Vector bundles and connections

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1 Vector bundles

In the following, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Smooth manifolds are assumed to be second countable. For a smooth manifold M we denote by $\mathcal{D}(M, \mathbb{K})$ the ring of smooth functions $M \to \mathbb{K}$, and we set $\mathcal{D}(M) := \mathcal{D}(M, \mathbb{R})$.

By definition, a **smooth $\mathbb{K}$-vector bundle** of rank r over a smooth manifold M consists of

- a smooth manifold E called the **total space** of the bundle,
- a smooth map $\pi : E \to M$ called the **projection**,
- for each $p \in M$ a $\mathbb{K}$-vector space structure on the **fibre** $E_p := \pi^{-1}(p)$,

such that the axiom of **local triviality** holds: For each $p \in M$ there should exist an open neighbourhood U of p and a diffeomorphism $\Phi : U \times \mathbb{K}^r \to \pi^{-1}U$ such that

- $\pi \circ \Phi(q, v) = q$ for all $(q, v) \in U \times \mathbb{K}^r$,
- for each $q \in U$ the map
$$\Phi_q : \mathbb{K}^r \to E_q, \quad v \mapsto \Phi(q, v)$$
is a linear isomorphism.

Such a diffeomorphism Φ is called a **local trivialization** of E. If $r = 1$ then E is called a (real or complex) **line bundle** over M.

Examples (i) The **product bundle**

$$M \times V \to M, \quad (q, v) \mapsto q,$$

where V is any finite-dimensional $\mathbb{K}$-vector space.

(ii) The tangent bundle $TM \to M$.

(iii) If $\pi : E \to M$ is a vector bundle and $N \subset M$ a submanifold, then $E|_N := \pi^{-1}N$ is a vector bundle over N.

(iv) More generally, if $\pi : E \to M$ is a vector bundle and $f : N \to M$ a smooth map, then

$$f^*E := \{(q, v) \in N \times E \,|\, f(q) = \pi(v)\}$$

is a vector bundle over N called the **pull-back** of E by f.

(v) To any vector space V we can associate its dual vector space V^*. This construction can also be applied fibrewise to vector bundles. Namely, if $\pi_E : E \to M$ is a vector bundle then

$$E^* := \bigcup_{p \in M} \{p\} \times E_p^*$$

has a unique structure of a vector bundle over M such that the projection is given by

$$\pi_{E^*} : E^* \to M, \quad (p, v) \mapsto p$$

and such that for any local trivialization $\Phi : U \times \mathbb{K}^r \to (\pi_E)^{-1}U$ of E the map

$$U \times (\mathbb{K}^r)^* \to (\pi_{E^*})^{-1}U, \quad (p, \xi) \mapsto \xi \circ \Phi_p^{-1}$$

is a local trivialization of E^*, where we agree that $(\mathbb{K}^r)^*$ can be canonically identified with $\mathbb{K}^r$. The bundle E^* is called the **dual bundle** of E. In the case $E = TM$ the dual bundle T^*M is called the **cotangent bundle** of M.

(vi) For any pair (V,W) of $\mathbb{K}$-vector spaces let $\mathrm{Hom}(V, W)$ denote the vector space of all linear maps $V \to W$. Applying this construction fibrewise to a pair (E, E') of vector bundles over a manifold M yields a vector bundle $\mathrm{Hom}(E, E') \to M$ whose fibre over $p \in M$ is

$$\mathrm{Hom}(E, E')_p = \mathrm{Hom}(E_p, E'_p).$$

In the case $E' = E$ we obtain the **endomorphism bundle** $\mathrm{End}(E) := \mathrm{Hom}(E, E)$.

Let $E \to M$ be a vector bundle of rank r. By a **subbundle** of E of rank s we mean a subset $E' \subset E$ such that for every point $p \in M$ there exist a local trivialization $\Phi : U \times \mathbb{K}^r \to \pi^{-1}U$ of E around p and an s–dimensional linear subspace $V \subset \mathbb{K}^r$ such that

$$\Phi(U \times V) = \pi^{-1}U \cap E'.$$

Then E' is in a natural way a vector bundle over M.

Example If $N \subset M$ is a submanifold then the tangent bundle TN is a subbundle of $TM|_N$.

By a **section** of a vector bundle $\pi : E \to M$ we mean a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_M$. The vector space $\Gamma(E)$ of all sections of E is a module over the ring $\mathcal{D}(M, \mathbb{K})$. Addition in $\Gamma(E)$ and multiplication by functions are defined pointwise by

$$(s+t)(p) := s(p) + t(p),$$
$$(fs)(p) := f(p)\, s(p)$$

for $s, t \in \Gamma(E)$ and $f \in \mathcal{D}(M, \mathbb{K})$.

