MAT4701 - Stochastic Analysis with Applications

Exercises about Conditonal Expectations, Stopping Times, Martingales

Exercise 1

Let $(X_n)_{n\in\mathbb{N}}$ be a submartingale w.r.t. the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}, \exists k \geq n : \mathbb{E}(X_k) \leq \mathbb{E}(X_n).$$

Show that $(X_n)_{n\in\mathbb{N}}$ is a martingale!

Exercise 2

Let $X_1, X_2, \ldots$ be i.i.d., integrable random variables with $\mathbb{E}(X_1) = 1$. Show that $M_n := \prod_{k=1}^n X_k$ is a martingale with respect to the filtration $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$

Exercise 3

Let $X_1, X_2, \ldots$ be i.i.d. with $P(X_1 = -\frac{1}{2}) = P(X_1 = \frac{1}{2}) = \frac{1}{2}$ and

$$M_n := \prod_{i=1}^n (1 + X_k).$$

(i) Show hat $(M_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. an adequate filtration.

(ii) Show that for $a \in \mathbb{R}$

$$\tau := \min\{k \geq 1 : X_k = a\}$$

is a stopping time w.r.t. the natural filtration.

Exercise 4

Let $X_1, X_2, \ldots$ be i.i.d. random variables which are square-integrable, but not constant. Let

$$M_n := \sum_{k=1}^n X_k X_{k+1}.$$

Under which conditions is $(M_n)_{n\in\mathbb{N}}$ w.r.t. the filtration $\mathcal{F}_n := \sigma(X_1, \ldots, X_{n+1})$ a martingale?

Exercise 5

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{N}(-a, 1)$ random variables and $a > 0$.

(i) For which value of $h \in \mathbb{R}$ is the process $Y_n = \exp(h \sum_{i=1}^n X_i)$ a martingale with respect to the natural filtration $\mathcal{F} := \sigma(X_1, \ldots, X_n)$?

(ii) For which values of h is it a sub- or supermartingale?

(iii) Assume $h = 2a$, let $x > 0$. Define $S_n := \sum_{i=1}^n X_i$. Show that

$$P(\sup_{n\geq 1} S_n > x) \leq e^{-2ax}.$$

Exercise 6
Let $(X_n)_{n\geq 1}$ i.i.d. random variables with density

$$f(x) = \frac{1}{2}e^{-|x|}, \;\; x \in \mathbb{R}.$$

Find $A(\theta)$ such that $\forall|\theta| < 1$ the sequence

$$M_n = M_n(\theta) = \exp\big[\theta \sum_{i=1}^{n} X_k - nA(\theta)\big]$$

is a martingale w.r.t. the natural filtration $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$.
Exercise 7
Let $(\Omega, \mathcal{F}, P)$ a probability space, $X \in L^2(\Omega, \mathcal{F}, P)$ a random variable and $\mathcal{A} \subset \mathcal{F}$ a sub-σ-algebra. The expected variance of X given $\mathcal{A}$ is defined by:

$$\mathrm{Var}[X|\mathcal{A}] := \mathbb{E}[(X - \mathbb{E}[X|\mathcal{A}])^2|\mathcal{A}].$$

Show that

$$\mathrm{Var}[X] = \mathbb{E}[\mathrm{Var}[X|\mathcal{A}]] + \mathrm{Var}[\mathbb{E}[X|\mathcal{A}]]$$

Exercise 8
Let $(X_k)_{k\geq 1}$ be i.i.d. random variables with $P(X_k = k) = \frac{1}{k}$ and $P(X_k = 0) = \frac{k-1}{k}$. Let $M_n = \prod_{k=1}^{n} X_k$, $M_0 = 0$. Is $(M_n)_{n\in\mathbb{N}}$ a martingale?
Exercise 9
Let $\xi_1, \xi_2, \dots$ i.di.d with $\mathbb{E}[\xi_j^2] = \sigma^2 < \infty$ and $\mathbb{E}[\xi_j] = 0$. Then, $X_n := \sum_{i=1}^{n} \xi_i$ is a martingale and X_n^2 is a submartingale. Find the increasing process A_n in the Doob decomposition of X_n^2, i.e. find A_n s.t. $X_n^2 - A_n$ is a martingale.

