MAT4720/9720 - Autumn 2019
Detailed program

<u>Foundations of probability and stochastic processes</u>

*Definition of filtration and information flow, right-continuous, left-continuous filtration (intersection and union of sigma-algebras)
*Probability measures
*P-augmented filtration
*Standard conditions for a filtration

*Stochastic processes, measurable processes, adapted processes
*Natural filtration generated by a stochastic process
Comparison of stochastic processes: equivalent processes in law, version/modification of a stochastic process, indistinguishable stochastic processes.
Discussion on the importance of P-augmentation in the standard conditions imposed for a filtration
Properties of the trajectories: left-, right- continuity, measurability
*Definition and comments on the Borel sigma-algebra: continuity and measurability
*Definition of progressively measurable stochastic processes, comments
[B] Prop 2.1 - with proof
Continuity and Kolmogorov theorem (without proof)

Ref. [B] Section 2.1 (with embedded reference to Section 1.1). [B] Section 2.1, 2.2

In the course we use all the different types of convergences: L^p- convergence, convergence in Probability, convergence in distribution, P-a.s. convergence. For a short summary see references given at the webpage of the course.

Exercises:
[B] 1.2, 1.3 (Prove the statement in the exercise), 1.9, 1.10, 1.19
[Ø] 2.1, 2.3, 2.5, 2.18

<u>Brownian motion</u>

Brownian motion w.r.t. any fixed filtration (embedded concept of independence of a random variable from a sigma-algebra)

Definition of p-variation of a function on a bounded close interval.
*Results on the quadratic variation of Brownian motion (convergence in $L^2(P)$ and P-a.s are discussed - see Levy theorem).
Ref [B] elements of sections 2.2 and 3.3.

***Conditional expectation**

*Definition, properties ([B] Propositions 4.1, 4.2 - proof left as exercise). Discussion around the implication coming from Jensen's inequality: conditional expectation as a contraction (linear bounded operator with norm less or equal to 1). The case of L^2 and the geometrical meaning of best prediction as orthogonal projection. Freezing Lemma ([B] Lemma 4.1 no proof).

Exercise in class [B Remark 4.5]

Ref: [B] Section 4.2

Suggested exercises [B] 4.5, 4.9, 4.11, 4.15

Martingales: discrete and continuous time

*Martingales (sub- and super-): definition and related comments, examples.

Discrete time martingales.
*Doob's decomposition theorem with proof ([B]Theorem 5.1).
Convergence result ([B] Theorem 5.4) no proof.
P-a.s convergence ([B] Theorem 5.4 no proof) with comments
*Bounded martingales in L^p (p>1) and related convergence ([B] Theorem 5.5 and 5.6 - with proof)
Here the *Doob's inequality has to be used.
*Definition of uniform integrability (u.i.) of a family of random variables and related characterisation ([B] Propositions 5.2, 5.3 no proofs).
*Relation between L^1 convergence and u.i. ([B] Theorem 5.7 no proof)
Martingale convergence theorem in L^1 and P-a-s ([B] Theorem 5.8 no proof) and study of the limit ([B] Proposition 5.5 no proof).

*Continuous time martingales.
Here we have reviewed some of the results seen in discrete time with all comments.
About inequalities [B] Theorem 5.9, 5.12 (no proofs)
About convergence [B] Theorem 5.10, 5.11 (no proofs)
We have put the results into practice studying in full detail the case exposed in [B] Example 5.2 and Exercise 5.9. With this we have used the Law of Iterated Logarithm, which gives the behaviour of the trajectories of Brownian motion, see [B] Theorem 3.2 and Corollary 3.2 in Section 3.4
Result on exponential martingales and Brownian motion (a characterisation) [B] Theorem 5.17- with proof.
We have discussed the existence of a right-continuous modification of super-martingales ([B] Theorem 5.14 - no proof)
We have discussed the relationship between martingales and processes of finite variation ([B] Theorem 5.15 - no proof)
We have presented the Doob's decomposition theorem and the compensator ([B] Theorem 5.16 - no proof). Comments and the concept of predictable process in continuous time were given.

Ref: [B] Elements of sections 5.1, 5.2, 5.3. 5.4, 5.5, 5.6; [Ø] Appendix B and C.

Suggested exercises [B] Example 5.4; Exercises 5.1, 5.4, 5.5, 5.7, 5.11, 5.12, 5.13, 5.15, 5.23;
[Ø] Exercises 3.4, 3.3

***Ito stochastic integral**

*The spaces of the integrands, initial focus on the L^2 setting (what we have called the "classical" framework)
*Construction of the integral starting from the elementary (simple) integrands. Properties of the integral ([B Lemma 7.1] with proof).
*Comments about the stochastic integral as an isometric operator (Ito isometry).
*Construction of the integral for general integrands in the L^2 setting ([B Lemma 7.2] no proof)

Ref: [B] Chapter 7; [Ø] Chapter 3

Random times, stopping times, local martingales.

