Exercise 1. We have a free resolution

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m \longrightarrow 0 .$$

Applying $\mathrm{Hom}(-,B)$ & noting that $\mathrm{Hom}(\mathbb{Z},B) \cong B$, we find that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, B)$ is the homology of the complex

$$0 \longleftarrow B \xleftarrow{m} B ,$$

i.e., $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, B) \cong B/mB$.

Exercise 2. This follows from the similar properties for Hom and H^*.

Exercise 3 By Exercise 2, $\mathrm{Ext}^1_{\mathbb{Z}}(\bigoplus_\alpha \mathbb{Z}, A) \cong \prod_\alpha \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, A) = 0$.

Exercise 4. a) Given an extension

$$\xi : \quad 0 \longrightarrow B \xrightarrow{\alpha} E \xrightarrow{\beta} A \longrightarrow 0,$$

we obtain, by applying $\mathrm{Hom}(A,-)$, an exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(A,E) \xrightarrow{\beta_*} \mathrm{Hom}(A,A) \xrightarrow{\partial} \mathrm{Ext}^1(A,B) \to \cdots$$

If $\mathrm{Ext}^1(A,B) = 0$, the map $\mathrm{Hom}(A,E) \xrightarrow{\beta_*} \mathrm{Hom}(A,A)$ is surjective and hence there is a map $s \in \mathrm{Hom}(A,E)$ such that

$$\beta_*(s) = \beta \circ s = id_A .$$

This means that β admits a section, or, in other words, that ξ is split.

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b) Part a) shows that in general there is a map $\Theta : \{\text{extension of } A \text{ by } B\}/_{iso} \longrightarrow \mathrm{Ext}^1(A,B)$ given by $\Theta(\xi) = \partial(id_A)$.

<u>Θ surjective</u> Let $x \in \mathrm{Ext}^1(A,B)$, and let F be a free module surjecting onto A, so that we have an exact sequence

$$0 \to K \xrightarrow{j} F \xrightarrow{p} A \to 0. \qquad (*)$$

Applying $\mathrm{Hom}(-,B)$ we get

$$\cdots \to \mathrm{Hom}(F,B) \to \mathrm{Hom}(K,B) \xrightarrow{\partial} \mathrm{Ext}^1(A,B) \to 0.$$

Thus we can find $\beta \in \mathrm{Hom}(K,B)$ s.t. $\partial(\beta) = x \in \mathrm{Ext}^1(A,B)$.

We now modify $(*)$ so that it starts at B instead of K.

Let $E = \mathrm{coker}\left(K \xrightarrow{y \mapsto (j(y), -\beta(y))} F \oplus B \right)$

Then there is a map $E \to A$ induced by $F \oplus B \xrightarrow{(z,b) \mapsto (p(z),0)} A$, along with a commutative diagram w/ exact rows

$$\begin{array}{ccccccccc}
 & 0 & \to & K & \xrightarrow{j} & F & \xrightarrow{p} & A & \to 0 \\
 & & & \beta \downarrow & & \downarrow & & \downarrow = & \\
\xi : & 0 & \to & B & \to & E & \to & A & \to 0
\end{array}$$

which yields $\Theta(\xi) = x$.

<u>Θ injective</u>. The above construction defines in fact a map

$$\psi : \mathrm{Ext}^1(A,B) \to \{\text{extensions}\}/_{iso}$$

satisfying $\psi(\Theta(\xi)) = \xi$, hence Θ is injective.

3.1.2. First recall that, by Hatcher, p. 195, any abelian group G has a "canonical" free resolution of the form $0 \to F_1 \to F_0 \to H \to 0$.

Let $\alpha: H \to H$ be multiplication by n.

We take the free resolution of H above and dualize:

$$\begin{array}{ccccccccc} 0 & \leftarrow & \mathrm{Hom}(F_1, G) & \leftarrow & \mathrm{Hom}(F_0, G) & \leftarrow & \mathrm{Hom}(H, G) & \leftarrow & 0 \\ & & \downarrow \alpha^* & & \downarrow \alpha^* & & \downarrow \alpha^* & & \\ 0 & \leftarrow & \mathrm{Hom}(F_1, G) & \leftarrow & \mathrm{Hom}(F_0, G) & \leftarrow & \mathrm{Hom}(H, G) & \leftarrow & 0 \end{array}$$

For $\varphi \in \mathrm{Hom}(F_i, G)$ we then have

$$\alpha^*(\varphi)(x) = \varphi(\alpha(x)) = \varphi(nx) = n\varphi(x),$$

i.e., α^* is multiplication by n.

Hence the induced map on homology is multiplication by n as well.

Similarly if we resolve G and apply $\mathrm{Hom}(H, -)$.

3.1.3. Take the free $\mathbb{Z}/4$ - resolution

$$\cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{p} \mathbb{Z}/2 \to 0,$$

where $p(x) = x \bmod 2$.

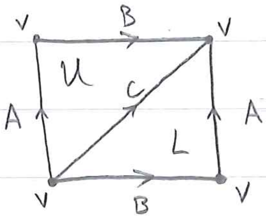
We dualize this sequence; i.e., apply $\mathrm{Hom}_{\mathbb{Z}/4}(-, \mathbb{Z}/2)$.
Note that $\mathrm{Hom}_{\mathbb{Z}/4}(\mathbb{Z}/4, \mathbb{Z}/2) \cong \mathbb{Z}/2$ (indeed, a generator of $\mathbb{Z}/4$ can be mapped to either 0 or 1).
Hence the dualized chain complex has the form

$$\cdots \leftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{id} \mathbb{Z}/2 \leftarrow 0, \qquad (\text{since } \cdot 2^* = 0)$$

hence $\mathrm{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2 \quad \forall\, n \geq 1$.