By definition, a **vector field** on M is a section of TM. We denote by $\mathcal{X}(M) = \Gamma(TM)$ the $\mathcal{D}(M)$–module of all vector fields on M.

Let $\pi : E \to M$ and $\pi' : E' \to M'$ be $\mathbb{K}$-vector bundles and $f : M \to M'$ a map. A smooth map $F : E \to E'$ is called a **bundle homomorphism over** f if F maps each fibre E_p linearly into $E'_{f(p)}$. (This implies that f is smooth.) If in addition $M = M'$ and f is the identity map then F is called a **bundle homomorphism.** A bundle homomorphism $F : E \to E'$ which is also a diffeomorphism is called a **bundle isomorphism**. A vector bundle is called **trivial** if it is isomorphic to a product bundle.

Examples (i) If $f : M \to M'$ is a smooth map between manifolds then the tangent map $Tf : TM \to TM'$ is a bundle homomorphism over f.

(ii) If $\pi : E \to M$ is a vector bundle and $f : N \to M$ a smooth map then there is a canonical bundle homomorphism

$$f^*E \to E$$

obtained by restricting the projection $N \times E \to E$ to f^*E.

(iii) Let $E = M \times \mathbb{K}^r$ and $E = M \times \mathbb{K}^s$ be product bundles over M. Then a bundle homomorphism $E \to E'$ has the form

$$M \times \mathbb{K}^r \to M \times \mathbb{K}^s, \quad (p, v) \mapsto (p, \alpha(p)v)$$

for some smooth map α from M into the space $M(s \times r, \mathbb{K})$ of $s \times r$ matrices with entries in $\mathbb{K}$.

Let $E \to M$ be a $\mathbb{K}$–bundle of rank r and let $U \subset M$ be an open subset. By a **frame** for E over U we mean an r–tuple $(s_1, \ldots, s_r)$ of sections of $E|_U$

such that $(s_1(p), \ldots, s_r(p))$ is a basis for E_p for all $p \in U$. If $U = M$ then $(s_1, \ldots, s_r)$ is called a **global frame**. In that case the map

$$M \times \mathbb{K}^r \to E, \quad (p, v) \mapsto \sum_{j=1}^{r} v^j s_j(p)$$

is a bijective bundle homomorphism and therefore an isomorphism. Conversely, such an isomorphism clearly gives rise to a global frame.

2 Connections

Given a section s of a vector bundle $E \to M$ and a vector field X on M we would like to have some kind of derivative $\nabla_X s$ of s with respect to X. This derivative should be a new section of E. Because there is no canonical isomorphism between the fibres of E at two different points, we are unable to define a canonical derivative of this kind. Instead, we will formulate some properties that we would like such a derivative to have.

By a **connection** (or **covariant derivative**) in a $\mathbb{K}$-vector bundle $E \to M$ we mean a map

$$\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s$$

such that for all $X, Y \in \mathcal{X}(M)$, $s, t \in \Gamma(E)$, and $f \in \mathcal{D}(M, \mathbb{K})$ one has

(i) $\nabla_{X+Y}(s) = \nabla_X s + \nabla_Y s$,

(ii) $\nabla_{fX}(s) = f\nabla_X s$,

(iii) $\nabla_X(s + t) = \nabla_X s + \nabla_X t$,

(iv) $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

Note that if f is constant, say $f \equiv \alpha$, then $Xf = 0$, so (iv) gives

$$\nabla_X(\alpha s) = \alpha\nabla_X s.$$

A section s is called **covariantly constant**, or **parallel**, if $\nabla_X s = 0$ for all vector fields X.

Example A product bundle $E = M \times V \to M$ has a canonical connection ∇ called the **product connection**. To define this, note that to any function $h : M \to V$ we can associate a section $\tilde{h}$ of E given by

$$\tilde{h}(p) = (p, h(p)),$$

and any section of E has this form. Now define $\nabla_X \tilde{h}$ to be the section corresponding to the function Xh, i.e.

$$\nabla_X \tilde{h} := \widetilde{Xh}.$$

Theorem 2.1 *Any vector bundle $E \to M$ admits a connection ∇.*

Proof. Choose an open cover $\{U_\alpha\}_{\alpha \in I}$ of M such that $E|_{U_\alpha}$ is trivial for each α. Choose a partition of unity $\{g_\alpha\}$ subordinate to this cover. By the example above, each bundle $E|_{U_\alpha}$ admits a connection ∇^α. Now define, for any section s of E and vector field X on M,

$$\nabla_X s := \sum_\alpha g_\alpha \nabla_X^\alpha(s),$$

where $\nabla_X^\alpha(s)$, defined initially on U_α, is extended trivially to all of M. Then ∇ clearly satisfies the first three axioms for a connection. To verify the fourth one, let $f \in \mathcal{D}(M, \mathbb{K})$. Then

$$\begin{aligned} \nabla_X(fs) &= \sum_\alpha g_\alpha (Xf \cdot s + f \nabla_X^\alpha(s)) \\ &= Xf \cdot s + f \sum_\alpha g_\alpha \nabla_X^\alpha(s) \\ &= Xf \cdot s + f \nabla_X(s). \qquad \square \end{aligned}$$

For an arbitrary connection ∇ we will now investigate the dependence of $\nabla_X s$ on the two variables X and s, beginning with s.

