$$D(R)(M_*, N_*) = \{P_* \longrightarrow N_*\}/\simeq$$

where $0 \to P_* \xrightarrow{\sim} M_*$ is a cofibrant complex that is quasi-isomorphic to M_*. If $M_* = M$ is concentrated in degree 0, $P_* \to M$ is a projective resolution, and $N_* = N[s]$ is concentrated in degree s, then

$$D(R)(M_*, N_*) \cong \mathrm{Ext}^s_R(M, N)\,.$$

25. November 18th lecture

25.1. **Connectivity of spaces and maps.** Let $n \geq 0$. Recall that a space X is n-connected if $\pi_k(X) = 0$ for all $0 \leq k \leq n$. A map $f\colon X \to Y$ is n-connected if $f_*\colon \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $0 \leq k < n$ and a surjection for $k = n$. For $n \geq 1$, this is equivalent to asking that the homotopy fiber Ff is $(n-1)$-connected. We say that a pair (X, A) is n-connected if the inclusion $i\colon A \to X$ is n-connected. This is equivalent to asking that $\pi_k(X, A) = 0$ for all $0 \leq k \leq n$, where we define $\pi_0(X, A)$ to be the cofiber of $\pi_0(A) \to \pi_0(X)$.

25.2. **The homotopy excision theorem.** Consider a CW complex X, with subcomplexes A and B such that $A \cup B = X$. Let $C = A \cap B$.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

Note that $A/C \cong X/B$. By the excision theorem in homology, we have the following square of isomorphisms.

$$\begin{tikzcd} H_*(A, C) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_*(X, B) \arrow[d, "\cong"] \\ \tilde{H}_*(A/C) \arrow[r, "\cong"] & \tilde{H}_*(X/B) \end{tikzcd}$$

By functoriality of relative homotopy groups, there is also a homomorphism

$$\pi_k(A, C) \longrightarrow \pi_k(X, B)$$

(for any choice of base point $x_0 \in C$), but this is not in general an isomorphism.

Example 25.1. Let $X = D^2 \cup_{S^1} D^2 \cong S^2$. Then $\pi_3(D^2, S^1) \cong \pi_2(S^1) = 0$ while $\pi_3(S^2, D^2) \cong \pi_3(S^2)$ maps onto $\mathbb{Z}$ by the Hopf invariant. Hence $\pi_3(D^2, S^1) \to \pi_3(S^2, D^2)$ is not an isomorphism.

However, there is a partial excision result for relative homotopy groups, valid in a "stable range" up to the degree where the Hopf invariant is defined. (The so-called EHP-sequence makes this more precise.)

Theorem 25.2 (Homotopy excision, Theorem 4.23 in Hatcher)**.** *Let $X = A \cup B$ be a CW complex given as the union of two subcomplexes, with connected intersection $C = A \cap B$. If (A, C) is m-connected and (B, C) is n-connected, where $m, n \geq 0$, then the map*

$$\pi_k(A, C) \longrightarrow \pi_k(X, B)$$

is an isomorphism for $0 \leq k < m + n$ and is surjective for $k = m + n$.

Sketch proof. See Hatcher's pages 361–363 for a full proof. The argument is due to Michael Boardman, according to the historical notes in Switzer's book "Algebraic Topology – Homology and Homotopy", where this result appears as Lemma 6.20 and Theorem 6.21.

We concentrate on the key inductive step, where $A = C \cup I^{m+1}$ and $B = C \cup I^{n+1}$ are obtained by attaching single $(m+1)$- and $(n+1)$-cells, respectively. Hence

$$X = C \cup I^{m+1} \cup I^{n+1} .$$

We discuss surjectivity of

$$\pi_k(A, C) \longrightarrow \pi_k(X, B)$$

for $k \leq m + n$. Hence we start with a map

$$f : (I \times I^{k-1}, \partial I^k, J^{k-1}) \longrightarrow (X, B, x_0)$$

and need to argue that we can deform it to a map into (A, C, x_0). If $p \in A - C \cong \operatorname{int} I^{m+1}$ and $q \in B - C \cong \operatorname{int} I^{n+1}$ are interior points in the attached cells, then $A \subset X - \{q\}$, $B \subset X - \{p\}$ and $C \subset X - \{p, q\}$ are deformation retracts, so we have the following commutative diagram with vertical isomorphisms.

$$\begin{tikzcd}
\pi_k(A, C) \arrow[r] \arrow[d, "\cong"'] & \pi_k(X, B) \arrow[d, "\cong"] \\
\pi_k(X - \{q\}, X - \{p, q\}) \arrow[r] & \pi_k(X, X - \{p\})
\end{tikzcd}$$

Hence it will suffice to show that

$$f : (I \times I^{k-1}, \partial I^k, J^{k-1}) \longrightarrow (X, X - \{p\}, x_0)$$

can be deformed to a map into $(X - \{q\}, X - \{p, q\})$. We thus wish to pull f away from q, while keeping its restriction to $1 \times I^{k-1} \cong I^{k-1}$ away from p.

Deforming f and choosing p and q is an application of the piecewise linear approximation Lemma 4.10 in Hatcher. Considering f as a map to $X = B \cup I^{m+1}$ we can deform f to f_1 relative to $f^{-1}(B)$ so that there is a linear simplex $\Delta^{m+1} \subset I^{m+1}$ with

$$f_1^{-1}(\Delta^{m+1}) = P_1 \cup \cdots \cup P_M$$

a finite union of convex polyhedra, such that each $f_1|P_i \to \Delta^{m+1}$ is the restriction of a linear surjection $\mathbb{R}^k \to \mathbb{R}^{m+1}$. Next we view (the deformed map) f_1 as a map to $X = A \cup I^{n+1}$, and deform it to f_2 relative to $f_1^{-1}(A)$ so that there is a linear simplex $\Delta^{n+1} \subset I^{n+1}$ with

$$f_2^{-1}(\Delta^{n+1}) = Q_1 \cup \cdots \cup Q_N$$

a finite union of convex polyhedra, such that each $f_2|Q_i \to \Delta^{n+1}$ is the restriction of a linear surjection $\mathbb{R}^k \to \mathbb{R}^{n+1}$. This did not change f_1 on $f_2^{-1}(\Delta^{m+1}) \subset f_2^{-1}(A)$, so f_2 is now piecewise-linear on the preimages of both Δ^{m+1} and Δ^{n+1}.

