(Elliptic inequality)

Theorem: Let E, E' be complex vector bundles over a compact manifold M and $P: \Gamma(E) \to \Gamma(E')$ a P.D.O. of order m and with injective symbol. Then for any integers k, l there is a constant $C < \infty$ such that for every $s \in \Gamma(E)$ one has

$$\|s\|_{(l)} \leq C \left(\|Ps\|_{(l-m)} + \|f\|_{(k)} \right)$$

Proof: It clearly suffices to prove this for $k = l-1$.

(i) We first establish such an inequality locally. Fix $p \in M$. Choose a neighbourhood U of p diffeomorphic to $\mathbb{R}^n$ and a compact subset $K \subset U$ containing a neighbourhood of p. Choose local presentations σ, ε of E, E' over U such that σ, ε involve the same embedding $\varphi: U \to \mathbb{T}^n$, where $\varphi(p) = 0$.

As we have seen before, we can choose a

presentation of E such that for the corresponding Sobolev norms one has

$$\|s\|_{(j)} = \|s_\delta\|_{(j)}$$

for every integer j and section s of E supported in K. Similarly for E'. Let Q be the differential operator on $U' := \varphi(U)$ given by

$$Q s_\delta = (Ps)_\varepsilon$$

for s as above. Choose $g \in C^\infty(\mathbb{T}^n)$ with $\mathrm{supp}(g) \subset U'$ and $g = 1$ on $\varphi(K)$. Applying the previous lemma to the operator gQ we see that there is a neighbourhood $V \subset K$ of p such that for all $s \in \Gamma(E)$ supported in V one has

$$\begin{aligned}\|s\|_{(\ell)} = \|s_\delta\|_{(\ell)} &\leq \mathrm{const}\left(\|gQs_\delta\|_{(\ell-m)} + \|s_\delta\|_{(\ell-2m)}\right)\\ &= \mathrm{const}\left(\|(Ps)_\varepsilon\|_{(\ell-m)} + \|s\|_{(\ell-2m)}\right)\\ &= \mathrm{const}\left(\|Ps\|_{(\ell-m)} + \|s\|_{(\ell-2m)}\right).\end{aligned}$$

(ii) Because M is compact we can cover M by finitely many open sets $V_1, \dots, V_r$ of the kind found in (i). Choose a partition of unity $\{\gamma_j\}$ on M subordinate to $\{V_j\}$. Then $P_j := [P, \gamma_j]$ has order at most $m-1$, so for all $s \in \mathcal{P}(E)$,

$$\begin{aligned}
\|s\|_{(\ell)} &\leq \sum_j \|\gamma_j s\|_{(\ell)} \\
&\leq \text{const} \sum_j \left(\|P\gamma_j s\|_{(\ell-m)} + \|\gamma_j s\|_{(\ell-2m)} \right) \\
&\leq \text{const} \sum_j \left(\|\gamma_j P s\|_{(\ell-m)} + \|P_j s\|_{(\ell-m)} + \|s\|_{(\ell-2m)} \right) \\
&\leq \text{const} \sum_j \left(\|Ps\|_{(\ell-m)} + \|s\|_{(\ell-1)} \right).
\end{aligned}$$

//.

Theorem: Let E, E' be complex vector bundles over a compact manifold M, and

$$P : \Gamma(E) \to \Gamma(E')$$

an elliptic operator of order m. Then for every integer l the induced operator

$$P_{(l)} : H_l(M, E) \to H_{l-m}(M, E')$$

is a Fredholm operator.

Proof: By the previous theorem, $P_{(l)}$ has finite-dimensional kernel and closed image. Choose a Riemannian metric on M and Hermitian metrics on E and E'. Then the formal adjoint $P^* : \Gamma(E') \to \Gamma(E)$ is also elliptic, and the duality pairing

$$H_k(M, E') \times H_{-k}(M, E') \to \mathbb{C}$$

for $k = l - m$ gives rise to a conjugate-linear isomorphism

$$\left(\operatorname{im} P_{(l)}\right)^{\perp} \simeq \ker P^{*}_{(m-l)} .$$

Hence $\operatorname{im} P_{(l)}$ has finite codimension. //

Def: Let $E \to M$ be a vector bundle, where M is compact, and $U \overset{open}{\subset} M$.

