The stochastic non-anticipating (NA) derivative and integral representations

by

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In the complete probability space $(\Omega, \mathcal{F}, P)$ we consider a standard Brownian motion $B = B_t$, $t \geq 0$ ($B_0 = 0$), generating the filtration $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$. Let $X = X_t$, $t \geq 0$, be a martingale with respect to the filtration $\mathbb{F}$ with

$$E|X_t|^2 < \infty, \quad \text{for all } t \geq 0,$$

i.e. X is a martingale in $L_2(\Omega)$. Here $L_2(\Omega) = L_2(\Omega, \mathcal{F}, P)$ is the standard L_2-space with the norm

$$\|\xi\| := \Big(E|\xi|^2\Big)^{1/2} = \Big(\int_\Omega |\xi(\omega)|^2 P(d\omega)\Big)^{1/2}.$$

For any $T > 0$, we write $L_2(\Omega\times[0,T])$ for the functional space $L_2(\Omega\times[0,T], \mathcal{F}\times\mathfrak{B}[0,T], P\times dt)$ of the stochastic functions

$$\varphi := \varphi_t, \qquad 0 \leq t \leq T,$$

where, for each t,

$$\varphi_t = \varphi_t(\omega), \qquad \omega \in \Omega,$$

is a random variable and

$$\|\varphi\|_{L_2} := \Big(E\int_0^T |\varphi_t|^2 dt\Big)^{1/2} = \Big(\int_{\Omega\times[0,T]} |\varphi_t(\omega)|^2 P(d\omega)\times dt\Big)^{1/2} < \infty.$$

Corresponding notation and meaning is adopted for the case $T = \infty$.

Here below we recall two fundamental results.

The Itô representation theorem. *For any $\mathcal{F}_T$-measurable random variable $\xi \in L_2(\Omega)$ there exists a unique $\mathbb{F}$-adapted (non-anticipating) stochastic function $\varphi = \varphi_t$, $0 \leq t \leq T$, in $L_2(\Omega \times [0,T])$ such that*

$$\xi = E\xi + \int_0^T \varphi_s dB_s. \tag{1}$$

The martingale representation theorem. *For any martingale $X = X_t$, $t \geq 0$, in $L_2(\Omega)$ with respect to the filtration $\mathbb{F}$ there exists a unique $\mathbb{F}$-adapted (non-anticipating)stochastic function function $\varphi = \varphi_t$, $t \geq 0$, in $L_2(\Omega \times [0,\infty))$ such that*

$$X_t = EX_0 + \int_0^t \varphi_s dB_s, \qquad t \geq 0. \tag{2}$$

We refer to [Ø], Chapter 4, for the details.

The theorems above consider only the existence of the integrand φ. But for many applications it is fundamental to know how to determine this φ explicitly - e.g. the application to hedging problems in mathematical finance. In the sequel the problem we are dealing with is how to determine φ in the above stochastic integral representations.

We are going to answer this question by means of the non-anticipating stochastic derivative in the framework of Itô stochastic calculus.

First of all recall the definition of the partitions of $[0, T)$.

Definition. The *partitions of* $(0, T]$ are the series of disjoint intervals of the form

$$[t^n_{k-1}, t^n_k), \quad k = 1, ..., K_n, \qquad (0 = t^n_0 < ... < t^n_{K_n} = T)$$

such that $\bigsqcup_{k=1}^{K_n} [t^n_{k-1}, t^n_k) = [0, T)$ and

$$\delta_n := \max_{k=1,\dots,K_n} |t^n_k - t^n_{k-1}| \longrightarrow 0, \quad n \to \infty.$$

Definition. For any random variable $\xi \in L_2(\Omega)$ we can define the *non-anticipating stochastic derivative*

$$\mathcal{D}\xi = \mathcal{D}_s\xi, \qquad 0 \leq s \leq T, \tag{3}$$

of ξ with respect to the integrator B_t, $t \geq 0$, as the element in $L_2(\Omega \times [0, T])$ given by the limit

$$\mathcal{D}\xi = \lim_{n\to\infty} \varphi^n \qquad \text{in} \qquad L_2(\Omega \times [0, T]) \tag{4}$$

with

$$\varphi^n(s) := \sum_{k=1}^{K_n} E\Big[\xi \cdot \frac{\Delta B_{nk}}{\Delta t_{nk}} \Big| \mathcal{F}_{t^n_{k-1}}\Big] 1_{[t^n_{k-1}, t^n_k)}(s), \qquad 0 \leq s \leq T, \tag{5}$$

where $\Delta B_{nk} := B_{t^n_k} - B_{t^n_{k-1}}$ and $\Delta t_{nk} := t^n_k - t^n_{k-1}$.