Proposition 2.1 *Let ∇ be a connection in $E \to M$ and X a vector field on M. Then ∇_X is a* **local operator***, i.e. if a section s of E vanishes on an open subset $U \subset M$ then $\nabla_X s$ vanishes on U, too.*

Proof. Suppose $s|_U = 0$ and let $p \in U$. Choose a smooth real function f on M which is supported in U and satisfies $f(p) = 1$. Then $fs = 0$, so

$$0 = \nabla_X(fs) = Xf \cdot s + f \nabla_X s.$$

Evaluating this equation at p gives $(\nabla_X s)(p) = 0$. $\square$

Proposition 2.2 *Let E, E' be $\mathbb{K}$-vector bundles over M and*

$$A : \Gamma(E) \to \Gamma(E')$$

a $\mathcal{D}(M, \mathbb{K})$–linear map. Then there exists a unique bundle homomorphism $a : E \to E'$ such that $As = as$ for all $s \in \Gamma(E)$.

Explicitly, the assumption on A is that

$$A(s+t) = As + At, \quad A(fs) = fAs$$

for all $s, t \in \Gamma(E)$ and $f \in \mathcal{D}(M, \mathbb{K})$. The main point in the proposition is that $(As)(p)$ depends only on the value of s at p.

Proof of proposition. Uniqueness follows from the fact that for any $p \in M$ and $v \in E_p$ there exists a section s of E with $s(p) = v$. To prove existence of a, let $p \in M$ and $v \in E_p$. Choose a section s of E with $s(p) = v$ and define

$$av := (As)(p).$$

To show that av is independent of the choice of s it suffices to verify that if t is any section of E with $t(p) = 0$ then $(At)(p) = 0$. Let r be the rank of E. Choose sections $s_1, \ldots, s_r$ of E which are linearly independent at every point in some neighbourhood U of p. Choose a smooth real function f on M which is supported in U and satisfies $f(p) = 1$. Then

$$ft = \sum_j g^j s_j$$

for some uniquely determined $\mathbb{K}$–valued functions $g^1, \ldots, g^r$ on M that vanish outside U. Clearly, $g^j(p) = 0$. Applying A to both sides of the above equation we obtain

$$fAt = \sum_j g^j As_j,$$

and evaluating both sides of this equation at p gives $(At)(p) = 0$ as required. □

Corollary 2.1 *Let ∇ be a connection in $E \to M$ and $s \in \Gamma(E)$, $p \in M$. If X, Y are vector fields on M satisfying $X_p = Y_p$ then $(\nabla_X s)(p) = (\nabla_Y s)(p)$.*

Proof. This follows from the proposition because for given s the map

$$X \mapsto \nabla_X s$$

is $\mathcal{D}(M)$–linear. □

The corollary allows us to define $\nabla_v s$ for any tangent vector $v \in T_pM$. Namely, choose any vector field X with $X_p = v$ and set

$$\nabla_v s := (\nabla_X s)(p).$$

Proposition 2.3 *Let $E_1, \ldots, E_m, E'$ be $\mathbb{K}$–vector bundles over M and*

$$B : \Gamma(E_1) \times \cdots \times \Gamma(E_m) \to \Gamma(E')$$

a $\mathcal{D}(M, \mathbb{K})$–multilinear map. Then there exists for each $p \in M$ a $\mathbb{K}$–multilinear map

$$B_p : (E_1)_p \times \cdots \times (E_m)_p \to E'_p$$

such that if $s_j \in \Gamma(E_j)$, $j = 1, \ldots, m$ then

$$B(s_1, \ldots, s_m)(p) = B_p(s_1(p), \ldots, s_m(p)). \tag{1}$$

Proof. If at least one of the sections s_j vanishes at p then, by Proposition 2.2, $B(s_1, \ldots, s_m)$ also vanishes at p. By repeated application of this we see that if for each j we are given a pair of sections s_j, t_j of E_j with the same value at p then

$$B(s_1, \ldots, s_m)(p) = B(t_1, \ldots, t_m)(p).$$

We can now define B_p by (1). $\square$

Proposition 2.4 *Let $E \to M$ be a $\mathbb{K}$–vector bundle.*

(i) *If ∇, ∇' are connections in E then there exists a unique bundle homomorphism $a : TM \to End_{\mathbb{K}}(E)$ such that for all vector fields X on M and sections s of E one has*

$$\nabla'_X s - \nabla_X s = a(X) \cdot s.$$

(ii) *If ∇ is any connection in E and $a : TM \to End_{\mathbb{K}}(E)$ a bundle homomorphism then there is a connection ∇' in E given by*

$$\nabla'_X s = \nabla_X s + a(X) \cdot s.$$

Proof. The second statement is easily verified. To prove the first one, set

$$B : \mathcal{X}(X) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla'_X s - \nabla_X s.$$

