# Solutions to the exam in MAT4720

**Problem 1**
(i) See e.g. Def. 2.9 in the lecture notes (LN).
(ii) See e.g. Def. 2.10, LN.
(iii) See e.g. Th. 9.1, LN.
(iv) See e.g. Def. 8.9, LN.
(v) See e.g. Th. 8.22, LN.

**Problem 2**
(i) Since $B_t - B_s$ is independent of events in $\mathcal{F}_s$ for $t \geq s$ and has mean zero, we obtain by using the properties of conditional expectations that

$$\begin{aligned} E[B_t + 3t \,|\mathcal{F}_s] &= E[B_t - B_s \,|\mathcal{F}_s] + E[B_s + 3t \,|\mathcal{F}_s] \\ &= E[B_t - B_s] + B_s + 3t \\ &= B_s + 3t \neq B_s + 3s \end{aligned}$$

for $t > s$. So $X_t, t \geq 0$ cannot be a martingale.

(ii) Since the above integrand processes are continuous transformations of the Brownian motion, the integrand processes are measurable and adapted.

We observe that

$$E[\int_0^T B_s^2 ds] = \int_0^T E[B_s^2] ds = \int_0^T s ds < \infty$$

and

$$E[\int_0^T \exp(2\sin(B_s)) ds] \leq E[\int_0^T e^2 ds] < \infty,$$

that is the integrand processes w.r.t. $X_t, Y_t, 0 \leq t \leq T$ are in $V(0,T)$. So a versions of $X_t, Y_t, 0 \leq t \leq T$ are by the properties of Itô integral processes square integrable martingales.

Assume that $Z_t, 0 \leq t \leq T$ is a square integrable martingale. Define the sequence of stopping times $\tau_k \nearrow \infty$ a.e. by

$$\tau_k := \inf\{t > 0 : \int_0^t \exp(4B_s^2) ds \geq k\}, k \in \mathbb{N}.$$

On the other hand, we have for $t \geq s$

$$E[Z_t^2 \,|\mathcal{F}_s] \geq (E[Z_t \,|\mathcal{F}_s])^2 = Z_s^2$$

since for $A \in \mathcal{F}_s$

$$1_A Z_t = E[1_A Z_t \,|\mathcal{F}_s] + Y,$$

where $Y$ is orthogonal to $E[1_A Z_t \,|\mathcal{F}_s]$ (see Lemma 7.6, LN), which implies

$$E[1_A Z_t^2] \geq E[1_A (E[Z_t \,|\mathcal{F}_s])^2]$$

for all $A \in \mathcal{F}_s$ and hence the above inequality (Jensen's inequality).

Thus $Z_t^2, 0 \leq t \leq T$ is a continuous submartingale. So by Problem 5 (ii), Itô's isometry and e.g. Exerc. 3, Prob. 5 we get

$$\begin{aligned} E[Z_T^2] &\geq E[Z_{T\wedge\tau_k}^2] = E[(\int_0^{T\wedge\tau_k} \exp(2B_s^2)dB_s)^2] \\ &= E[\int_0^T 1_{[s,\infty)}(T\wedge\tau_k)\exp(4B_s^2)ds]. \end{aligned}$$

Using Fatou's Lemma, we see that

$$E[\int_0^T \exp(4B_s^2)ds] \leq E[Z_T^2] < \infty.$$

However,

$$\begin{aligned} \int_0^T E[\exp(4B_s^2)]ds &= \int_0^T \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s}} \exp(-\frac{y^2}{2s}) \exp(4y^2)dyds \\ &\geq \int_1^T \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s}} \exp(-\frac{y^2}{2s}) \exp(4y^2)dyds \\ &\geq \int_1^T \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s}} \exp(-\frac{y^2}{2}) \exp(4y^2)dyds \\ &= \int_1^T \frac{1}{\sqrt{2\pi s}}ds \int_{\mathbb{R}} \exp(\frac{7y^2}{2})dy = \infty, \end{aligned}$$

which is a contradiction.