Remarks.

i) The above definition does not depend on the choice of the partitions, thus it is well-posed.

ii) The sochastic functions $\varphi_n(s)$, $0 \leq s \leq T$, are simple functions in $L_2(\Omega \times [0, T])$ and they are $\mathbb{F}$-adapted (thanks to the properties of conditional expectation), thus the stochastic functions φ^n are simple integrands.

iii) The stochastic function $\mathcal{D}\xi = \mathcal{D}_s\xi$, $0 \le s \le T$, is then the limit of simple integrands in $L_2(\Omega \times [0,T])$ and thus it is an integrand itself (recall the scheme of construction of the Itô integration).

Theorem. *For any $\mathcal{F}_T$-measurable random variable $\xi \in L_2(\Omega)$ the integrand $\varphi = \varphi_s$, $0 \le s \le T$, appearing in the Itô representation* (1)

$$\xi = E\xi + \int_0^T \varphi_s dB_s$$

can be determined by the stochastic non-anticipating derivative:

$$\varphi_s = \mathcal{D}_s\xi, \qquad 0 \le s \le T.$$

Proof. For a better understanding we split the proof in several steps, here marked with a bullet.

- From the Ito representation theorem, we have that

  $$\xi = E\xi + \int_0^T \varphi_s dB_s,$$

  then taking the conditional expectation we define the process

  $$\xi_t := E\big[\xi|\mathcal{F}_t\big] = E\xi + \int_0^t \varphi_s dB_s, \qquad 0 \le t \le T,$$

  which is a martingale with respect to $\mathbb{F}$ (see also Martingale representation theorem). Also, we can observe that

  $$E\big[\xi\Delta B_{nk}|\mathcal{F}_{t^n_{k-1}}\big] = E\big[\Delta\xi_{nk}\Delta B_{nk}|\mathcal{F}_{t^n_{k-1}}\big]$$

  for $\Delta\xi_{nk} := \xi_{t^n_k} - \xi_{t^n_{k-1}}$. In fact, we have

  $$\begin{aligned} E\Big[\xi\Delta B_{nk}|\mathcal{F}_{t^n_{k-1}}\Big] &= E[\xi]\, E\Big[\Delta B_{nk}|\mathcal{F}_{t^n_{k-1}}\Big] \\ &+ \int_0^{t^n_{k-1}} \varphi_s dB_s E\Big[\Delta B_{nk}|\mathcal{F}_{t^n_{k-1}}\Big] \\ &+ E\Big[\int_{t^n_{k-1}}^{t^n_k} \varphi_s dB_s \Delta B_{nk}|\mathcal{F}_{t^n_{k-1}}\Big] \\ &+ E\Big[E\Big[\int_{t^n_k}^{T} \varphi_s dB_s|\mathcal{F}_{t^n_k}\Big]\Delta B_{nk}|\mathcal{F}_{t^n_{k-1}}\Big] \\ &= E\Big[\Delta\xi_{nk}\Delta B_{nk}|\mathcal{F}_{t^n_{k-1}}\Big]. \end{aligned}$$

Hence the simple function in (5) can be rewritten in the equivalent form

$$\varphi^n(s) := \sum_{k=1}^{K_n} E\Big[\Delta\xi_{nk} \cdot \frac{\Delta B_{nk}}{\Delta t_{nk}}\Big|\mathcal{F}_{t^n_{k-1}}\Big] 1_{[t^n_{k-1},t^n_k)}(s), \qquad 0 \le s \le T. \tag{6}$$

- Being the integral $\int_0^T \varphi_s dB_s$ an Itô integral, then by construction there exists a sequence of simple integrands ψ^n, $n = 1, 2, ...,$ with

$$\psi^n_s := \sum_{k=1}^{K_n} e^n_{k-1} 1_{[t^n_{k-1},t^n_k)}(s) + e^n_T 1_{\{T\}}(s), \quad 0 \le s \le T,$$

  (with bounded $\mathcal{F}_{t^n_{k-1}}$-measurable random variables e^n_{k-1}), such that

$$\varphi = \lim_{n\to\infty} \psi^n \text{ in } L_2(\Omega \times [0,T]), \quad \text{i.e. } \|\varphi - \psi^n\|_{L_2} \longrightarrow 0,\ n \to \infty, \tag{7}$$

  and thus

$$\int_0^T \varphi_s dB_s = \lim_{n\to\infty} \int_0^T \psi^n_s dB_s \quad \text{in } L_2(\Omega).$$

- Note that in the construction of the Itô integral there is no statement of uniqueness of the sequence of simple integrands. We are going to exploit this fact and prove that the sequence φ^n, $n = 1, 2...,$ of simple integrands (see Remark (iii)) also characterizes the integral, i.e.

$$\int_0^T \varphi_s dB_s = \lim_{n\to\infty} \int_0^T \varphi^n_s dB_s \quad \text{in } L_2(\Omega).$$

