UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in MAT4720 — Stochastic analysis and stochastic differential equations.

Day of examination: Friday, December 07, 2018.

Examination hours: 14.30 – 18.30.

This problem set consists of 3 pages.

Appendices: Appendix

Permitted aids: none.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1. (10 points)

(i) Give the definition of a martingale and a submartingale.
(ii) Give the definition of a 1-dimensional Brownian motion.
(iii) What is the statement of Girsanov's theorem ?
(iv) What is the definition of a stopping time ?
(v) State Dynkin's formula.

Problem 2. (10 points)

In the following assume that $\mathcal{F}_t, t \geq 0$ is the natural filtration of a Brownian motion.
(i) Use the definition of a martingale to check whether

$$X_t := B_t + 3t$$

is a martingale or not ($B_t, t \geq 0$ is a 1-dimensional Brownian motion).
(ii) Are the processes

$$X_t := \int_0^t B_s dB_s,$$

$$Y_t := \int_0^t \exp(\sin(B_s)) dB_s,$$

$$Z_t := \int_0^t \exp(2B_s^2) dB_s,$$

$0 \leq t \leq T$ for $T > 1$ (square integrable) martingales ?

(iii) Compute $E[\exp(B_t - B_s) B_s^2 \,|\mathcal{F}_s]$ for $t \geq s$.

Hint: If Y is normally distributed with mean zero, then $E[\exp(Y)] = \exp(-\frac{1}{2} Var[Y])$

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(iv) Use Itô's formula to check whether the process $X_t := (B_t + t)\exp(-B_t - \frac{1}{2}t), t \geq 0$ is a martingale or not.

Problem 3. (10 points)

Let $B_t, 0 \leq t \leq T$ be a 1−dimensional Brownian motion and denote by $\mathcal{F}_t, 0 \leq t \leq T$ its natural filtration.
(i) Show that the 1−dimensional SDE

$$dX_t = \log(1 + X_t^2)dt + X_t dB_t, X_0 = x, 0 \leq t \leq T$$

has a unique (strong) solution.
(ii) Let $\mu_t, \sigma_t, 0 \leq t \leq T$ be bounded, measurable and $\mathcal{F}_t$−adapted processes.
Use Itô's formula to show that the process

$$X_t := x\exp(\int_0^t (\mu_s - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dB_s)$$

is a solution to the SDE

$$dX_t = \mu_t X_t dt + \sigma_t X_t dB_t, X_0 = x, 0 \leq t \leq T.$$

Is this solution unique ?

Problem 4. (10 points)

Consider a 1−dimensional Brownian motion $B_t, 0 \leq t \leq T$ and denote by $\mathcal{F}_t, 0 \leq t \leq T$ its natural filtration.
(i) Prove that the process $X_t := B_t^3 - 3tB_t$ is martingale by using Itô's formula.
(ii) Show that the process $X_t := B_t^3 - 3tB_t$ is a martingale by only using the properties of the Brownian motion and of conditional expectations.

Hint: If $Y \sim \mathcal{N}(0,1)$, then $E[Y^3] = 0$.

Problem 5. (10 points)

(i) Assume that $B_t, t \geq 0$ is a 1−dimensional Brownian motion and let $\mathcal{F}_t, t \geq 0$ be its natural filtration.
Show that the process $X_t := B_t^2, t \geq 0$ is a submartingale with respect to $\mathcal{F}_t, t \geq 0$.
(ii) Let $Y_t, t \geq 0$ be a continuous $\mathcal{H}_t$-adapted process with $E[|Y_t|] < \infty$ for all t.
Prove that $Y_t, t \geq 0$ is a submartingale with respect to the filtration $\mathcal{H}_t, t \geq 0$ *if and only if* for every pair of bounded stopping times $S \leq T$ with respect to $\mathcal{H}_t, t \geq 0$ we have

$$E[Y_T] \geq E[Y_S].$$

Hint: Use in connection with (ii) the *optional sampling theorem* in the Appendix.

Problem 6. (10 points)

Let $B_t, t \geq 0$ be a 1−dimensional Brownian motion. Consider the process $B_t^x := B_t + x, t \geq 0$ for $x > 0$. Define

$$\tau = \inf\{t > 0 : B_t^x = 0\},$$

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where $\inf \varnothing := \infty$ by convention.
(i) Show that $\tau < \infty$ with probability 1 for all $x > 0$ and that $E[\tau] = \infty$ for all $x > 0$.
(ii) Find an explicit expression of the *Laplace transform* of τ defined by

$$g(\lambda) := E[\exp(-\lambda \tau)], \lambda > 0.$$

End

Appendix

Optional Sampling theorem: Let $M_t, t \geq 0$ be a submartingale with continuous paths with respect to the filtration $\mathcal{H}_t, t \geq 0$ on a probability space $(\Omega, \mathcal{H}, P)$ and let τ_1 and τ_2 be two stopping times with respect to $\mathcal{H}_t, t \geq 0$ such that $\tau_1 \leq \tau_2 \leq K < \infty$ for a constant K. Then

$$E[M_{\tau_2} | \mathcal{H}_{\tau_1}] \geq M_{\tau_1}$$

with probability 1, where $\mathcal{H}_\tau$ is the stopping time $\sigma-$algebra defined by

$$\mathcal{H}_\tau = \{A \in \mathcal{H} : A \cap \{\tau \leq t\} \in \mathcal{H}_t \text{ for all } t\}.$$

Remark: The above result also holds true for continuous martingales $M_t, t \geq 0$. In this case, we have equality, that is

$$E[M_{\tau_2} | \mathcal{H}_{\tau_1}] = M_{\tau_1}$$

for $\tau_1 \leq \tau_2 \leq K < \infty$.