(i) Let $\Gamma_U(E) := \{ s \in \Gamma(E) \mid supp(s) \subset U \}$

(ii) For any $\ell \in \mathbb{Z}$ let $\mathring{H}_\ell(U, E)$ denote the closure of $\Gamma_U(E)$ in $H_\ell(M, E)$.

Lemma: If U, V are open subsets of M with $\bar{V} \subset U$, and k, ℓ are integers with $k \leq \ell$, then

$$\mathring{H}_k(V, E) \cap H_\ell(M, E) \subset H_\ell(U, E),$$

where all spaces are thought of as linear subspaces of $H_k(M, E)$.

Proof: Let $s \in \mathring{H}_k(V, E) \cap H_\ell(M, E)$.

Choose a sequence $\{t_j\}$ in $\Gamma(E)$ with

$$\| t_j - s \|_{(\ell)} \to 0 \text{ as } j \to \infty.$$

Choose $\beta \in C^\infty(M)$ with $\mathrm{supp}(\beta) \subset U$ and $\beta = 1$ on V. Set $s_j := \beta t_j$. Then $s_j - s = \beta(t_j - s)$, so

$$\|s_j - s\|_{(\ell)} \leq \mathrm{const}\, \|t_j - s\|_{(\ell)} \to 0.$$

Therefore, $s \in \mathring{H}_\ell(U, E)$. //

(Elliptic regularity)

<u>Theorem</u>: Let E, E' be complex vector bundles over a compact manifold M, and $P: \Gamma(E) \to \Gamma(E')$ a differential operator of order m and with injective symbol. Let k, l be integers.

(i) If $s \in H_l(M,E)$ and $P_{(l)} s \in H_{k-m}(M,E')$ then $s \in H_k(M,E)$.

(ii) If $s \in H_l(M,E)$ and $P_{(l)} s$ is smooth, then s is smooth.

<u>Proof</u>: First observe that (ii) follows from (i), since $\bigcap_k H_k(M,E) = \Gamma(E)$. To prove (i), it clearly suffices to consider $k = l-1$.

<u>Step 1</u>: We first show that any point $p \in M$ has a neighborhood V such that if $s \in \mathring{H}_l(V,E)$ and $P_{(l)} s \in H_{l-m+1}(M,E)$ then $s \in H_{l+1}(M,E)$.

As shown earlier we can find

* a neighbourhood U_0 of p,
* local presentations δ, ε of E, E' over U_0 which involve the same embedding $\varphi: U_0 \to \mathbb{T}^n$,
* a differential operator

$$Q: C^\infty(\mathbb{T}^n, \mathbb{C}^d) \to C^\infty(\mathbb{T}^n, \mathbb{C}^{d'})$$

of order m such that for any $s \in \Gamma(E)$ with support in U_0 one has $(Ps)_\varepsilon = Q s_\delta$, and such that $Q^* Q + 1$ induces an isomorphism

$$H_k(\mathbb{T}^n, \mathbb{C}^d) \xrightarrow{\cong} H_{k-2m}(\mathbb{T}^n, \mathbb{C}^d)$$

for $k = l, l+1$.

Choose neighbourhoods U, V of p such that

$$\bar{V} \subset U, \quad \bar{U} \subset U_0.$$

and write $U' := \varphi(U)$, $V' := \varphi(V)$.