First assume $\mathbb{K} = \mathbb{R}$. It is easy to see that B is $\mathcal{D}(M)$–bilinear, so by Proposition 2.2 it is given by a collection of bilinear maps

$$T_pM \times E_p \to E_p, \quad p \in M,$$

or equivalently, by a collection of linear maps

$$T_pM \to \mathrm{End}(E_p).$$

Together, these maps make up the desired bundle homomorphism a. If $\mathbb{K} = \mathbb{C}$ just observe that a takes values in $\mathrm{End}_{\mathbb{C}}(E)$. $\square$

Example Let $E \to M$ be a trivial vector bundle and $(s_1, \ldots, s_r)$ a global frame for E. Then there is a 1-1 correspondence between connections in E and $r \times r$ matrices $\omega = (\omega^i_j)$ of $\mathbb{K}$–valued 1–forms on M specified by the formula

$$\nabla_X s_j = \sum_{i=1}^{r} \omega^i_j(X) s_i \qquad (2)$$

for vector fields X on M. To deduce this from the proposition, let ∇^0 be the product connection in E given by $\nabla_X s_j = 0$. Then any other connection in E has the form $\nabla^0 + a$, where in this case a is given by an $r \times r$ matrix of 1–forms.

Note that ω can be thought of as a matrix-valued 1–form on M. We call ω the **connection form** of ∇ with respect to the given global frame.

3 Pull-back connections

Let $\pi : E \to M$ be a vector bundle and $f : \tilde{M} \to M$ a smooth map. By a **section of** E **along** f we mean a smooth map $t : \tilde{M} \to E$ such that $\pi \circ t = f$. We will usually identify such a map t with the corresponding section $\tilde{s}$ of the pull-back bundle f^*E given by $\tilde{s}(p) = (p, t(p))$. Note that if s is a section of E then $f^*s := s \circ f$ is a section of E along f.

Proposition 3.1 *Let ∇ be a connection in $E \to M$ and $f : \tilde{M} \to M$ a smooth map. Then there exists a unique connection $\tilde{\nabla} = f^*\nabla$ in $\tilde{E} := f^*E$ such that for all $s \in \Gamma(E)$, $p \in \tilde{M}$, and $v \in T_p\tilde{M}$ we have*

$$\tilde{\nabla}_v(f^*s) = \nabla_{f_*v}(s) \quad \text{in } \tilde{E}_p = E_{f(p)}, \qquad (3)$$

where $f_* : T_p\tilde{M} \to T_{f(p)}M$ is the tangent map of f.

Proof. Uniqueness follows from the fact that if $\{s_j\}$ is a frame for E over an open set $U \subset M$ then $\{f^*s_j\}$ is a frame for $\tilde{E}$ over $f^{-1}U$. We prove existence in three steps.

(i) First suppose $\tilde{M}$ is an open subset of M and f the inclusion map, so that $\tilde{E}$ is just the restriction of E to $\tilde{M}$. In this case the existence of $\tilde{\nabla}$ follows from Proposition 2.1 and Corollary 2.1.

(ii) Next we consider the case when E is trivial. Let $(s_1, \ldots, s_r)$ be a global frame for E and (ω^i_j) the matrix of 1–forms on M with

$$\nabla_Y s_j = \sum_i \omega^i_j(Y) s_i$$

for vector fields Y on M. Set $\tilde{s}_j := f^* s_j$ and $\tilde{\omega}^i_j := f^* \omega^i_j$, and let $\tilde{\nabla}$ be the unique connection in $\tilde{E}$ such that

$$\tilde{\nabla}_X \tilde{s}_j = \sum_i \tilde{\omega}^i_j(X) \tilde{s}_i$$

for vector fields X on $\tilde{M}$. We now verify that $\tilde{\nabla}$ satisfies (3). Let $p \in \tilde{M}$, $v \in T_p\tilde{M}$, and $s \in \Gamma(E)$. Then $s = \sum_j h^j s_j$ for some functions $h^j \in \mathcal{D}(M, \mathbb{K})$. Set $q := f(p)$, $w := f_* v$, and $g^j := h^j \circ f$. Then $f^* s = \sum_j g^j \tilde{s}_j$, and