(iii) It follows from the independent and normally distributed increments of the Brownian motion, the properties of conditional expectations and the fact that $B_s^2$ is $\mathcal{F}_s-$measurable that

$$\begin{aligned} E[\exp(B_t - B_s)B_s^2 \,|\mathcal{F}_s] &= B_s^2 E[\exp(B_t - B_s) \,|\mathcal{F}_s] \\ &= B_s^2 E[\exp(B_t - B_s)] \\ &= B_s^2 \exp(-\frac{1}{2}Var[B_t - B_s]) \\ &= B_s^2 \exp(-\frac{1}{2}(t-s)), t \geq s. \end{aligned}$$

(iv) Choosing the $C^2$-function $g(t,x) := (x+t)\exp(-x-\frac{1}{2}t)$ gives

$$\begin{aligned} X_t &= g(t,B_t) \\ &= 0 + \int_0^t (\exp(-B_s - \frac{1}{2}s) - \frac{1}{2}X_s)ds + \int_0^t (\exp(-B_s - \frac{1}{2}s) - X_s)dB_s \\ &\quad + \frac{1}{2}\int_0^t (-\exp(-B_s - \frac{1}{2}s) - \exp(-B_s - \frac{1}{2}s) + X_s)ds \\ &= \int_0^t (\exp(-B_s - \frac{1}{2}s) - X_s)dB_s. \end{aligned}$$

Since the integrand process is in $V(0,T)$ for all $T$, the stochastic integral process has a continuous martingale version.

**Problem 3**

The derivative of the drift function is given by $\frac{2x}{1+x^2}$, which is a bounded function. On the other hand,

$$\log(1+x^2) \leq \log((1+|x|)^2) = 2\log(1+|x|) \leq 2(1+|x|).$$

So the drift function satisfies the linear growth and Lipschitz condition (globally). Obviously, the same applies to the diffusion coefficient. Hence, it follows from the existence and uniqueness theorem for strong solutions of SDE's that the above equation has a unique stong solution on $[0,T]$.

(ii) Let $\mu_t, \sigma_t, 0 \leq t \leq T$ be bounded, measurable and $\mathcal{F}_t-$adapted processes.

Using Itô's formula applied to the Itô process $Y_t := \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dB_s$ and the function $g(y) := x\exp(y)$, we find that

$$\begin{aligned} X_t &= g(Y_t) = x + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)X_s ds + \int_0^t \sigma_s X_s dB_s \\ &\quad + \frac{1}{2}\int_0^t \sigma_s^2 X_s ds \\ &= x + \int_0^t \mu_s X_s ds + \int_0^t \sigma_s X_s dB_s \end{aligned}$$

So $X_t, 0 \leq t \leq T$ is a solution to the SDE

$$dX_t = \mu_t X_t dt + \sigma_t X_t dB_t, X_0 = x, 0 \leq t \leq T.$$

The coefficients of the SDE are stochastic. Therefore we cannot directly apply the existence and uniqueness theorem for SDE's.

It follows from the mandatory assignment, Prob. 6, that

$$E[\int_0^T X_t^2 dt] < \infty.$$

Let us assume there is another solution $Z_t, 0 \leq t \leq T$ satisfying the same integrability condition. Then, using Itô's isometry and Hölder's inequality that

$$\begin{aligned} E[|X_t - Z_t|^2] &\leq 2E[(\int_0^t \mu_s (X_s - Z_s) ds)^2 + (\int_0^t \sigma_s (X_s - Z_s) dB_s)^2] \\ &\leq C \int_0^t E[|X_s - Z_s|^2] ds \end{aligned}$$

for a constant $C$ depending of the size of the processes $\mu_t, \sigma_t, 0 \leq t \leq T$ and the time horizon $T$.

So Gronwall's Lemma shows that

$$E[|X_t - Z_t|^2] \leq 0 \exp(C) = 0$$

for all $t$. So $X_t = Z_t$ with prob. 1 for all $t$. Hence the solution is unique.

