MAT4720/9720
Stochastic Analysis and Stochastic Differential Equations

ASSIGNMENT

Evaluation: passed/not passed.

Exercise 1.
Consider the following finite probability space:
$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$
$\mathcal{F} = \mathcal{P}(\Omega)$ (the family of all possible subsets
$P(\omega_1) = P(\omega_2) = P(\omega_3) = P(\omega_4) = 1/4$
Consider the following random variables:
$X(\omega_1) = X(\omega_2) = 8; X(\omega_3) = X(\omega_4) = 4$
$Y(\omega_1) = 9; Y(\omega_2) = 6; Y(\omega_3) = 6; Y(\omega_4) = 3.$

1. Find the σ-algebra $\sigma(X)$ generated by X and the σ-algebra $\sigma(Y)$ generated by Y.
2. Is Y measurable with respect to $\sigma(X)$?
3. Compute the conditional expectation:

$$E\big[Y|\sigma(X)\big].$$

Exercise 2.
Let X, Z be given square integrable random variable independent of each other. Let $\mathcal{T} := \{\emptyset, \Omega\}$. Let $(B_t)_t$ be a Brownian motion and consider $0 \leq r \leq s \leq t$. Apply the properties of conditional expectation to compute further (or simplify the expression):

(i) $E[Z|X]$

(ii) $E[X^2Z + Z^2 - X|X]$

(iii) $E\big[E[X^2Z|X]|\mathcal{T}\big]$

(iv) $E[B_tB_s]$

(v) $E[B_s^2B_t - B_r^3]$

(vi) $E[(B_t - B_s)^2(B_t - B_r)]$

Give full detail of your reply.

Exercise 3.
Let us consider the game of the roulette. Recall that a roulette has 37 numbers (18 red, 18 black, and 1 green, standing for the number 0).

We study the following simple strategy, under the assumption that there are no liquidity problems for the players and no restrictions on the amount played.
The player has an initial amount $e > 0$ (which represents the initial endowment).
At the beginning the player bets the full amount e on the red. Independently of the outcome of the roulette, he will continue betting on red, each time doubling the amount.

(i) Denote the outcome of the gain/loss of the bet at time k by Y_k. Define p_k the probability that the game k was a gain. Describe the sequence Y_k, $k = 1, 2, ...$ and deduce the distribution of Y_k.

(ii) Denote $X_n := \sum_{k=1}^{n} Y_k$, $n = 1, 2, ...$, be the process representing the total gain/loss over time.

- Describe the natural filtration generated by $(X_n)_n$ in terms of $(Y_k)_k$?
- Is $(X_n)_n$ a martingale, super-martingale, sub-martingale with respect to its natural filtration?
- Is this sequence converging P-a.s.?
  (Hint: Search for, state, and use the Kolmogorov's three series theorem.)

Exercise 4.
Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with the filtration $(\mathcal{F}_t)_{t \geq 0}$.

1. Let $(B_t)_t$ be a Brownian motion. Determine if if the process $X_t = B_t^3 - 3tB_t$, $t \geq 0$, is a martingale, super-martingale, sub-martingale.

2. Let $(N_t)_t$ be a Poisson process, i.e. $N_0 = 0$ P-a.s., it has independent increments and $N_t - N_s$ has Poisson distribution with expectation $E[N_t - N_s] = \lambda(t - s)$ where ($\lambda > 0$ is call the *intensity* of the Poisson process).

   - Find a deterministic process $(\nu_t)_t$ with $\nu_0 = 0$, such that $(N_t - \nu_t)_t$ is a martingale. Denote $M_t := N_t - \nu_t$.
   - Show that $(M_t^2 - \lambda t)_t$ is a martingale
   - Show that $(M_t^2 - N_t)_t$ is a martingale

Exercise 5.
Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with the filtration $(\mathcal{F}_n)_{n=0,1,...}$. Let $(M_n)_n$ is a martingale converging in $L^1(P)$ to the constant value $m \in \mathbb{R}$. Prove that $M_n = m$ for all $n = 0, 1, ...$