  From the Itô isometry and the construction of the integral, we conclude that it is enough to prove that

$$\varphi = \lim_{n\to\infty} \varphi^n \text{ in } L_2(\Omega \times [0,T]), \quad \text{i.e. } \|\varphi - \varphi^n\|_{L_2} \longrightarrow 0,\ n \to \infty. \tag{8}$$

- Moreover, since

$$\|\varphi - \varphi^n\|_{L_2} \le \|\varphi - \psi^n\|_{L_2} + \|\psi^n - \varphi^n\|_{L_2}, \tag{9}$$

  we only need to show that

$$\|\psi^n - \varphi^n\|_{L_2} \longrightarrow 0, \quad n \to \infty. \tag{10}$$

- In fact, we can see that

$$
\begin{aligned}
\|\psi^n - \varphi^n\|_{L_2}^2 &= E\Big[\int_0^T \Big(\sum_{k=1}^{K_n}\Big\{e_{k-1}^n - \frac{1}{\Delta t_{nk}} E\big[\Delta\xi_{nk}\Delta B_{nk}|\mathcal{F}_{t_{k-1}^n}\big]\Big\}1_{[t_{k-1}^n, t_k^n)}(s)\Big)^2 ds\Big] \\
&= E\Big[\int_0^T \sum_{k=1}^{K_n} \frac{1}{(\Delta t_{nk})^2}\big(e_{k-1}^n \Delta t_{nk} - E[\Delta\xi_{nk}\Delta B_{nk}|\mathcal{F}_{t_{k-1}^n}]\big)^2 1_{[t_{k-1}^n, t_k^n)}(s) ds\Big] \\
&= \sum_{k=1}^{K_n} \frac{1}{\Delta t_{nk}} E\Big[\Big(E\big[e_{k-1}^n \Delta t_{nk} - \int_0^T \varphi_s 1_{[t_{k-1}^n, t_k^n)}(s) ds \big| \mathcal{F}_{t_{k-1}^n}\big]\Big)^2\Big] \\
&= \sum_{k=1}^{K_n} \frac{1}{\Delta t_{nk}} E\Big[\Big(\int_0^T (e_{k-1}^n - \varphi_s) 1_{[t_{k-1}^n, t_k^n)}(s) ds\Big)^2\Big] \\
&\le \sum_{k=1}^{K_n} \frac{1}{\Delta t_{nk}} E\Big[\int_0^T (\psi_s^n - \varphi_s)^2 1_{[t_{k-1}^n, t_k^n)}(s) ds \cdot \int_0^T 1_{[t_{k-1}^n, t_k^n)}(s) ds\Big] \\
&= \|\psi^n - \varphi\|_{L_2}^2 \longrightarrow 0, \quad n \to \infty.
\end{aligned}
$$

  Here above we have applied Hölder inequality.

- From (9) we have that $\|\varphi - \varphi^n\|_{L_2}^2 \le 2\|\varphi - \psi^n\|_{L_2}^2 \longrightarrow 0$, $n \to \infty$.

By this we end the proof. □

Corollary. *The non-anticipating stochastic derivative $\mathcal{D}\xi$ is continuous with respect to ξ in $L_2(\Omega)$. Namely,*

$$\xi = \lim_{n\to\infty} \xi_n\,, \quad \text{i.e.} \qquad \|\xi - \xi_n\| \longrightarrow 0, \quad n \to \infty,$$

implies

$$\mathcal{D}\xi = \lim_{n\to\infty} \mathcal{D}\xi_n\,, \quad \text{i.e.} \qquad \|\mathcal{D}\xi - \mathcal{D}\xi_n\|_{L_2} \longrightarrow 0, \quad n \to \infty.$$

Proof. This can be proved applying the Itô isometry and the stochastic integral representation from the main theorem. □

Corollary. *Any martingale $X = X_t$, $0 \le t \le T \quad (T < \infty)$, in $L_2(\Omega)$ with respect to the filtration $\mathbb{F}$ admits the following (unique) stochastic integral representation*

$$X_t = EX_0 + \int_0^t \mathcal{D}_s X_T dB_s, \qquad t \ge 0. \tag{11}$$

(Recall that for martingales $EX_t = EX_0$, for all t).

References.

[Ø] "Stochastic Differential Equations" (by B. Øksendal). Springer 2003.

Further reading on the non-anticipating derivative: original material.

[1] "On stochastic integration and differentiation" (by G. Di Nunno and Yu.A. Rozanov). Acta Applicandae Mathematicae (1999), 58, pp. 231-235.

[2] "Stochastic integral representation, stochastic derivatives and minimal variance hedging" (by G. Di Nunno). Stochastics and Stochastics Reports (2002), 73, pp. 181-198.