Solution of Exercise 1
We consider the Doob composition $X_n = M_n + A_n$ with M_n a martingale and $A_n \geq 0$ an increasing predictable process. It follows

$$\mathbb{E}[X_n] = \underbrace{\mathbb{E}[M_n]}_{=\mathbb{E}[M_{n+1}]} + \underbrace{\mathbb{E}[A_n]}_{\geq \mathbb{E}[A_{n-1}]} \geq \mathbb{E}[X_{n-1}]. \tag{1}$$

Hence, the sequence of $(\mathbb{E}[X_n])_{n\in\mathbb{N}}$ is an increasing sequence of real numbers. On the other hand, by

$$\forall n \in \mathbb{N}, \exists k \geq n : \mathbb{E}(X_k) \leq \mathbb{E}(X_n)$$

we can a extract a decreasing a subsequence $(\mathbb{E}[X_{n_k}])_{k\in\mathbb{N}}$. Hence, the whole sequence $(\mathbb{E}[X_n])_{n\in\mathbb{N}}$ must be constant. By (1), this is only possible if $\mathbb{E}[A_n] = 0$ for all n. Since $A_n \geq 0$, it follows $A_n = 0$. Hence, $X_n = M_n$ is a martingale.
Solution of Exercise 2
By definition, M_n is adapted to $\mathcal{F}_n$.
Furthermore by independece, we have

$$\mathbb{E}[|M_n|] = \mathbb{E}[|\prod_{i=1}^{n} X_i|] = \prod_{i=1}^{n} \mathbb{E}[|X_i|] < \infty.$$

It's left to check the Martingale property:

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] = \mathbb{E}[\underbrace{\prod_{i=1}^{n-1} X_i}_{\mathcal{F}_{n-1}-\text{meas.}} \cdot X_n|\mathcal{F}_{n-1}] = \prod_{i=1}^{n-1} X_i \cdot \underbrace{\mathbb{E}[X_n]}_{\text{ind. of } \mathcal{F}_{n-1}} \mathcal{F}_{n-1}] = \prod_{i=1}^{n-1} X_i = M_{n-1}.$$

Solution of Exercise 3(i)
$\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$ is a filtration which (M_n) is adapted to. As a finite random variable M_n is also integrable. Furthermore, we have

$$\begin{aligned}\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}|M_n(1 + X_{n+1})|\mathcal{F}_n] = M_n\mathbb{E}[1 + X_{n+1}|\mathcal{F}_n]M_n\mathbb{E}[1 + X_{n+1}] \\ &= M_n(\frac{1}{2} \cdot \frac{1}{2} + \frac{3}{2} \cdot \frac{1}{2}) = M_n.\end{aligned}$$

Solution of Exercise 3(ii)

$$\begin{aligned}\{\tau = n\} &= \{X_1, \ldots, X_{n-1} \neq a, X_n = a\} \\ &= \{M_1 - 1 \neq a, \frac{M_2}{M_1} - 1 \neq a, \ldots, \frac{M_n}{M_{n-1}} - 1 \neq a\} \in \sigma(M_1, \ldots, M_n)\end{aligned}$$

Solution of Exercise 4
M_n is obviously adapted.

By Cauchy-Schwarz, each summand $X_k X_{k+1}$ is in L^1 and hence M_n is integrable. (This can also be seen by exploiting independence of X_k.)
Hence, M_n is and integral if and only if

$$M_n \overset{!}{=} \mathbb{E}[\underbrace{M_n}_{\mathcal{F}_n-\text{meas.}} + \underbrace{X_{n+1}}_{\mathcal{F}_n-\text{meas.}} X_{n+2}|\mathcal{F}_n] = M_n + X_{n+1}\mathbb{E}[X_{n+2}|\mathcal{F}_n] = M_n + X_{n+1}\mathbb{E}[X_{n+2}]$$

Since X_i are not constant, it follows that $\mathbb{E}[X_i] = 0$ must hold.