For any $p \in \Delta^{m+1} \subset A - C$ the preimage $f_2^{-1}(p)$ is a subcomplex of I^k of dimension $\leq k - m - 1$, which is $< n$ since $k \leq m + n$.

Likewise, for any $q \in \Delta^{n+1} \subset B - C$ the preimage $f_2^{-1}(q)$ is a subcomplex of dimension $\leq k - n - 1$. These are disjoint, but we need to deform $f_2^{-1}(q)$ horizontally in $I^k = I \times I^{k-1}$, to the face $1 \times I^{k-1} \cong I^{k-1}$, without meeting $f_2^{-1}(p)$. Hence let $\pi : I^k \to I^{k-1}$ be projection to the last $(k-1)$ coordinates, so that

$$T = \pi^{-1}\pi f^{-1}(q) \subset I^k$$

is a subcomplex of dimension $\leq k - n$. We claim that for a generic $p \in \Delta^{m+1}$ we have $f^{-1}(p) \cap T = \emptyset$. This is equivalent to $p \notin f(T)$. Since f_2 is piecewise-linear on $f_2^{-1}(\Delta^{m+1})$, and piecewise-linear maps cannot increase dimension, the part $\Delta^{m+1} \cap f_2(T) \subset I^{m+1}$ of its image is a subcomplex of dimension $\leq k - n$. By the assumption that $k \le m+n$, this dimension is $\le m$. Hence almost every point $p \in \Delta^{m+1}$ does not meet $f_2(T)$, as claimed.

Since $\pi f_2^{-1}(p)$ and $\pi f_2^{-1}(q) = \pi(T)$ are closed and disjoint in I^{k-1} we can separate them by open neighborhoods, and find a function $\phi\colon I^{k-1} \to I$ with $f_2^{-1}(p)$ to the left and $f_2^{-1}(q)$ to the right of the (sideways) graph

$$\Gamma = \{(\phi(s_1,\dots,s_{k-1}), s_1,\dots,s_{k-1}) \subset I \times I^{k-1}\,.$$

We then precompose f_2 with the maps

$$(s_0, s_1, \dots, s_{k-1}) \longmapsto ((1-t+t\phi(s_1,\dots,s_{k-1}))s_0, s_1,\dots,s_{k-1})$$

for $0 \le t \le 1$. For $t=0$ this is the identity map, and for $t=1$ it is given by

$$(s_0, s_1, \dots, s_{k-1}) \longmapsto (\phi(s_1,\dots,s_{k-1})s_0, s_1,\dots,s_{k-1})\,.$$

This way we get a homotopy from f_2 to a map f_3 with image on and to the left of Γ. The homotopy is through maps taking ∂I^k to $X - \{p\}$, since $f_2^{-1}(p)$ lies to the left of Γ. Furthermore, f_3 takes values in $X - \{q\}$, since $f_2^{-1}(q)$ lies to the right of Γ. Thus $[f] = [f_3]$ is in the image from $\pi_k(A,C) \cong \pi_k(X - \{q\}, X - \{p,q\})$, proving surjectivity.

The proof of injectivity for $k < m+n$ is similar. Suppose given maps

$$f, g\colon (I^k, \partial I^k, J^{k-1}) \longrightarrow (A, C, x_0)$$

and a homotopy

$$F\colon (I \times I^k, I \times \partial I^k, I \times J^{k-1}) \longrightarrow (X, B, x_0)$$

between if and ig, where $i\colon (A,C) \to (X,B)$ denotes the inclusion. By deformation to a generic piecewise-linear situation, using the hypothesis that $k+1 \le m+n$, we can deform F relative to f and g to avoid an interior point $q \in B - C$, and thus give a homotopy F_3 between f and g. □

25.3. **Freudenthal's suspension theorem.** We recall how the suspension isomorphism in homology follows from the excision isomorphism. For any CW complex X, we can decompose its suspension as a union of two cones:

$$\Sigma X = C_+X \cup_X C_-X\,.$$

The suspension isomorphism is the composite

$$\tilde H_*(X) \xleftarrow{\cong} H_{*+1}(C_+X, X) \xrightarrow{\cong} H_{*+1}(\Sigma X, C_-X) \xleftarrow{\cong} \tilde H_{*+1}(\Sigma X)$$

where the left and right isomorphisms follow from the long exact sequence in homology and the contractibility of C_+X and C_-X, respectively, while the middle arrow is the excision isomorphism.

For homotopy groups, the suspension map

$$E\colon \pi_k(X) \longrightarrow \pi_{k+1}(\Sigma X)$$

can be factored as the composite

$$\pi_k(X) \xleftarrow{\cong} \pi_{k+1}(C_+X, X) \longrightarrow \pi_{k+1}(\Sigma X, C_-X) \xleftarrow{\cong} \pi_{k+1}(\Sigma X)$$

taking the homotopy class of $f\colon S^k \to X$ via the class of $(Cf, f)\colon (CS^k, S^k) \to (CX, X)$ to that of $Cf/f\colon CS^k/S^k \to CX/X$ and $\Sigma f\colon \Sigma S^k \to \Sigma X$.

The example $X = S^1$, $k=2$ shows that E is not generally an isomorphism. However, in the range of the homotopy excision theorem, we get the following "Suspension Theorem" of Hans Freudenthal.

Theorem 25.3 (Freudenthal (1938))**.** *Let X be an $(n-1)$-connected CW complex. Then*

$$E\colon \pi_k(X) \longrightarrow \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k \le 2n-2$ and is surjective for $k = 2n-1$.

Proof. We may assume $n \ge 1$, so X is connected. The homotopy excision theorem for $\Sigma X = C_+X \cup C_-X$ with $C_+X \cap C_-X = X$ shows that

$$\pi_{k+1}(C_+X, X) \longrightarrow \pi_{k+1}(\Sigma X, C_-X)$$

is an isomorphism for $k+1 < n+n$ and is surjective for $k+1 = n+n$. □

Corollary 25.4. *The suspension homomorphism*

$$E\colon \pi_k(S^n) \longrightarrow \pi_{k+1}(S^{n+1})$$

is an isomorphism for $k \le 2n-2$ and is surjective for $k = 2n-1$.