$$\tilde{\nabla}_v \tilde{s}_j = \sum_i \tilde{\omega}^i_j(v) \tilde{s}_i(p) = \sum_i \omega^i_j(w) s_i(q) = \nabla_w s_j,$$

so

$$\begin{aligned} \tilde{\nabla}_v(f^* s) &= \sum_j [v(g^j) \cdot \tilde{s}_j(p) + g^j(p) \cdot \tilde{\nabla}_v \tilde{s}_j] \\ &= \sum_j [w(h^j) \cdot s_j(q) + h^j(q) \cdot \nabla_w s_j] \\ &= \nabla_w s. \end{aligned}$$

(iii) In the general case, choose an open cover $\{U_\alpha\}_{\alpha \in I}$ of M such that $E|_{U_\alpha}$ is trivial for each α. Set $\tilde{U}_\alpha := f^{-1} U_\alpha$. By case (ii), we have a pull-back connection $\tilde{\nabla}^\alpha$ in $\tilde{E}|_{\tilde{U}_\alpha}$. By uniqueness, $\tilde{\nabla}^\alpha$ and $\tilde{\nabla}^\beta$ restrict to the same connection on $\tilde{U}_\alpha \cap \tilde{U}_\beta$ for each $\alpha, \beta \in I$. Hence, the connections $\tilde{\nabla}^\alpha$ patch together to give the desired connection in $\tilde{E}$. □

Example Let $E \to M$ be a $\mathbb{K}$–vector bundle with connection ∇ and

$$\gamma : I \to M, \quad t \mapsto \gamma(t)$$

a smooth path in M, where $I \subset \mathbb{R}$ is an interval. Proposition 3.1 provides a $\mathbb{K}$–linear operator $\frac{D}{dt} := (\gamma^* \nabla)_{\frac{d}{dt}}$ acting on sections of E along γ with the follwing properties.

- For all smooth functions $f : I \to \mathbb{K}$ and sections σ of E along γ one has
$$\frac{D}{dt}(f\sigma) = \frac{df}{dt} \cdot \sigma + f \cdot \frac{D\sigma}{dt}.$$
- If $\sigma = f^*s$ is the pull-back of a section s of E then
$$\frac{D\sigma}{dt} = \nabla_{\frac{d\gamma}{dt}}(s).$$

4 Holonomy

Proposition 4.1 *Let $E \to M$ be a $\mathbb{K}$–vector bundle with connection ∇, and let $\gamma : I \to M$ be a smooth curve. Let $t_0 \in I$ and $v \in E_{\gamma(t_0)}$. Then there exists a unique parallel section σ of E along γ such that $\sigma(t_0) = v$.*

Here, σ is called parallel if $\frac{D\sigma}{dt} = 0$.

Proof. It is a simple exercise to show that any vector bundle over an interval is trivial. Hence, there exists a global frame $(\sigma_1, \dots, \sigma_r)$ of the pull-back bundle γ^*E. In terms of this frame, the operator $\frac{D}{dt}$ is given by an $r \times r$ matrix (c^i_j) of functions $I \to \mathbb{K}$ such that
$$\frac{D\sigma_j}{dt} = \sum_i c^i_j \sigma_i.$$
We are therefore seeking smooth $\mathbb{K}$–valued functions $f^1, \dots, f^r$ on the interval I with specified values at t_0 such that
$$\begin{aligned} 0 = \frac{D}{dt} \sum_j f^j \sigma_j &= \sum_j \left(\frac{df^j}{dt} \cdot \sigma_j + f^j \sum_i c^i_j \sigma_i \right) \\ &= \sum_i \left(\frac{df^i}{dt} + \sum_j c^i_j f^j \right) \sigma_i, \end{aligned}$$
or, equivalently,
$$\frac{df^i}{dt} + \sum_j c^i_j f^j = 0, \quad i = 1, \dots, r. \tag{4}$$
Because this is a *linear* system of ordinary differential equations, it has a unique solution with given values at t_0. □

The proposition allows us to define for any $a, b \in I$ the **holonomy map**, or **parallel transport**,

$$\mathrm{Hol}_a^b := \mathrm{Hol}_a^b(\nabla, \gamma) : E_{\gamma(a)} \overset{\approx}{\to} E_{\gamma(b)}$$

of ∇ along γ. This is the linear isomorphism characterized by the property that if σ is any parallel section of E along γ then

$$\mathrm{Hol}_a^b(\sigma(a)) = \sigma(b).$$

Clearly, if $a, b, c \in I$ then

$$\mathrm{Hol}_a^c = \mathrm{Hol}_b^c \circ \mathrm{Hol}_a^b.$$

We can exploit this property to define Hol_a^b even if γ is only piecewise smooth. Namely, if

$$a = a_0 < a_1 < \cdots < a_n = b$$

and $\gamma_{[a_{i-1}, a_i]}$ is smooth for $i = 1, \ldots, n$ we define Hol_a^b to be the composite of the holonomies along each subinterval, i.e.

$$\mathrm{Hol}_a^b := \mathrm{Hol}_{a_{n-1}}^{a_n} \circ \cdots \circ \mathrm{Hol}_{a_1}^{a_2} \circ \mathrm{Hol}_{a_0}^{a_1}. \tag{5}$$

The following property expresses a connection in terms of its holonomy.