Solution of Exercise 5

i)We have $e^{hS_n} = \prod_{1\le i\le n} e^{hX_i}$ which is adapted to the given filtration. Because of the independence of X_i, we have

$$\mathbb{E}[e^{hS_n}] = \prod_{1\le i\le n} \mathbb{E}[e^{hX_i}]$$

and it suffices to show the integrability of e^{hX_1}:

$$\begin{aligned} \mathbb{E}[e^{hX_1}] &= \int e^{hx} \frac{1}{\sqrt{2\pi}} e^{-(x+a)^2/2} dx \\ &= e^{-ha} e^{h^2/2} \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-(x-h)^2/2} dx}_{=1, \text{ (density)}} \\ &= e^{-ha+h^2/2} < \infty. \end{aligned} \tag{2}$$

Hence, it is left to prove that

$$\mathbb{E}[e^{hS_n}|\mathcal{F}_{n-1}] = e^{hS_{n-1}} \cdot \underbrace{\mathbb{E}[e^{hX_n}|\mathcal{F}_{n-1}]}_{=\mathbb{E}[e^{hX_n}]} \overset{!}{=} e^{hS_{n-1}} \tag{3}$$

holds, i.e. $\mathbb{E}[e^{hX_n}] \overset{\text{i.i.d.}}{=} \mathbb{E}[e^{hX_1}] \overset{!}{=} 1$. From (2), we see that this is the case if and only if $h = 2a$ or $h = 0$.

ii) For $h \in [0, 2a]$ we get a supermartingale, for $h \le 0$ or $h \ge 2a$ we get a submartingale (which we can see by plugging in these values in the equation (3) above):

$$Y_n \text{ supermartingale } \Leftrightarrow \mathbb{E}[e^{hX_1}] \le 1 \Leftrightarrow h \in [0, 2]$$

and

$$Y_n \text{ submartingale } \Leftrightarrow \mathbb{E}[e^{hX_1}] \ge 1 \Leftrightarrow h \in (-\infty, 0] \cup [2, \infty).$$

iii) We apply Doob's martingale inequality:

$$\begin{aligned} P(\sup_{n\ge 1} S_n > x) &\overset{\text{MCT}}{=} \lim_{k\to\infty} P(\sup_{n\le k} X_n > x) = \lim_{k\to\infty} P(\sup_{n\le\infty} e^{hS_n} < e^{hx}) \\ &\le \lim_{k\to\infty} \frac{\mathbb{E}[e^{hS_n}]}{e^{hx}} = \frac{1}{e^{hx}} = e^{-2ax}. \end{aligned}$$

Solution of Exercise 6
Measurability is obvious.
From

$$M_n = \exp(\theta \sum_{k=1}^{n} X_k - nA(\theta)) = \exp(\sum_{k=1}^{n} (\theta X_k - A(\theta))) = \prod_{k=1}^{n} \underbrace{\exp((\theta X_k - A(\theta)))}_{=:Y_k} \tag{4}$$

we obtain

$$\begin{aligned}
\mathbb{E}[M_n] &:= \prod_{k=1}^{n} \int_0^\infty \frac{1}{2} e^{-|x|} \exp(\theta x - A(\theta)) dx \\
&= \prod_{k=1}^{n} \frac{1}{2} (\int_0^\infty e^{-x} \exp(\theta x - A(\theta)) dx + \int_0^\infty \exp(-(\theta+1)x - A(\theta)) dx) \\
&= \prod_{k=1}^{n} \frac{1}{2} (\int_0^\infty \exp[(\theta - 1)x - A(\theta)] ds + \int_0^\infty \exp[-(\theta+1)x - A(\theta)] ds).
\end{aligned} \tag{5}$$