Note that this corollary only depends on the special case of the homotopy excision theorem that we discussed, with $X = S^{n+1}$ the union of $A = D^{n+1}_+ \cong S^n \cup I^{n+1}$ and $B = D^{n+1}_- \cong S^n \cup I^{n+1}$ meeting along $C = S^n$.

Theorem 25.5. *The Hurewicz homomorphism*

$$h_n \colon \pi_n(S^n) \xrightarrow{\cong} H_n(S^n) \cong \mathbb{Z}$$

is an isomorphism for each $n \ge 1$.

Proof. We have commutative squares

$$\begin{tikzcd}
\pi_n(S^n) \arrow[r, "E"] \arrow[d, "h_n"'] & \pi_{n+1}(S^{n+1}) \arrow[d, "h_{n+1}"] \\
H_n(S^n) \arrow[r, "\cong"] & H_{n+1}(S^{n+1})
\end{tikzcd}$$

for all $n \ge 1$. When $n = 1$ we know that h_1 is an isomorphism, and Freudenthal's suspension theorem tells us that $E\colon \pi_1(S^1) \to \pi_2(S^2)$ is surjective. By the suspension isomorphism in homology it follows that E is also injective, so all four maps in the diagram are isomorphisms.

For $n \ge 2$ the suspension theorem tells us that $E\colon \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism. We know by induction that h_n is an isomorphism, so again the suspension isomorphism in homology tells us that all four maps in the diagram are isomorphisms. □

This proves that the stable 0-stem $\pi_0(S) = \operatorname{colim}_n \pi_n(S^n)$ is isomorphic to $\mathbb{Z}$, via the Brouwer degree of a map $S^n \to S^n$.

26. November 23th lecture

26.1. The complex Hopf fibration.

Corollary 26.1. *The Hopf invariant*

$$H\colon \pi_3(S^2) \xrightarrow{\cong} \mathbb{Z}$$

is an isomorphism.

Proof. The (complex Hopf) fiber bundle

$$S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2$$

induces a long exact sequence in homotopy, with

$$\pi_3(S^1) \longrightarrow \pi_3(S^3) \xrightarrow{\eta} \pi_3(S^2) \longrightarrow \pi_2(S^1)\,.$$

Since $\pi_k(S^1) = 0$ for $k \ge 2$, it follows that η induces an isomorphism from $\pi_3(S^3) \cong \mathbb{Z}$ generated by the identity map to $\pi_3(S^2)$, which therefore must be isomorphic to $\mathbb{Z}$ and generated by the homotopy class of η. Since $H(\eta) = 1$, it follows that the Hopf invariant is an isomorphism in this case. □

Corollary 26.2. *The suspension homomorphism*

$$E\colon \pi_{n+1}(S^n) \to \pi_{n+2}(S^{n+1})$$

is a surjection (from $\pi_3(S^2) \cong \mathbb{Z}$) for $n = 2$ and an isomorphism for $n \ge 3$.

Corollary 26.3. *The suspension homomorphism*

$$E\colon \pi_{n+2}(S^n) \to \pi_{n+3}(S^{n+1})$$

is a surjection (from $\pi_5(S^3)$) for $n = 3$ and an isomorphism for $n \ge 4$.

26.2. **The J-homomorphism.** Let $SO(n)$ denote the rotation group of orientation-preserving isometries of $\mathbb{R}^n$. In particular, $SO(2)$ is the circle group and $SO(3) \cong \mathbb{R}P^3$. The rule that to an isometry $A\colon \mathbb{R}^n \to \mathbb{R}^n$ associates the induced map of one-point compactifications $S^n \to S^n$, gives a map

$$J_n\colon SO(n) \longrightarrow \Omega^n S^n = \mathrm{Map}(S^n, S^n)\,.$$

The induced homomorphism

$$\pi_k(J_n)\colon \pi_k(SO(n)) \longrightarrow \pi_k(\Omega^n S^n) \cong \pi_{k+n}(S^n)$$

was introduced by George Whitehead (1942), as a generalization of the Hopf construction. Around 1958 it was usually called the J-homomorphism, e.g., by John Milnor. Note that we take the identity map of S^n as the base point in $\Omega^n S^n$, not the constant map, but this does not affect the higher homotopy groups, since the different path components of $\Omega^n S^n$ are homotopy equivalent.

For $n = 2$ and $k = 1$, the homomorphism

$$\pi_1(J_2)\colon \pi_1(SO(2)) \longrightarrow \pi_1(\Omega^2 S^2) \cong \pi_3(S^2)$$

is an isomorphism, taking the simple loop in $SO(2)$ to the class of Hopf's fiber bundle η.

There is an inclusion $SO(n) \to SO(n+1)$ that takes a matrix A to the block matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Similarly, there is a stabilization map $E\colon \Omega^n S^n \to \Omega^{n+1} S^{n+1}$ taking $f\colon S^n \to S^n$ to its suspension $\Sigma f\colon S^{n+1} \to S^{n+1}$. These are compatible under the J-homomorphism, and the latter induces the Freudenthal suspension. In other words, the following diagram commutes.

$$\begin{tikzcd}
\pi_k(SO(n)) \arrow[r, "\pi_k(J_n)"] \arrow[d] & \pi_{k+n}(S^n) \arrow[d, "E"] \\
\pi_k(SO(n+1)) \arrow[r, "\pi_k(J_{n+1})"] & \pi_{k+n+1}(S^{n+1})
\end{tikzcd}$$

We consider the following particular case.

$$\begin{tikzcd}
\pi_1(SO(2)) \arrow[r, "\pi_1(J_2)"] \arrow[d] & \pi_3(S^2) \arrow[d, "E"] \\
\pi_1(SO(3)) \arrow[r, "\pi_1(J_3)"] & \pi_4(S^3)
\end{tikzcd}$$

Since $SO(3) \cong \mathbb{R}P^3$ has S^3 as a double covering space, $\pi_1(SO(3)) = \mathbb{Z}/2$, which proves that $2 \cdot E\eta = 0$ in $\pi_4(S^3)$.