Proposition 4.2 *Let ∇ be a connection in a vector bundle $E \to M$ and s a section of E. Let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve, where $\epsilon > 0$. For $-\epsilon < t < \epsilon$ set*

$$h_t := Hol_t^0(\nabla, \gamma) : E_{\gamma(t)} \to E_{\gamma(0)}.$$

Then

$$\nabla_{\gamma'(0)}(s) = \frac{d}{dt}\bigg|_0 h_t(s(\gamma(t))).$$

Note that $t \mapsto h_t(s(\gamma(t)))$ is a curve in the finite-dimensional vector space $E_{\gamma(0)}$, and we are differentiating this curve at 0 in the usual sense.

Proof. Let $(\sigma_1, \ldots, \sigma_r)$ be a global frame for $\gamma^* E$ consisting of parallel sections. Then $\gamma^* s = \sum_j f^j \sigma_j$ for some functions $f^j : (-\epsilon, \epsilon) \to \mathbb{K}$. Now,

$$h_t(s(\gamma(t))) = \sum_j f^j(t) \cdot \sigma_j(0),$$

and

$$\nabla_{\gamma'(0)}(s) = \frac{D\gamma^* s}{dt}(0) = \sum_j (f^j)'(0) \cdot \sigma_j(0) = \frac{d}{dt}\bigg|_0 h_t(s(\gamma(t))). \qquad \square$$

5 Curvature

The **curvature** $F = F^\nabla$ of a connection ∇ in a $\mathbb{K}$–vector bundle $E \to M$ associates to any pair X, Y of vector fields on M the map $\Gamma(E) \to \Gamma(E)$ given by

$$F(X,Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

If $F(X,Y) = 0$ for all X, Y then ∇ is called **flat**. We will sometimes use the notation

$$[\nabla_X, \nabla_Y] := \nabla_X \nabla_Y - \nabla_Y \nabla_X.$$

Example Let ∇ be the product connection in $E := M \times \mathbb{K}^r \to M$. Identifying sections of E with functions $h : M \to \mathbb{K}^r$ we have $\nabla_X h = Xh$, so

$$\nabla_X \nabla_Y h - \nabla_Y \nabla_X h = XYh - YXh = [X,Y]h = \nabla_{[X,Y]} h.$$

Therefore, the product connection is flat.

Proposition 5.1 *Let ∇ be a connection in a vector bundle $E \to M$. Then The map*

$$\mathcal{X}(M) \times \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X,Y,s) \mapsto F(X,Y)s$$

is $\mathcal{D}(M,\mathbb{K})$–multilinear. Hence, F defines for each $p \in M$ a skew-symmetric bilinear map

$$F_p : T_pM \times T_pM \to End_{\mathbb{K}}(E_p).$$

Proof. We prove linearity in X (and hence in Y because of the skew-symmetry), leaving the linearity in s as an exercise for the reader.

Let $g \in \mathcal{D}(M,\mathbb{K})$. Then

$$[gX, Y] = g[X,Y] - (Yg)X,$$

so

$$\begin{aligned} F(gX,Y) &= \nabla_{gX} \nabla_Y s - \nabla_Y \nabla_{gX} s - \nabla_{[gX,Y]} s \\ &= g\nabla_X \nabla_Y s - [(Yg)\nabla_X s + g\nabla_Y \nabla_X s] - \left[g\nabla_{[X,Y]} s - (Yg)\nabla_X s\right] \\ &= gF(X,Y)s. \qquad \square \end{aligned}$$