**Problem 4** Consider a 1−dimensional Brownian motion $B_t, 0 \leq t \leq T$ and denote by $\mathcal{F}_t, 0 \leq t \leq T$ its natural filtration.

(i) Prove that the process $X_t := B_t^3 - 3tB_t$ is martingale using Itô's formula.

Using Itô's formula applied to $g(t, x) := x^3 - 3tx$, we get that

$$\begin{aligned} X_t &= g(t, B_t) = \int_0^t (-3B_s) ds + \int_0^t (3B_s^2 - 3s) dB_s \\ +\frac{1}{2}\int_0^t 6B_s ds &= \int_0^t (3B_s^2 - 3s) dB_s, 0 \leq t \leq T. \end{aligned}$$

The above integrand process is in the class $V(0, T)$. So the integral process and therefore $X_t, 0 \leq t \leq T$ has a continuous martingale version.

(ii) It follows from the properties of the Brownian motion (independent stationary increments) and those of conditional expectations that

$$
\begin{aligned}
& E[B_t^3 - 3tB_t \,|\mathcal{F}_s] \\
= \;& E[(B_t - B_s + B_s)^3 \,|\mathcal{F}_s] - 3tB_s \\
= \;& E[(B_t - B_s)^3 + 2(B_t - B_s)^2 B_s + (B_t - B_s)B_s^2 + (B_t - B_s)^2 B_s \\
& +2(B_t - B_s)B_s^2 + B_s^3 \,|\mathcal{F}_s] - 3tB_s \\
= \;& E[(B_t - B_s)^3] + 3B_s E[(B_t - B_s)^2] + 3B_s^2 E[(B_t - B_s)] \\
& +B_s^3 - 3tB_s \\
= \;& 3B_s(t-s) + B_s^3 - 3tB_s = B_s^3 - 3sB_s.
\end{aligned}
$$

So $X_t, 0 \leq t \leq T$ is a martingale.

**Problem 5**

(i) Using Itô's formula we find that

$$
\begin{aligned}
B_t^2 &= \int_0^t 2B_s dB_s + \frac{1}{2}\int_0^t 2ds \\
&= \int_0^t 2B_s dB_s + \frac{1}{2}\int_0^t 2ds.
\end{aligned}
$$

Since the integral process is a square integrable martingale, we see for $t \geq s$ that

$$
\begin{aligned}
E[B_t^2 \,|\mathcal{F}_s] &= E[\int_0^t 2B_s dB_s \,|\mathcal{F}_s] + t \\
&= \int_0^s 2B_s dB_s + t \\
&\geq \int_0^s 2B_s dB_s + s \\
&= B_s^2.
\end{aligned}
$$

So $X_t, t \geq 0$ is a submartingale w.r.t. $\mathcal{F}_t, t \geq 0$.

(ii) If $Y_t, t \geq 0$ is a submartingale, it follows from the optional sampling theorem $\mathcal{H}_t, t \geq 0$ by taking expectation that for every pair of bounded stopping times $S \leq T$ with respect to $\mathcal{H}_t, t \geq 0$ we have

$$E[Y_T] = E[E[Y_T \,|\mathcal{H}_S]] \geq E[X_S].$$

Let us show the converse statement: To this end, let $t > s$ and $A \in \mathcal{H}_s$ Define the random time

$$\tau = s1_A + 1_{A^c}t.$$

Since

$$\{\tau \leq l\} = \begin{cases} \Omega & , \text{if } l \geq t \\ A & , \text{if } s \leq l < t \\ \varnothing & , \text{if } l < s \end{cases},$$

$\tau$ is a (bounded) stopping time w.r.t. $\mathcal{H}_t, t \geq 0$. So

$$E[Y_s] \leq E[Y_\tau] = E[1_A Y_s] + E[1_{A^c} Y_t]$$

and therefore

$$E[1_{A^c} Y_s] = E[(1 - 1_A) Y_s] \leq E[1_{A^c} Y_t] = E[1_{A^c} E[Y_t \,|\mathcal{H}_s]]$$

for all $A^c \in \mathcal{H}_s$. Hence

$$Y_s \leq E[Y_t \,|\mathcal{H}_s] \text{ a.e.}$$

for all $t > s$.