We need another invariant to detect that $E\eta$ is nontrivial. One such is the Hopf–Steenrod invariant, which extends the Hopf invariant to a stable invariant, using Steenrod's power operations in place of the cohomology cup product. Granting this, we can deduce that $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ is generated by $E^{n-2}\eta$ for all $n \geq 3$, and that $\pi_4(S^2) \cong \mathbb{Z}/2$ is generated by $\eta \circ E\eta$. Hence $\pi_1(S) = \operatorname{colim}_n \pi_{n+1}(S^n) \cong \mathbb{Z}/2$, with the stable class of the Hopf fibration η as the only nonzero element.

$\pi_k(S^n)$	$n=1$	2	3	4	5	6
$k=1$	$\mathbb{Z}$	0	0	0	0	0
2	0	$\mathbb{Z}$	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0
4	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0	0
5	0	$(\mathbb{Z}/2)$	$(\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
6	0	$(\mathbb{Z}/12)$	$(\mathbb{Z}/12)$	$(\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}$

26.3. **The Hurewicz theorem.**

Definition 26.4. Let G be an abelian group and $n \geq 2$. A Moore space of type (G, n) is a simply-connected CW complex $M(G, n)$ with homology groups

$$\tilde{H}_k(M(G,n)) \cong \begin{cases} G & \text{for } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Any two Moore spaces of type (G, n) are homotopy equivalent. See Hatcher's Example 4.34. It follows that there is a homotopy equivalence

$$\Sigma M(G,n) \simeq M(G, n+1)$$

inducing the identity isomorphism

$$G = \tilde{H}_n(M(G,n)) \cong \tilde{H}_{n+1}(\Sigma M(G,n)) \cong \tilde{H}_{n+1}(M(G,n+1)) = G\,.$$

Theorem 26.5 (Hurewicz)**.** *Let X be an $(n-1)$-connected CW complex, with $n \geq 2$. Then the Hurewicz homomorphism*

$$h_n \colon \pi_n(X) \longrightarrow H_n(X)$$

is an isomorphism.

Proof. Let $G = \pi_n(X)$. We must show that $H_n(X) \cong G$, and that the Hurewicz homomorphism induces this isomorphism. Choose a resolution

$$0 \to \bigoplus_\beta \mathbb{Z} \xrightarrow{\partial} \bigoplus_\alpha \mathbb{Z} \xrightarrow{\epsilon} G \to 0$$

of G by free abelian groups. Let $W^n = \bigvee_\alpha S^n$. The Hurewicz homomorphism

$$h_n \colon \pi_n(W^n) \xrightarrow{\cong} H_n(W^n) \cong \bigoplus_\alpha \mathbb{Z}$$

is an isomorphism, by a variation of Theorem 25.5. Choose a map

$$f \colon W^n = \bigvee_\alpha S^n \longrightarrow X$$

inducing the surjection ϵ on π_n. Also choose a map

$$\phi \colon \bigvee_\beta S^n \longrightarrow W^n$$

inducing the injection ∂ on π_n. The composite ϕf induces $\epsilon\partial = 0$, hence is null-homotopic. Use ϕ as attaching maps for $(n+1)$-cells in a CW complex

$$V^{n+1} = C\phi = (\bigvee_\alpha S^n) \cup_\phi C(\bigvee_\beta S^n)$$

and use the null-homotopy to extend f to a map $g \colon V^{n+1} \to X$. Here V^{n+1} is a simply-connected CW complex with a single homology group:

$$\tilde{H}_k(V^{n+1}) \cong \begin{cases} G & \text{for } k = n, \\ 0 & \text{else.} \end{cases}$$

Hence $V^{n+1} = M(G, n)$ is a Moore space of type (G, n). By cellular approximation we know that

$$\pi_n(g) \colon \pi_n(V^{n+1}) \xrightarrow{\cong} \pi_n(X) \cong G$$

is an isomorphism.

We claim that $H_n(g) \colon G \cong H_n(V^{n+1}) \to H_n(X)$ is also an isomorphism, so that it suffices to verify that h_n is an isomorphism for the Moore space V^{n+1}. To see this, choose a map

$$\psi \colon \bigvee_\gamma S^{n+1} \longrightarrow X$$

inducing a surjection on π_{n+1}, let

$$W^{n+1} = V^{n+1} \vee \bigvee_\gamma S^{n+1}$$

and let $f = g \vee \psi \colon W^{n+1} \to X$. Then $\pi_k(f) \colon \pi_k(W^{n+1}) \to \pi_k(X)$ is an isomorphism for $k \le n$ and a surjection for $k = n+1$. Note that $H_n(V^{n+1}) \cong H_n(W^{n+1})$. Finally, extend $W^{n+1} \to X$ to a weak homotopy equivalence $W \to X$, by attaching cells of dimension $\ge n+2$. This does not affect the cellular homology in degree n, so $H_n(W^{n+1}) \cong H_n(X)$. By Whitehead's theorem W is homotopy equivalent to X. Hence we have the following commutative diagram with horizontal isomorphisms.

$$\begin{tikzcd}
\pi_n(V^{n+1}) \arrow[r, "\cong"] \arrow[d, "h_n"'] & \pi_n(W^{n+1}) \arrow[r, "\cong"] \arrow[d] & \pi_n(W) \arrow[r, "\cong"] \arrow[d] & \pi_n(X) \arrow[d, "h_n"] \\
H_n(V^{n+1}) \arrow[r, "\cong"] & H_n(W^{n+1}) \arrow[r, "\cong"] & H_n(W) \arrow[r, "\cong"] & H_n(X)
\end{tikzcd}$$

Having reduced to the case of V^{n+1}, we use the following commutative diagram.