It follows from the proposition that if E is trivial and $(s_1, \ldots, s_r)$ is a global frame for E then there is an $r \times r$ matrix (Ω^i_j) of 2–forms on M, called the **curvature form** of ∇ with respect to the given frame, such that for all vector fields X, Y on M one has

$$F(X,Y)s_j = \sum_i \Omega^i_j(X,Y) s_i.$$

Theorem 5.1 *Let ∇ be a connection in a trivial vector bundle $E \to M$, and let $(s_1, \dots, s_r)$ be a global frame for E. Let $\omega = (\omega^i_j)$ and $\Omega = (\Omega^i_j)$ be the corresponding connection form and curvature form of ∇, respectively. Then for all i, k one has*

$$\Omega^i_k = d\omega^i_k + \sum_j \omega^i_j \wedge \omega^j_k.$$

In matrix notation,

$$\Omega = d\omega + \omega \wedge \omega.$$

Proof. Recall that for any 1–forms α, β on M one has

$$\begin{aligned}(\alpha \wedge \beta)(X, Y) &= \alpha(X)\beta(Y) - \alpha(Y)\beta(X), \\ (d\alpha)(X, Y) &= X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).\end{aligned}$$

Now

$$\begin{aligned}\nabla_X \nabla_Y s_k &= \sum_j \nabla_X(\omega^j_k(Y) s_j) \\ &= \sum_j \left(X(\omega^j_k(Y) \cdot s_j + \omega^j_k(Y) \sum_i \omega^i_j(X) s_i \right) \\ &= \sum_i \left(X(\omega^i_k(Y) + \sum_j \omega^i_j(X)\omega^j_k(Y) \right) s_i,\end{aligned}$$

and of course there is a similar formula with X, Y switched. Combining this with

$$\nabla_{[X,Y]} s_k = \sum_i \omega^i_k([X, Y]) s_i$$

we obtain

$$F(X, Y) s_k = \sum_i \left(d\omega^i_k + \sum_j \omega^i_j \wedge \omega^j_k \right) (X, Y) \cdot s_i. \qquad \square$$

Corollary 5.1 *In the situation of the theorem, let $f : \tilde{M} \to M$ be a smooth map and $(\tilde{\Omega}^i_j)$ the curvature form of the pull-back connection $f^*\nabla$ with respect to the global frame $(f^* s_j)$ for f^*E. Then*

$$\tilde{\Omega}^i_j = f^* \Omega^i_j. \qquad \square$$

6 Orthogonal connections

By a **Euclidean metric** on a real vector bundle $E \to M$ we mean a choice of scalar product $\langle \cdot, \cdot \rangle$ on each fibre E_p such that for every pair s, t of sections of E the function

$$M \to \mathbb{R}, \quad p \mapsto \langle s(p), t(p) \rangle$$

is smooth. A real vector bundle equipped with a Euclidean metric is called a **Euclidean vector bundle**.

Example The product bundle $M \times \mathbb{R}^k \to M$ where each fibre has the standard Euclidean metric.

Theorem 6.1 *Every real vector bundle admits a Euclidean metric.*

Proof. Glue together local Euclidean metrics using a partition of unity. □

Let $E \to M$ be a Euclidean vector bundle of rank k. By an **orthonormal frame** for E over an open set $U \subset M$ we mean a k–tuple $(s_1, \dots, s_k)$ of sections of $E|_U$ such that $(s_1(p), \dots, s_k(p))$ is an orthonormal basis for E_p for every $p \in U$. Note that the Gram-Schmidt process applied pointwise to an arbitrary frame for E over U yields an orthonormal frame.

Let $E \to M$ be a real vector bundle equipped with a connection ∇ as well as a Euclidean metric. Then ∇ is called **orthogonal**, or **compatible with the Euclidean metric**, if for all sections s, t of E and vector fields X on M one has

$$X\langle s, t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle.$$

Example If each fibre of a product bundle $E = M \times \mathbb{R}^k \to M$ has the same scalar product, then the product connection in E is orthogonal.

Theorem 6.2 *Every Euclidean vector bundle admits an orthogonal connection.*

Proof. This is proved just as Theorem 2.1. □

For any finite-dimensional Euclidean space V let $\mathrm{so}(V)$ denote the Lie algebra of skew-symmetric linear endomorphisms of V. If $E \to M$ is a Euclidean vector bundle then we can apply this construction fibrewise to obtain a subbundle

$$\mathrm{so}(E) := \bigcup_{p \in M} \mathrm{so}(E_p)$$

of the endomorphism bundle $\mathrm{End}(E)$.

Proposition 6.1 *Let ∇ be an orthogonal connection in a Euclidean vector bundle $E \to M$, and $a : TM \to End(E)$ a bundle homomorphism. Then the connection $\nabla + a$ is orthogonal if and only if a takes values in $so(E)$.*

Proof. Set $\nabla' := \nabla + a$. Then for any vector field X on M and sections s, t of E one has

$$\langle \nabla'_X s, t\rangle + \langle s, \nabla'_X t\rangle = X\langle s, t\rangle + \langle a(X)s, t\rangle + \langle s, a(X)t\rangle.$$

From this the proposition follows immediately. □

Theorem 6.3 *Let $E \to M$ be a real vector bundle equipped with both a connection ∇ and a Euclidean metric. Then the following are equivalent.*

(i) *∇ is orthogonal.*

(ii) *For every piecewise smooth curve $\gamma : [a, b] \to M$ the holonomy $E_{\gamma(a)} \to E_{\gamma(b)}$ of ∇ along γ is an orthogonal linear map.*

Proof. Suppose ∇ is orthogonal and let $\gamma : [a, b] \to M$ be a piecewise smooth curve. We will show that the holonomy of ∇ along γ is orthogonal. Because of the composition law (5) for the holonomy we may assume γ is smooth. We may in fact also assume E is trivial, since for any sufficiently fine partion of the interval $[a, b]$ the image of each subinterval under γ will be contained in an open subset of M over which E is trivial.

