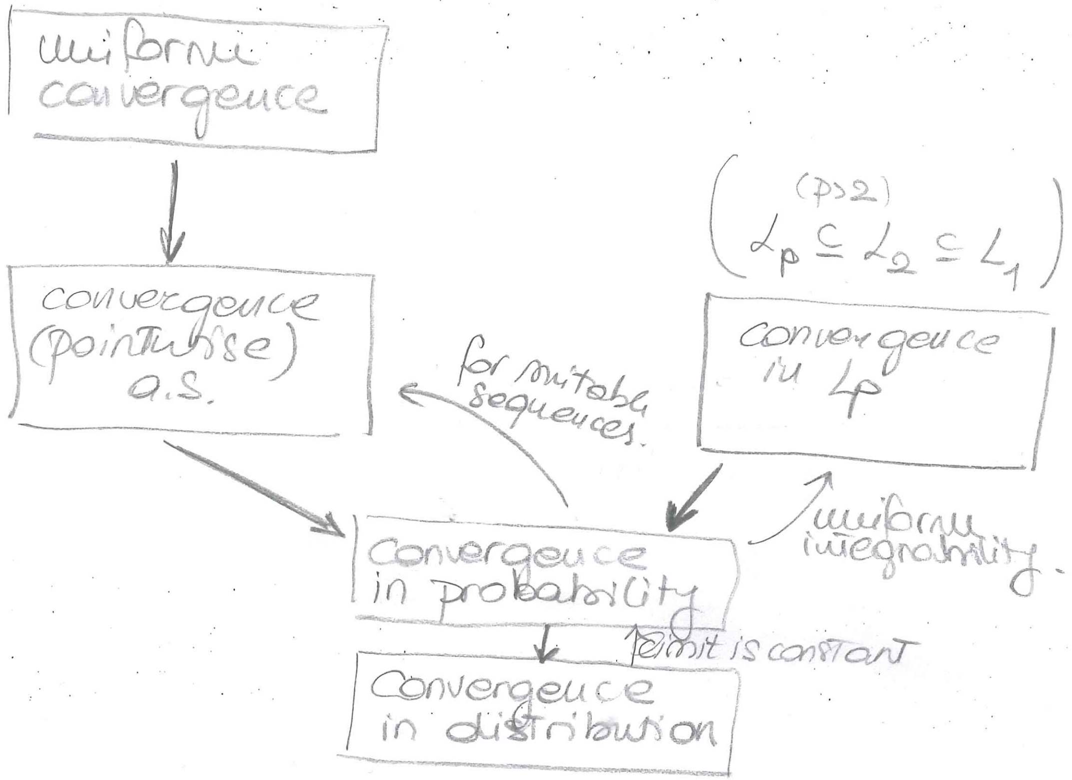

Appendix

A1 Modes of Convergence

The following theory can be found for instance in Feller (1968), Karr (1993) or Loève (1978).

We introduce the main modes of convergence for a sequence of random variables $A, A_1, A_2, \ldots$.

Convergence in Distribution

The sequence (A_n) *converges in distribution* or *converges weakly to the random variable* A ($A_n \xrightarrow{d} A$) if for all bounded, continuous functions f the relation

$$Ef(A_n) \to Ef(A), \quad n \to \infty,$$

holds.

Notice: $A_n \xrightarrow{d} A$ holds if and only if for all continuity points x of the distribution function F_A the relation

$$F_{A_n}(x) \to F_A(x), \quad n \to \infty, \tag{A.1}$$

is satisfied. If F_A is continuous then (A.1) can even be strenghtened to uniform convergence:

$$\sup_x |F_{A_n}(x) - F_A(x)| \to 0, \quad n \to \infty.$$

It is also well known that convergence in distribution is equivalent to pointwise convergence of the corresponding characteristic functions:

$$A_n \xrightarrow{d} A \quad \text{if and only if} \quad Ee^{itA_n} \to Ee^{itA} \quad \text{for all } t.$$

Example A1.1 (Convergence in distribution of Gaussian random variables)
Assume that (A_n) is a sequence of normal $N(\mu_n, \sigma_n^2)$ random variables.
First suppose that $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$, where μ and σ^2 are finite numbers. Then the corresponding characteristic functions converge for every $t \in \mathbb{R}$:

$$Ee^{itA_n} = e^{it\mu_n - 0.5\sigma_n^2 t^2} \to e^{it\mu - 0.5\sigma^2 t^2}.$$

The right-hand side is the characteristic function of an $N(\mu, \sigma^2)$ random variable A. Hence $A_n \xrightarrow{d} A$.

Also the converse is true. If we know that $A_n \xrightarrow{d} A$, then the characteristic functions $e^{it\mu_n - 0.5\sigma_n^2 t^2}$ necessarily converge for every t. From this fact we conclude that there exist real numbers μ and σ^2 such that $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$. This implies that A is necessarily a normal $N(\mu, \sigma^2)$ random variable. □

Convergence in Probability

The sequence (A_n) *converges in probability to the random variable* A ($A_n \xrightarrow{P} A$) if for all positive ε the relation

$$P(|A_n - A| > \varepsilon) \to 0, \quad n \to \infty,$$

holds.

Convergence in probability implies convergence in distribution. The converse is true if and only if $A = a$ for some constant a.

Almost Sure Convergence

The sequence (A_n) *converges almost surely* (*a.s.*) or *with probability* 1 *to the random variable* A ($A_n \xrightarrow{a.s.} A$) if the set of ωs with

$$A_n(\omega) \to A(\omega), \quad n \to \infty,$$

has probability 1.

This means that

$$P(A_n \to A) = P(\{\omega : A_n(\omega) \to A(\omega)\}) = 1.$$

Convergence with probability 1 implies convergence in probability, hence convergence in distribution. Convergence in probability does not imply convergence a.s. However, $A_n \xrightarrow{P} A$ implies that $A_{n_k} \xrightarrow{a.s.} A$ for a suitable subsequence (n_k).

L^p-Convergence

Let $p > 0$. The sequence (A_n) *converges in* L^p or *in pth mean to* A ($A_n \xrightarrow{L^p} A$) if $E[|A_n|^p + |A|^p] < \infty$ for all n and

$$E|A_n - A|^p \to 0, \quad n \to \infty.$$

By Markov's inequality, $P(|A_n - A| > \varepsilon) \le \varepsilon^{-p} E|A_n - A|^p$ for positive p and ε. Thus $A_n \xrightarrow{L^p} A$ implies that $A_n \xrightarrow{P} A$. The converse is in general not true.

For $p = 2$, we say that (A_n) *converges in mean square to* A. This notion can be extended to stochastic processes; see for example Appendix A4. Mean square convergence is convergence in the Hilbert space

$$L^2 = L^2[\Omega, \mathcal{F}, P] = \{X : EX^2 < \infty\}$$

endowed with the inner product $< X, Y >= E(XY)$ and the norm $\|X\| = \sqrt{(X, X)}$. The symbol X stands for the equivalence class of random variables Y satisfying $X \stackrel{d}{=} Y$.