$$\begin{tikzcd}
\pi_n(\bigvee_\beta S^n) \arrow[r, "\pi_n(\phi)"] \arrow[d, "h_n"', "\cong"] & \pi_n(W^n) \arrow[r, "\pi_n(f)"] \arrow[d, "h_n"', "\cong"] & \pi_n(V^{n+1}) \arrow[d, "h_n"] \\
H_n(\bigvee_\beta S^n) \arrow[r, "H_n(\phi)"] & H_n(W^n) \arrow[r, "H_n(f)"] & H_n(V^{n+1})
\end{tikzcd}$$

Both $\pi_n(f)$ and $H_n(f)h_n$ are surjections. The image of $\pi_n(\phi)$ is contained in the kernel of $\pi_n(f)$, which is contained in the kernel of $H_n(f)h_n$, which equals the image of $\pi_n(\phi)$. Hence these are all equal, and $h_n \colon \pi_n(V^{n+1}) \to H_n(V^{n+1})$ must be an isomorphism. □

Corollary 26.6. *Let X be a 1-connected CW complex, and let $n \ge 1$. Then X is n-connected if and only if $\tilde{H}_k(X) = 0$ for all $k \le n$.*

Proof. This follows by induction on $n \ge 1$. When $n = 1$, we know that $\tilde{H}_0(X) = H_1(X) = 0$. For $n \ge 2$ we may assume that X is $(n-1)$-connected, and then $\pi_n(X) \cong H_n(X)$, so X is n-connected if and only if $H_n(X) = 0$. □

Theorem 26.7 (Whitehead)**.** *Let $f \colon X \to Y$ be a map of 1-connected CW complexes. Then f is a (weak) homotopy equivalence if and only if $f_* \colon H_*(X) \to H_*(Y)$ is an isomorphism.*

Proof. Replacing Y by the mapping cylinder Mf, we may instead consider a CW pair (Y, X). We suppose that $H_*(Y, X) = 0$, and must show that $\pi_n(Y, X) = 0$ for all $n \ge 1$. This is clear when $n = 1$ from the long exact sequence in homotopy for the pair (Y, X), since $\pi_1(Y) = \pi_0(X) = 0$.

Suppose that (Y, X) is n-connected, for some $n \ge 1$. The pair (CX, X) is 2-connected, so by the homotopy excision theorem for $Y \cup CX$ the homomorphism

$$\pi_k(Y, X) \longrightarrow \pi_k(Y \cup CX, CX) \cong \pi_k(Y \cup CX)$$

is an isomorphism for $k \le n+1$ (and a surjection for $k = n+2$). For $k = 1$ this tells us that $Y \cup CX$ is 1-connected. Since $\tilde{H}_*(Y \cup CX) \cong H_*(Y, X) = 0$, it follows from the corollary to the Hurewicz theorem that $\pi_*(Y \cup CX) = 0$. The isomorphism for $k = n+1$ then tells us that $\pi_{n+1}(Y, X) = 0$, so (Y, X) is $(n+1)$-connected.

By induction, it follows that $\pi_*(Y, X) = 0$ in all degrees. Therefore $X \to Y$ is a weak homotopy equivalence, hence also a homotopy equivalence. The converse is well-known. □

27. November 25th lecture

27.1. **Postnikov towers.** Let X be a connected CW complex. For each $n \ge 0$ we can "kill" the homotopy groups $\pi_k(X)$ for $k > n$ to obtain a space $P^n X$ with

$$\pi_k(P^n X) \cong \begin{cases} \pi_k(X) & \text{for } k \le n, \\ 0 & \text{for } k > n. \end{cases}$$

Moreover, there is a map $i_n\colon X \to P^nX$ inducing the isomorphisms on π_k for $k \le n$. The space P^nX (with the structure map i_n) is called the n-th Postnikov section of X.

To kill $\pi_{n+1}(X)$, choose a map

$$\phi\colon \bigvee_\alpha S^{n+1} \longrightarrow X$$

that induces a surjection on π_{n+1}, and let

$$P^nX_1 = X \cup_\phi C(\bigvee_\alpha S^{n+1})$$

be obtained from X by attaching $(n+2)$-cells. Then $X \to P^nX_1$ induces an isomorphism on π_k for $k \le n$ and $\pi_{n+1}(P^nX_1) = 0$, by cellular approximation. Inductively, suppose that P^nX_{m-1} is obtained from X by attaching cells of dimension $n+2, \dots, n+m$, so that $X \to P^nX_{m-1}$ induces an isomorphism on π_k for $k \le n$ and $\pi_k(P^nX_{m-1}) = 0$ for $n+1 \le k < n+m$. Let

$$\psi\colon \bigvee_\beta S^{n+m} \to P^nX_{m-1}$$

induce a surjection on π_{n+m}, and let

$$P^nX_m = P^nX_{m-1} \cup_\psi C(\bigvee_\alpha S^{n+m})\,.$$

Then P^nX_m satisfies the inductive hypothesis for m, by cellular approximation. Let P^nX be the colimit (increasing union) of the sequence

$$i_n\colon X \subset P^nX_1 \subset \cdots \subset P^nX_{m-1} \subset P^nX_m \subset \cdots \subset P^nX\,.$$

Then $i_n\colon X \to P^nX$ is the n-th Postnikov section of X.

If $j_n\colon X \to Q^nX$ is any other map killing the homotopy groups $\pi_k(X)$ for $k > n$ then we can inductively extend j_n over P^nX_m for each $m \ge 1$, since the obstruction $\bigvee_\beta S^{n+m} \to P^nX_{m-1} \to Q^nX$ to extending the map from P^nX_{m-1} to P^nX_m lies in $\prod_\beta \pi_{n+m}(Q^nX)$, which vanishes.

$$\begin{tikzcd}
\bigvee_\beta S^{n+m} \arrow[r, "\psi"] \arrow[dr] & P^nX_{m-1} \arrow[r] \arrow[d] & P^nX_m \arrow[dl, dashed] \\
& Q^nX &
\end{tikzcd}$$

The extension is unique up to homotopy, because $\prod_\beta \pi_{n+m+1}(Q^nX)$ also is trivial. The fully extended map $e\colon P^nX \to Q^nX$ must induce an isomorphism on π_k for all k. Hence any two n-th Postnikov sections of X are (weakly) homotopy equivalent, in a way that is compatible with the structure maps from X.