*Definition of stopping time [B Section 3.5], example: the first exit time of an open set by a Brownian motion.
*Definition of optional sigma-algebra.
Properties of stopping times [B Proposition 3.5] - proof left as exercise. Exercise: given 2 stopping times of which one is bounded, show that their infimum is a bounded stopping time.
Stopping times are approximated from the right [B Lemma 3.3] no proof. Stopping times and processes [B Proposition 3.6, Example 2.3 - no proof].
Comment: The importance of the filtration right continuity when dealing with first exit times [B Proposition 3.7 - no proof].
The stopping theorem [B Theorem 5.13, 5.2, Corollary 5.1 - no proofs] Application to right continuous martingales: the stopped process [B Proposition 5.6 - no proof].
The increments of the Brownian motion and random times [B Theorem 3.3 - no proof]
*Rules of calculus of stochastic integral in presence of stopping times [B Theorem 7.5 - no proof].

*Definition of local martingale [B Section 7.6]

Suggested exercises: [B] 3.5, 5.6, 5.8, 5.22; 7.2, 7.3, 7.4, 7.5, 7.6, 7.8; [Ø] 3.1, 3.3, 3.4, 3.6, 3.15

Extended stochastic Ito integral

*Definition of extended Ito integrals via stopping time.
Recognise the integral as a local martingale.
Proof that the extended integral is indeed an extension of the classical Ito integral.
The extended Ito integral admits a continuous modification (no proof)
*Some properties of the integral [B Remark 7.4]
*Approximations (continuity in probability [B Proposition 7.3, Lemma 7.3] and, consequently, the interpretation of the extended Ito integral as limit (in probability) of Riemann sums for continuous integrands [B Proposition 7.4]
Multiplication by a random variable and rule of calculus [B Theorem 7.6 - no proof only a hint]

Ref: [B] Section 7.5; [Ø] Section 3.3

Ito processes

*Definition of Ito process.
Additional comments on its connection with semi-martingales.
Examples in class from the exercises earlier suggested.
The increasing associated process (predictable compensator) of an Ito process

[B] Section 8.1 Introduction; [Ø] Section 4.1

Ito formula (basis of Ito calculus)

*Theorem presenting the formula [Ø] Theorem 4.1.2 with proof in dimensions 1. Statement only for the multi-dimension case.
*Application for the product rule
Exercises from exponential martingales

Suggested exercises: [B] 8.1, 8.2, 8.3, 8.7, 8.18, 8.19; [Ø] 4.1, 4.2, 4.5, 4.11, 4.13, 4.14, 4.15

Integral representations and stochastic NA-derivative

Stochastic integral representation theorem with proof [B Theorem 12.4]; [Ø Theorem 4.3.3 + lemma 4.3.1, lemma 4.3.2];
*Martingale representation theorem [B Theorem 12.5 with proof] and [Ø Theorem 4.3.4 with proof];
Local martingales representation theorem [B Theorem 12.6 - no proof].
*Stochastic Non-Anticipating derivative (NA-derivative): definition and associated theorem, which gives and explicit stochastic integral representation theorem. Notes are available on the webpage of the course.

Stochastic differential equations (SDEs)

*Concepts of Strong and Weak solutions.
*Concepts of uniqueness of solutions: strong (pathwise uniqueness), weak (uniqueness in law)
Examples of explicit solutions of SDEs obtained by the use of Ito formula: Ornstein-Uhlembeck process, geometric Brownian motion.
*Exercise: solution of all linear SDEs. See notes available at the webpage of the course
*Theorem sufficient condition for the existence of a strong solution and its strong uniqueness. With proof. Part 1: Proof of uniqueness. Part 2: Proof of existence [B] Theorem 9.2 (also [Ø] Theorem 5.2.1).
Result on the the localisation of the Lipschitz condition for the existence. Statement of the result [B Theorem 9.4]

Ref: [B] Section 9.1, 9.2 [Ø] Section 5.3

Suggested exercises: [Ø] Exercises 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.11, 5.16, 5.18. [B] Exercises 3.4, 9.2, 9.4, 9.7

Markov processes (some elements) and Markov property of SDEs

*Definition of Markov transition function, Markov process, comments about the use of the Markov transition function in the computation of conditional expectation, expectation, finite dimensional distribution.
The canonical space for a Markov process.
*Time homogeneous transition function.
Feller property and Feller processes.
*Strong Markov property.
Semigroup of a Markov process and semigroup for a homogeneous Markov process
*Markov property of SDEs

Ref: [B] Chapter 6, Section 6.1, Section 6.2 , Section 9.7

Suggested exercises [Ø] Exercise 7.1, 7.2, 7.3, 7.10; [B] Exercise 6.2

Girsanov theorem and change of measures

*Absolutely continuous probability measures and equivalent probability measures. Radon-Nikodym density.
Conditions for the martingality of the Radon-Nikodym density [B Proposition 12.1, Theorem 12.2, Corollary 12.1 - no proofs] [B Example 12.2]
*Girsanov theorem [B Theorem 12.1] - sketch of proof.
*Some exercises of application of Girsanov theorem to Ito processes [Ø Theorem 8.6.6, Example 8.6.7]

Suggested exercises [Ø] Exercises 8.11, 8.12, 8.18

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* Marks definitions and topics central to this course and of importance for follow up courses of next semester.