Geometry and Analysis, Fall 2018

Problem sheet 1, to be discussed Friday the 31th August.

Problem 1. Let E, E' be vector bundles over M and $F : E \to E'$ a bijective bundle homomorphism. Show that F is a bundle isomorphism.

Problem 2. Let E, E' be vector bundles over M and $F : E \to E'$ a bundle homomorphism which has *constant rank*, i.e. $F(E_p)$ has the same dimension for all $p \in M$. Show that the kernel

$$\ker(F) := \bigcup_{p \in M} \ker(F_p)$$

and image

$$\mathrm{im}(F) := \bigcup_{p \in M} \mathrm{im}(F_p)$$

of F are subbundles of E and E', respectively.

Problem 3. For $n \geq 1$ let $\mathbb{CP}^n$ be complex projective n–space, which is the space of all 1–dimensional linear subspaces of $\mathbb{C}^{n+1}$. Recall that $\mathbb{CP}^n$ is in a natural way a smooth $2n$–manifold. Show that

$$L := \{(\lambda, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid v \in \lambda\}$$

is a rank 1 subbundle of the product bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1} \to \mathbb{CP}^n$.

Problem 4. Let E and E' be $\mathbb{K}$–vector bundles over M equipped with connections ∇ and ∇', respectively. Show that there exists exactly one connection $\tilde{\nabla}$ in the bundle $\mathrm{Hom}(E, E')$ over M such that for all vector fields X on M and sections $u \in \Gamma(\mathrm{Hom}(E, E'))$ and $s \in \Gamma(E)$ one has

$$\nabla'_X(us) = \tilde{\nabla}_X u \cdot s + u \cdot \nabla_X s.$$