$$\begin{tikzcd}
X \arrow[r, "i_n"] \arrow[dr, "j_n"'] & P^nX \arrow[d, "e"', "\simeq"] \\
& Q^nX
\end{tikzcd}$$

Likewise, the map $i_n\colon X \to P^nX$ can be extended, uniquely up to homotopy, over $P^{n+1}X_m$ for each $m \ge 1$, hence also over $i_{n+1}\colon X \to P^{n+1}X$. This follows from the vanishing of $\pi_{n+1+m}(P^nX)$ for each

$m \geq 1$. The resulting diagram

$$\begin{tikzcd}
X \arrow[dr, "i_{n+1}"] \arrow[ddr, "i_n"'] & \vdots \arrow[d, "p_{n+1}"] \\
& P^{n+1}X \arrow[d, "p_n"] \\
& P^n X \arrow[d, "p_{n-1}"] \\
& \vdots \arrow[d, "p_0"] \\
& P^0 X
\end{tikzcd}$$

(with $P^0X \simeq *$) is called the Postnikov tower of X.

The homotopy limit, or microscope, of the Postnikov tower can be defined by replacing each map p_n by a fibration, and forming the limit of the resulting tower of fibrations. There are then isomorphisms

$$\pi_k(X) \xrightarrow{\cong} \pi_k(\operatorname*{holim}_n P^nX) \xrightarrow{\cong} \lim_n \pi_k(P^nX)$$

for all k, so that X is (weakly) homotopy equivalent to the homotopy limit of its Postnikov sections.

((ETC: Partial duality between Postnikov tower and skeletal filtration. Is there a better analogy given by attaching Moore spaces than spheres?))

27.2. **Eilenberg–Mac Lane spaces.** Let G be an abelian group, and let $n \geq 2$. The n-th Postnikov section of a Moore space of type $M(G,n)$ is an Eilenberg–Mac Lane space of type $K(G,n)$.

$$i_n \colon M(G,n) \longrightarrow P^nM(G,n) = K(G,n)$$

In other words, build $M(G,n)$ with n- and $(n+1)$-cells from a presentation of G, and then attach m-cells for $m \geq n+2$ to kill the homotopy groups in degrees $\geq n+1$.

It is characterized, up to homotopy equivalence, by its homotopy groups

$$\pi_k(K(G,n)) = \begin{cases} G & \text{for } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

To see this, suppose that L is a CW complex with these homotopy groups. Use the presentation of G to construct a map $M(G,n) \to L$ that induces an isomorphisms on π_n. Thereafter extend this map over the cells of $K(G,n)$ of dimension $\geq n+2$, which is always possible because $\pi_k(L) = 0$ for $k \geq n+1$. The extended map $K(G,n) \to L$ is then a (weak) homotopy equivalence. This characterization also applies for $n \in \{0,1\}$. We therefore write $K(G,n)$ for any CW complex of this homotopy type.

Example 27.1. There are Moore and Eilenberg–Mac Lane spaces

$$\begin{aligned} M(\mathbb{Z},1) = S^1 &\xrightarrow{\simeq} S^1 = K(\mathbb{Z},1) \\ M(\mathbb{Z}/2,1) = \mathbb{R}P^2 &\longrightarrow \mathbb{R}P^\infty = K(\mathbb{Z}/2,1) \\ M(\mathbb{Z},2) = \mathbb{C}P^1 &\longrightarrow \mathbb{C}P^\infty = K(\mathbb{Z},2)\,. \end{aligned}$$

Corollary 27.2. *There is a homotopy equivalence*

$$\tilde\sigma \colon K(G,n) \simeq \Omega K(G,n+1)$$

inducing the identity isomorphism

$$G = \pi_n(K(G,n)) \cong \pi_n(\Omega K(G,n+1)) \cong \pi_{n+1}(K(G,n+1)) = G\,.$$

The left adjoint map

$$\sigma \colon \Sigma K(G,n) \longrightarrow K(G,n+1)$$

extends the equivalence

$$\Sigma M(G,n) \xrightarrow{\simeq} M(G,n+1)$$

(which we can take to be the identity).

27.3. **The homotopy characterization of cohomology.**

Lemma 27.3. *There are natural isomorphisms*

$$H^n(K(G,n);G) \xrightarrow{\cong} \mathrm{Hom}(H_n(K(G,n)),G) \xrightarrow{\cong} \mathrm{Hom}(G,G)\,.$$

Proof. The first isomorphism is given by the universal coefficient theorem, since $H_{n-1}(K(G,n)) = 0$. The second isomorphism is induced by the Hurewicz isomorphism $\pi_n(K(G,n)) \cong H_n(K(G,n))$. □

Let $\iota_n \in H^n(K(G,n);G)$ correspond to the identity homomorphism $1 : G \to G$.

Theorem 27.4 (Eilenberg–Mac Lane)**.** *There is a natural isomorphism*

$$[X, K(G,n)] \cong H^n(X;G)$$

sending the homotopy class of $f \colon X \to K(G,n)$ to the cohomology class $f^*(\iota_n)$.

The notation $[X, K(G,n)]$ assumes that X is a CW complex, or that f is a map from a CW approximation to X. We are working in the category of based spaces, so to include the case $n = 0$ we should use reduced cohomology.