**Problem 6**

(i) Define the first exit times

$$\tau_k := \inf\{t > 0 : B_t^x = 0 \text{ or } B_t^x = k\}, k > x > 0.$$

We know from the examples of the lecture notes that $E[\tau_k] < \infty$.

Consider the probability that the Bm exits at the point $k$:

$$p_k := P(B_{\tau_k}^x = k).$$

Then Dynkin's formula applied to $f(y) = y^2$ on $[0, k]$ gives

$$E[f(B_{\tau_k}^x)] = f(x) + E[\tau_k].$$

We observe that

$$\begin{aligned} E[f(B_{\tau_k}^x)] &= E[1_{\{B_{\tau_k}^x = k\}} f(B_{\tau_k}^x) + 1_{\{B_{\tau_k}^x = 0\}} f(B_{\tau_k}^x)] \\ &= p_k k^2. \end{aligned}$$

On the other hand, Dynkin's formula applied to $f(y) = y$ on $[0, k]$ yields

$$p_k k = x.$$

So

$$E[\tau_k] = p_k k^2 - x^2 = x(k-x).$$

Thus monotone convergence in connection with $\tau_k, k \in \mathbb{N}$, $k > x$ implies for $k \longrightarrow \infty$ that $E[\lim_k \tau_k] = \infty$. We also see that $\lim_k \tau_k = \tau$ on $\{\tau < \infty\}$ a.e.

Further, we have

$$\bigcup_{k \in \mathbb{N}, k > x} Q_k = \{\tau < \infty\},$$

where $Q_k := \{B_{\tau_k} = 0\}$ the event that the Bm exits a zero. $Q_k \subset Q_{k+1}$ for such $k$. So

$$P(\tau < \infty) = \lim_k (1 - p_k) = 1.$$

Hence $\lim_k \tau_k = \tau$ with probability one. Therefore we have that $E[\tau] = E[\lim_k \tau_k] = \infty$.

(ii) Define for $\lambda > 0$ the process

$$\begin{aligned} Z_t &= \exp\left(\int_0^t (-\sqrt{2\lambda}) dB_s - \frac{1}{2} \int_0^t (-\sqrt{2\lambda})^2\right) \\ &= \exp(-\sqrt{2\lambda}) B_t - \lambda t), t \geq 0. \end{aligned}$$

We know from the assignment, Prob. 6 (or Girsanov's theorem) that $Z_t, t \geq 0$ is continuous square integrable martingale. Then it follows from the optional sampling theorem for martingales that $Z_{t \wedge \tau}, t \geq 0$ is a martingale w.r.t. $\mathcal{F}_{t \wedge \tau}, t \geq 0$. So in particular

$$E[Z_{t \wedge \tau}] = E[Z_t] = 1.$$

On the other hand,

$$E[Z_{t \wedge \tau}] = E[\exp(-\sqrt{2\lambda}) B_{t \wedge \tau} - \lambda(t \wedge \tau)].$$

We know that

$$\begin{aligned} \lim_{t \longrightarrow \infty} Z_{t \wedge \tau} &= \exp(-\sqrt{2\lambda}) B_\tau - \sqrt{2\lambda} x + \sqrt{2\lambda} x - \lambda \tau) \\ &= \exp(-\sqrt{2\lambda}) B_\tau^x + \sqrt{2\lambda} x - \lambda \tau) = \exp(\sqrt{2\lambda} x - \lambda \tau). \end{aligned}$$

Since $B_{t \wedge \tau}^x \in [0, \infty)$ a.e. we see that $Z_{t \wedge \tau} \leq \exp(\sqrt{2\lambda} x)$. Thus dominated convergence yields

$$E[\exp(\sqrt{2\lambda} x - \lambda \tau)] = 1,$$

which gives

$$g(\lambda) = E[\exp(-\lambda \tau)] = \exp(-\sqrt{2\lambda} x).$$