Geometry and Analysis, Fall 2018

Compulsory assignment, to be returned through Devilry by Friday the 26th October.

Problem 1. Let $E \to M$ be a Euclidean or Hermitian vector bundle and $K \subset E$ a subbundle. For each $p \in M$ let $K_p^\perp$ denote the orthogonal complement of the fibre K_p in E_p. Show that

$$K^\perp := \bigcup_{p \in M} K_p^\perp$$

is a subbundle of E.

Problem 2. Given a vector bundle $E \to M$ and a subbundle $K \subset E$, show that

$$E/K := \bigcup_{p \in M} E_p/K_p$$

has a unique structure of vector bundle over M such that the projection map $E \to E/K$ is a bundle homomorphism.

Problem 3. Let ∇ be a connection in a vector bundle $E \to M$. A subbundle $K \subset E$ is *preserved by* ∇ if for every vector field X on M the operator ∇_X maps sections of K to sections of K. Show that if E carries a Euclidean metric compatible with ∇ then $\mathrm{so}(E)$ is preserved by the connection in $\mathrm{End}(E)$ induced by ∇.

Problem 4. Let E, E' be vector bundles over M equipped with connections ∇, ∇', respectively, and T a section of $\mathrm{Hom}(E, E')$ which is parallel with respect to the induced connection. (In that case we say that ∇ and ∇' are T*–related.*)

(i) Prove that for any curve $\gamma : [a, b] \to M$ the holonomies of ∇ and ∇' along γ are related by

$$T_{\gamma(b)} \cdot \mathrm{Hol}_\gamma(\nabla) = \mathrm{Hol}_\gamma(\nabla') \cdot T_{\gamma(a)}.$$

Here, both sides are maps $E_{\gamma(a)} \to E'_{\gamma(b)}$.

(ii) Prove that T has constant rank.

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Problem 5. Let E, E' be vector bundles over M and $\pi : E \to E'$ a surjective bundle homomorphism. Let ∇ be a connection in E which preserves the kernel of π. Show that E' has a unique connection which is π–related to ∇.

Problem 6. Let Σ be a compact, connected, oriented 2–manifold whose boundary $\partial\Sigma$ has one component. Let $L \to \Sigma$ be a (necessarily trivial) complex line bundle equipped with a connection ∇. Show that the holonomy of ∇ along $\partial\Sigma$ in the positive direction is given by

$$\mathrm{Hol}_{\partial\Sigma}(\nabla) = \exp\left(-\int_\Sigma F(\nabla)\right).$$