Sketch proof. See Hatcher's Theorem 4.57. One proof proceeds by arguing that

$$h^n(X,A) = [X/A, K(G,n)]$$

defines a cohomology theory on the category of CW pairs, cf. Definition 10.3 or page 202 in Hatcher's book. The coboundary homomorphism

$$\delta \colon h^n(A) \longrightarrow h^{n+1}(X/A)$$

is the composite of the suspension isomorphism

$$h^n(A) = [A, K(G,n)] \cong [A, \Omega K(G,n+1)] \cong [\Sigma A, K(G,n+1)] = h^{n+1}(\Sigma A)$$

and the natural map

$$h^{n+1}(\Sigma A) \longrightarrow h^{n+1}(X/A)$$

induced by

$$X/A \simeq X \cup CA \xrightarrow{j} \Sigma A\,.$$

Homotopy invariance, excision and the sum axiom are then easily verified. The exactness axiom follows from the coexactness of the Puppe sequence

$$A \longrightarrow X \xrightarrow{i} X \cup CA \xrightarrow{j} \Sigma A\,.$$

The coefficients of this cohomology theory are given by the groups $h^k(S^0) = [S^0, K(G,k)]$, which are G for $k = 0$ and 0 otherwise. Hence $h^k(D^n, \partial D^n)$ is G for $k = n$ and 0 otherwise. For any CW complex X the skeletal filtration then gives a cochain complex

$$\dots \to h^n(X^n, X^{n-1}) \xrightarrow{\delta} h^{n+1}(X^{n+1}, X^n) \to \dots$$

that is isomorphic to the cellular complex

$$\dots \to W^n(X^n, X^{n-1};G) \xrightarrow{\delta} W^{n+1}(X^{n+1}, X^n;G) \to \dots$$

and whose cohomology calculates

$$h^n(X) \cong H^n(X;G)\,,$$

as claimed. □

The homotopy characterization of homology is a little more complicated. The structure maps

$$\sigma\colon \Sigma K(G,n) \longrightarrow K(G,n+1)$$

in the Eilenberg–Mac Lane spectrum

$$HG = \{n \mapsto K(G,n)\}$$

induce homomorphisms

$$\pi_{k+n}(K(G,n)\wedge X) \xrightarrow{E} \pi_{k+n+1}(\Sigma K(G,n)\wedge X) \xrightarrow{\sigma} \pi_{k+n+1}(K(G,n+1)\wedge X)\,.$$

Theorem 27.5 (Lima/Whitehead)**.** *There is a natural isomorphism*

$$\operatorname*{colim}_n \pi_{k+n}(K(G,n)\wedge X) \cong H_k(X;G)\,.$$

The proof comes down to checking that

$$h_k(X,A) = \operatorname*{colim}_n \pi_{k+n}(K(G,n)\wedge X/A)$$

defines a homology theory on the category of CW pairs, with coefficients $h_k(S^0) \cong G$ for $k = 0$ and 0 otherwise. Homotopy excision gives exactness for $(X,A) \mapsto \pi_{k+n}(K(G,n)\wedge X/A)$ in a range of degrees k that increases to infinity as n grows. Passing to the colimit over n hence gives exactness in all degrees.

These results suggest that to any sequence of spaces

$$E = \{n \mapsto E_n\}$$

with homotopy equivalences $\tilde\sigma\colon E_n \simeq \Omega E_{n+1}$ and adjoint maps $\sigma\colon \Sigma E_n \to E_{n+1}$ we can form a (generalized) cohomology theory

$$E^n(X) = [X, E_n]$$

and a generalized homology theory

$$E_k(X) = \operatorname*{colim}_n \pi_{k+n}(E_n \wedge X)\,.$$

This is indeed the case, and these objects E are the (sequential) spectra of stable homotopy theory.

27.4. ***k*-invariants.**

$$\begin{tikzcd}
 & X \arrow[d] & \\
 & \vdots \arrow[d] & \\
K(\pi_{n+1}(X),n+1) \arrow[r] & P^{n+1}X \arrow[d, "p_n"] & \\
 & P^nX \arrow[r, "k_n"] \arrow[d] & K(\pi_{n+1}(X),n+2) \\
 & \vdots \arrow[d] & \\
 & * &
\end{tikzcd}$$

We return to the Postnikov tower of a connected CW complex X. The projection $p_n\colon P^{n+1}X \to P^nX$ induces an isomorphism on π_k for all $k \neq n+1$, and the surjection $\pi_{n+1}(X) \to 0$ for $k = n$, so its homotopy fiber is an Eilenberg–Mac Lane space of type $K(\pi_{n+1}(X), n+1)$:

$$K(\pi_{n+1}(X),n+1) \longrightarrow P^{n+1}X \xrightarrow{p_n} P^nX\,.$$

For simply-connected X these are principal fibrations, in the sense that p_n is the inclusion of the homotopy fiber of a map

$$k_n \colon P^n X \longrightarrow K(\pi_{n+1}(X), n+2)$$

called the n-th Postnikov k-invariant of X. This also works for connected X with trivial action by $\pi_1(X)$ on $\pi_n(X)$ for all $n \geq 2$. See Hatcher's Theorem 4.69. By the homotopy representability of cohomology, there is a natural bijection

$$[P^n X, K(\pi_{n+1}(X), n+2)] \cong H^{n+2}(P^n X, \pi_{n+1}(X)) .$$

Hence the n-th k-invariant amounts to a class in the $(n+2)$-th cohomology of $P^n X$ with coefficients in $\pi_{n+1}(X)$, and the next Postnikov section, $P^{n+1}X$, is given by the homotopy fiber of k_n. This k-invariant is also often denoted k^{n+2}, since it lies in degree $(n+2)$ in cohomology.

For example, if each k-invariant is zero, then X is a product of Eilenberg–Mac Lane spaces

$$X \simeq \prod_{n \geq 1} K(\pi_n(X), n) .$$

Conversely, if a k-invariant is nonzero, then X is not a product of Eilenberg–Mac Lane spaces, but a more complicated extension of such spaces.

In principle, the weak homotopy type of a simply-connected space X is thus specified by the sequence of homotopy groups

$$\pi_2(X), \dots, \pi_n(X), \pi_{n+1}(X), \dots$$

together with the k-invariants

$$k^{n+2} \in H^{n+2}(P^n X; \pi_{n+1}(X))$$

for $n \geq 2$, where $P^2 X = K(\pi_2(X), 2)$ and $P^{n+1}X$ is inductively defined as the homotopy fiber of the map $P^n X \to K(\pi_{n+1}(X), n+2)$ representing k^{n+2}. A similar statement applies for path-connected spaces with trivial action of π_1 on the higher π_n. In practice, it is difficult to calculate $H^{n+2}(P^n X; \pi_{n+1}(X))$, but the Serre and Eilenberg–Moore spectral sequences are useful tools for this.