Since $\theta - 1, -\theta - 1 < 0$, this is integrable for every choice of $A(\theta)$.
By independence, we get

$$\begin{aligned}
\mathbb{E}[M_{n+1} | \underbrace{\sigma(M_1, \ldots, M_n)}_{=\sigma(X_1, \ldots, X_n)}] &= M_n \cdot \mathbb{E}[\exp[\theta X_{n+1} - A(\theta))] \\
&= M_n \cdot \mathbb{E}[\exp[(\theta X_1 - A(\theta))]].
\end{aligned}$$

So, we have to chose $A(\theta)$ such that

$$\begin{aligned}
1 &= \mathbb{E}[\exp(\theta X_1 - A(\theta))] \\
&= \frac{1}{2} (\int_0^\infty \exp[(\theta - 1)x - A(\theta)] dx + \int_0^\infty \exp[-(\theta+1)x - A(\theta)] dx) \\
&= \frac{1}{2} (\frac{1}{\theta - 1} \exp((\theta - 1)x - A(\theta)) \Big|_0^\infty - \frac{1}{\theta + 1} \exp(-(\theta+1)x - A(\theta)) \Big|_0^\infty) \\
&= \frac{\exp(-A(\theta))}{(1 - \theta^2)}
\end{aligned}$$

holds. Hence, we have to chose $A(\theta) = -\ln(1 - \theta^2)$.
Solution of Exercise 7
We define $\bar{X} := \mathbb{E}[X|\mathcal{A}]$:

$$
\begin{aligned}
\mathbb{E}[\mathrm{Var}[X|]] + \mathrm{Var}[\bar{X}] &= \mathbb{E}[\mathbb{E}[(X-\bar{X})^2|\mathcal{A}]] + \mathrm{Var}[\bar{X}] \\
&= \mathbb{E}[\mathbb{E}[(X-\bar{X})^2|\mathcal{A}]] + \mathbb{E}[\bar{X}^2] + \mathrm{Var}[\bar{X}] \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[\mathbb{E}[X \underbrace{\bar{X}}_{\mathcal{A}-\text{ meas.}} |\mathcal{A}]] + \mathbb{E}[\bar{X}^2] + \mathrm{Var}[\bar{X}] \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[\bar{X}\cdot\bar{X}] + \mathbb{E}[\bar{X}^2] + \mathbb{E}[\bar{X}^2] - \mathbb{E}[\bar{X}]^2 \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[\bar{X}^2] + 2\mathbb{E}[\bar{X}^2] - \mathbb{E}[X^2]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathrm{Var}[X] \qquad (6)
\end{aligned}
$$

Solution of Exercise 8

Yes, M_n is a martingale with respect to $\mathcal{F}_n := \sigma(M_0, \ldots, M_n)$. It is adapted by definition and integrable because of

$$\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \prod_{i=1}^{n} \mathbb{E}[X_i] = 1.$$

Finally, we have:

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \prod_{i=1}^{n} \mathbb{E}[X_i|\mathcal{F}_n] \cdot \mathbb{E}[X_{n+1}|\mathcal{F}_n] = \prod_{i=1}^{n} X_i \mathbb{E}[X_{n+1}] = M_n \cdot 1 = M_n.$$

Solution of Exercise 9

Claim: $A_n = n\sigma^2$

Proof. A_n is positive, increasing and as a constant random variable $\mathcal{F}_{n-1}$ measurable. Furthermore,

$$
\begin{aligned}
\mathbb{E}[X_n^2 - n\sigma^2|\mathcal{F}_{n-1}] &= \sum_{1\le i,j\le n-1} \underbrace{\mathbb{E}[\xi_i\xi_j|\mathcal{F}_{n-1}]}_{=\xi_i\xi_j} + \underbrace{\mathbb{E}[\xi_n^2]}_{=\sigma^2} + 2\sum_{j=1}^{n-1} \underbrace{\mathbb{E}[\xi_j\xi_n|\mathcal{F}_{n-1}]}_{=\mathbb{E}[\xi_j]\mathbb{E}[\xi_n]=0} - n\sigma^2 \\
&= X_{n-1}^2 - (n-1)\cdot\sigma^2
\end{aligned}
$$

□