As shown earlier, we can find presentations of E, E' such that for the corresponding Sobolev norms one has, for any $k \in \mathbb{Z}$ and any $s \in \Gamma(E)$, $t \in \Gamma(E')$ supported in U,

$$\|s\|_{(k)} = \|s_\delta\|_{(k)}$$

$$\|t\|_{(k)} = \|t_\varepsilon\|_{(k)}.$$

For any open subset $W \subset U$ the local presentation δ induces an isomorphism

$$\mathring{H}_k(W, E) \xrightarrow{\cong} H_k(\varphi(W), \mathbb{C}^d), \quad s \mapsto \delta s,$$

for any $k \in \mathbb{Z}$, and similarly for E'.

Now suppose $s \in \mathring{H}_\ell(V, E)$ and $Ps \in H_{\ell-m+1}(M, E)$ (where we write P instead of $P_{(\ell)}$ for simplicity).

By the previous lemma, $Ps \in H_{\ell-m+1}(U, E')$.

Let $t := \delta s \in \mathring{H}_\ell(V, \mathbb{C}^d)$. Then

$$Qt = \varepsilon Ps \in \mathring{H}_{\ell-m+1}(U', \mathbb{C}^{d'}).$$

Here we made use of the commutative diagram

$$\begin{tikzcd} \mathring{H}_\ell(V,E) \arrow[r, "P"] \arrow[d, "\delta", "\cong"'] & \mathring{H}_{\ell-m}(\mathcal{U},E') \arrow[r, phantom, "\supset"] \arrow[d, "\varepsilon", "\approx"'] & H_{\ell-m+1}(\mathcal{U},E') \arrow[d, "\varepsilon", "\approx"'] \\ \mathring{H}_\ell(V',\mathbb{C}^d) \arrow[r, "Q"] & \mathring{H}_{\ell-m}(\mathcal{U}',\mathbb{C}^{d'}) \arrow[r, phantom, "\supset"] & H_{\ell-m+1}(\mathcal{U}',\mathbb{C}^d) \end{tikzcd}$$

This yields

$$(Q^*Q+1)t \in \mathring{H}_{\ell-2m+1}(\mathcal{U}', \mathbb{C}^{d'})$$

$$\Rightarrow \; t \in H_{\ell+1}(\Pi^n, \mathbb{C}^d)$$

$$\Rightarrow \; t \in \mathring{H}_{\ell+1}(\mathcal{U}', \mathbb{C}^d).$$

Now observe that we have a commutative diagram

$$\begin{tikzcd}
\mathring{H}_{\ell+1}(\mathcal{U}, E) \arrow[r, "\cong"] \arrow[d] & \mathring{H}_{\ell+1}(\mathcal{U}', \mathbb{C}^d) \arrow[d] \\
\mathring{H}_{\ell}(\mathcal{U}, E) \arrow[r, "\cong"] & \mathring{H}_{\ell}(\mathcal{U}', \mathbb{C}^d)
\end{tikzcd},$$

where the horizontal isomorphisms are induced by δ, and the vertical maps are the canonical ones. Because $s \in \mathring{H}_\ell(V, E) \subset \mathring{H}_\ell(\mathcal{U}, E)$ and $t = \delta s$ lifts to $\mathring{H}_{\ell+1}(\mathcal{U}', \mathbb{C}^d)$, it follows that s lifts to $\mathring{H}_{\ell+1}(\mathcal{U}, E)$.

Step 2: Since M is compact, we can cover it by finitely many open sets V_j as provided by Step 1. Let $\{g_j\}$ be a partition of unity on M subordinate to $\{V_j\}$.

Suppose $s \in H_\ell(M,E)$ and $Ps \in H_{\ell-m+1}(M,E')$.

Because $[P,g_j]$ has order $\leq m-1$,

$$Pg_js = [P,g_j]s + g_jPs \in H_{\ell-m+1}(M,E').$$

But $g_js \in \mathring{H}_\ell(V_j,E)$, so $g_js \in H_{\ell+1}(M,E)$.

It follows that

$$s = \sum_j g_js \in H_{\ell+1}(M,E). \qquad //$$