27.5. **Whitehead towers.** Let the n-connected Whitehead cover $W_{n+1}X \to X$ be the homotopy fiber of the n-th Postnikov section $i_n \colon X \to P^n X$. Then

$$\pi_k(W_{n+1}X) \cong \begin{cases} \pi_k(X) & \text{for } k \geq n+1, \\ 0 & \text{otherwise,} \end{cases}$$

with the isomorphisms induced by the map to X. If $\tilde{X} \to X$ is a universal cover, then $\tilde{X} \simeq W_2 X$, so $W_n X$ generalizes this construction. We get a tower of homotopy fiber sequences

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d, "="] & \vdots \arrow[d] \\
W_{n+2}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "="] & P^{n+1}X \arrow[d, "p_n"] \\
W_{n+1}X \arrow[r] \arrow[d, "w_n"'] & X \arrow[r] \arrow[d, "="] & P^n X \arrow[d] \\
W_n X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "="] & P^{n-1}X \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d, "="] & \vdots \arrow[d] \\
W_1 X \arrow[r, "\simeq"] & X \arrow[r] & P^0 X
\end{tikzcd}$$

The homotopy limit

$$\operatorname*{holim}_n W_{n+1}X \simeq *$$

is contractible, and we get a tower of fibrations

$$\begin{tikzcd}
 & * \arrow[d] & \\
 & \vdots \arrow[d] & \\
K(\pi_n(X), n-1) \arrow[r] & W_{n+1}X \arrow[d, "w_n"'] & \\
 & W_nX \arrow[r] \arrow[d] & K(\pi_n(X), n) \\
 & \vdots \arrow[d] & \\
 & X &
\end{tikzcd}$$

where each map $W_nX \to K(\pi_n(X), n)$ is given by killing homotopy groups, and $W_{n+1}X$ is the homotopy fiber of this map.

Serre's computations of homotopy groups of spheres proceeded by calculating the (co-)homology of the Whitehead covers $W_{n+1}X$ of spheres, and reading off π_{n+1} as H_{n+1} of these n-connected spaces.

27.6. **The 2-connected cover of S^2.** The Whitehead towers of S^2 and S^3 agree, after the first stage. The inclusion $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^\infty$ is a map $S^2 \longrightarrow K(\mathbb{Z}, 2)$ that exhibits $\mathbb{C}P^\infty$ as the Postnikov section P^2S^2, so its homotopy fiber is the Whitehead cover W_3S^2. We get a map from the Hopf fiber bundle to the Puppe sequence

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & S^3 \arrow[r, "\eta"] \arrow[d] & S^2 \arrow[d, "="] & \\
K(\mathbb{Z}, 1) \arrow[r] & W_3S^2 \arrow[r, "w_2"] & S^2 \arrow[r] & K(\mathbb{Z}, 2)
\end{tikzcd}$$

where the left hand map is a homotopy equivalence, which implies that the middle vertical map is also a homotopy equivalence. Hence $\eta\colon S^3 \to S^2$ is the 2-connected cover of S^2. It follows that the $(n-1)$-connected covers of S^3 and S^2 are equivalent for all $n \geq 3$:

$$W_nS^3 \simeq W_nS^2 .$$

27.7. **The 3-connected cover of S^3.** What can we say about the 3-connected cover W_4S^3 of S^3, sometimes denoted $S^{\langle 3\rangle}$? Since $\pi_3(S^3) \cong \mathbb{Z}$, it is the homotopy fiber of a map $S^3 \to K(\mathbb{Z}, 3)$ that is an isomorphism on π_3 and represents the generator of $H^3(S^3; \mathbb{Z})$. We get a Puppe sequence

$$K(\mathbb{Z}, 2) \longrightarrow W_4S^3 \xrightarrow{w_3} S^3 \longrightarrow K(\mathbb{Z}, 3) ,$$

where $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$. Since W_4S^3 is 3-connected, we have a Hurewicz isomorphism

$$\pi_4(S^3) \cong H_3(W_4S^3) ,$$

so we would like to calculate the homology of the 3-connected total space in a homotopy fiber sequence

$$\mathbb{C}P^\infty \longrightarrow W_4S^3 \longrightarrow S^3 .$$

Since the base space is a sphere, this can be done by a special case of the Serre spectral sequence due to Hsien-Chung Wang (1949), which shows that there is a long exact sequence

$$\dots \to H^{m-3}(\mathbb{C}P^\infty) \longrightarrow H^m(W_4S^3) \longrightarrow H^m(\mathbb{C}P^\infty) \xrightarrow{\delta} H^{m-2}(\mathbb{C}P^\infty) \to \dots .$$

This can be obtained from the Mayer–Vietoris sequence associated to the open cover $W_4S^3 = U \cup V$ where U and V are the preimages of $S^3 - \{p\}$ and $S^3 - \{q\}$, for any points $p \neq q \in S^3$. Note that $\mathbb{C}P^\infty \simeq U \simeq V$ since $S^3 - \{p\}$ and $S^3 - \{q\}$ are contractible, and $U \cap V \simeq S^2 \times \mathbb{C}P^\infty$.

Moreover, $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$ with $|y| = 2$ and δ is a derivation with $\delta(y) = 1$. The latter claim implies that $\delta(y^n) = ny^{n-1}$, so that

$$H^m(W_4S^3) \cong \begin{cases} \mathbb{Z} & \text{for } m = 0, \\ \mathbb{Z}/n & \text{for } m = 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

In low degrees:

$$H^*(W_4S^3) = (\mathbb{Z}, 0, 0, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}/3, \dots) .$$

Granting that W_4S^3 has finite type, the universal coefficient theorem implies that

$$H_*(W_4S^3) = (\mathbb{Z}, 0, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}/3, 0, \dots) .$$

Hence $\pi_4(S^3) \cong \pi_4(W_4S^3) \cong H_4(W_4S^3) \cong \mathbb{Z}/2$. By Freudenthal's suspension theorem, this group is generated by $E\eta$, and more generally $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ is generated by $E^{n-2}\eta$ for each $n \geq 3$. Thus

$$\pi_1(S) = \operatorname*{colim}_n \pi_{n+1}(S^n) \cong \mathbb{Z}/2$$

is generated by the stable class of η.