3.1.6. I'll do $H^q(S^1 \times S^1; \mathbb{Z})$ to illustrate the technique.
Let $T^2 = S^1 \times S^1$, then we can identify T^2 with the unit cube w/ the opposite sides identified:

T^2:



We now write down the corresponding cochain complex.
For this, we first recall that if F is a free abelian group w/ finite basis $\mathcal{B}$, then the dual $F^* = \mathrm{Hom}(F, \mathbb{Z})$ is free abelian on basis $\mathcal{B}^* = \{ b^* \mid b \in \mathcal{B} \}$, where $b^*: F \to \mathbb{Z}$ is the function

$$b^*(b') = \begin{cases} 1, & b = b' \\ 0, & b \neq b' \end{cases}$$

So let $v = V^*$, $a = A^*$, $b = B^*$, $c = C^*$, $u = U^*$, $l = L^*$.
Then the terms in the cochain complex are as follows:

- $C^0(T, \mathbb{Z}) = \mathbb{Z}\{v\}$
- $C^1(T, \mathbb{Z}) = \mathbb{Z}\{a, b, c\}$
- $C^2(T, \mathbb{Z}) = \mathbb{Z}\{u, \ell\}$.

The coboundary maps are:

- $\delta^0 v \in C^1(T, \mathbb{Z})$ acts as:

  $(\delta^0 v)(A) = v(\partial A) = v(V) - v(V) = 0$,

  similarly $(\delta^0 v)(B) = 0 = (\delta^0 v)(C)$.

  Hence $\delta^0 v = 0$.

- $(\delta^1 a)(U) = a(\partial U) = a(B - C + A) = 1$

  $(\delta^1 a)(L) = a(A - B + C) = 1$

  $(\delta^1 b)(U) = b(\partial U) = b(B - C + A) = 1$, sim. for $(\delta^1 b)(L)$.

  $(\delta^1 c)(U) = c(B - C + A) = -1$,

  $(\delta^1 c)(L) = -1$.

  Hence $\delta^1 a = \delta^1 b = -\delta^1 c = u + \ell$.

So the cochain complex is

$$0 \longrightarrow \mathbb{Z}\{v\} \xrightarrow{0} \mathbb{Z}\{a, b, c\} \xrightarrow{\substack{a \mapsto u+\ell \\ b \mapsto u+\ell \\ c \mapsto -(u+\ell)}} \mathbb{Z}\{u, \ell\} \longrightarrow 0.$$

Hence

- $H^0(T; \mathbb{Z}) = \mathbb{Z}\{v\}$
- $H^1(T; \mathbb{Z}) = \mathbb{Z}\{a - b, a + c\} \cong \mathbb{Z} \oplus \mathbb{Z}$.
- $H^2(T; \mathbb{Z}) = \mathbb{Z}\{u\}$.

With $\mathbb{Z}/2$ - coefficients, we get $\underbrace{\mathbb{Z}/2\{v\}}_{= H^0(T; \mathbb{Z}/2)}$, $\underbrace{\mathbb{Z}/2\{a - b, a + c\}}_{= H^1(T; \mathbb{Z}/2)}$, $\underbrace{\mathbb{Z}/2\{u\}}_{= H^2(T; \mathbb{Z}/2)}$.

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3.1.11. a) From Example 2.40 we have

$$\tilde{H}_p(M(\mathbb{Z}/m, n)) = \begin{cases} 0, & m \neq n \\ \mathbb{Z}/m, & m = n \end{cases}.$$

Let $X = M(\mathbb{Z}/m, n)$, and let $p: X \to X/S^n = S^{n+1}$ be the quotient map.
Then p_* is trivial on all homology groups, since there are never any two groups $\neq 0$ in the appropriate dimensions.
On cohomology, however, we have

$$H^{n+1}(X) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}) = \mathbb{Z}/m \quad \text{by UCT.}$$

The long exact sequence for the pair (X, S^n) reads:

$$\begin{array}{ccccccccccc} \cdots \to & H^n(X) & \to & H^n(S^n) & \xrightarrow{\delta} & H^{n+1}(S^{n+1}) & \xrightarrow{p^*} & H^{n+1}(X) & \to & H^{n+1}(S^n) & \to \cdots \\ & \| & & \| & & \| & & \| & & \| & \\ & 0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z}/m & \to & 0 & \end{array}$$

Hence the map $p^*: H^{n+1}(S^{n+1}) \to H^{n+1}(X)$ is surjective.
This implies that the splitting in UCT is not natural. Indeed, in the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1(H_n(X), \mathbb{Z}) \arrow[r] & H^{n+1}(X) \arrow[r] & \mathrm{Hom}(H_{n+1}(X), \mathbb{Z}) \arrow[r] \arrow[l, bend left, dashed, "s_1"] & 0 \\ 0 \arrow[r] & \mathrm{Ext}^1(H_n(S^{n+1}), \mathbb{Z}) \arrow[r] \arrow[u] & H^{n+1}(S^{n+1}) \arrow[r] \arrow[u, "p^*"] & \mathrm{Hom}(H_{n+1}(S^{n+1}), \mathbb{Z}) \arrow[r] \arrow[u, "(p_*)^*"'] \arrow[l, bend left, dashed, "s_2"] & 0 \end{tikzcd}$$

the square involving the sections s_1, s_2 cannot commute, since $p^* s_2 \neq 0$ while $s_1 (p_*)^* = 0$.