Let (ω^i_j) be the connection form of ∇ with respect to a global frame $(s_1, \ldots, s_k)$ for E. Then

$$\begin{aligned}
0 &= X\langle s_i, s_j\rangle \\
&= \langle \nabla_X s_i, s_j\rangle + \langle s_i, \nabla_X s_j\rangle \\
&= \left\langle \sum_k \omega^k_i(X)s_k, s_j \right\rangle + \left\langle s_i, \sum_k \omega^k_j(X)s_k \right\rangle \\
&= \omega^j_i(X) + \omega^i_j(X),
\end{aligned}$$

so that $\omega^i_j = -\omega^j_i$. Now let σ be a parallel section of E along γ. We will show that the pointwise norm $|\sigma|$ is a constant function on $[a, b]$, which implies that the holonomy of ∇ along γ is orthogonal. Since $(\gamma^* s_i)$ is a global orthonormal frame for $\gamma^* E$ we have $\sigma = \sum_i f^i \gamma^* s_i$ for some real-valued functions f^i on the interval $[a, b]$, and

$$|\sigma|^2 = \sum_i (f^i)^2.$$

Let c^i_j be the functions given by $\gamma^*\omega^i_j = c^i_j dt$. Equation (4) now yields

$$\frac{d}{dt}|\sigma(t)|^2 = 2\sum_i f^i \cdot \frac{df^i}{dt} = -2\sum_{ij} f^i f^j c^i_j = 0$$

by the skew-symmetry of the matrix (c^i_j). Hence, σ has constant length as claimed.

To prove the reverse implication in the proposition, suppose (ii) holds and let s_1, s_2 be a pair of sections of E and X a vector field on M. Given a point $p \in M$ we can find $\epsilon > 0$ and a smooth curve

$$\gamma : (-\epsilon, \epsilon) \to M$$

such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. For $-\epsilon < t < \epsilon$ let

$$h_t = \mathrm{Hol}^0_t(\nabla, \gamma) : E_{\gamma(t)} \to E_p$$

be the holonomy of ∇, which is orthogonal by assumption. Using Proposition 4.2 we obtain

$$\begin{aligned} X_p\langle s_1, s_2\rangle &= \frac{d}{dt}\bigg|_0 \langle s_1(\gamma(t)), s_2(\gamma(t))\rangle \\ &= \frac{d}{dt}\bigg|_0 \langle h_t s_1(\gamma(t)), h_t s_2(\gamma(t))\rangle \\ &= \left\langle \frac{d}{dt}\bigg|_0 h_t s_1(\gamma(t)), s_2(p)\right\rangle + \left\langle s_1(p), \frac{d}{dt}\bigg|_0 h_t s_2(\gamma(t))\right\rangle \\ &= \langle \nabla_X s_1, s_2\rangle_p + \langle s_1, \nabla_X s_2\rangle_p, \end{aligned}$$

which shows that ∇ is orthogonal. □

Proposition 6.2 *Let ∇ be an orthogonal connection in a Euclidean vector bundle $E \to M$ and F the curvature of ∇. Then $F(X, Y)$ is a section of $so(E)$ for all vector fields X, Y on M.*

Proof. For any sections s, t of E we have

$$XY\langle s, t\rangle = \langle \nabla_X\nabla_Y s, t\rangle + \langle \nabla_Y s, \nabla_X t\rangle + \langle \nabla_X s, \nabla_Y t\rangle + \langle s, \nabla_X\nabla_Y t\rangle.$$

Combining this with the corresponding equality with X and Y interchanged we obtain

$$\begin{aligned} \langle [\nabla_X, \nabla_Y]s, t\rangle + \langle s, [\nabla_X, \nabla_Y]t\rangle &= [X, Y]\langle s, t\rangle \\ &= \langle \nabla_{[X,Y]} s, t\rangle + \langle s, \nabla_{[X,Y]} t\rangle. \end{aligned}$$

Therefore,

$$\langle F(X, Y)s, t\rangle + \langle s, F(X, Y)t\rangle = 0. \qquad \square$$