MAT4720/9720
Stochastic Analysis and Stochastic Differential Equations

ASSIGNMENT

Evaluation: passed/not passed.

Exercise 1.
During classes, we have presented the construction of the Itô stochastic integral with respect to a $(\mathbb{F}, P)$-Brownian motion as integrator. We also have commented on the extension of these arguments to general continuous $(\mathbb{F}, P)$-martingales in $L^2(P)$ as integrator. In this exercise you are asked to formalise part of this construction.

Let $M = M_t$, $t \in [0, T]$, be a continuous martingale with values in $L^2(P)$ and $< M >$ be its compensator, i.e. it is the continuous adapted increasing process such that $M^2 - < M >$ is a martingale. The

$$L^2(P \times d < M >) = L^2(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}[0, T], P \times d < M >),$$

so that

$$E\Big[\int_0^T \varphi(t) d < M > \Big] = \int_\Omega \int_0^T \varphi(\omega, t)\, d < M > (\omega)\, P(d\omega) < \infty,$$

is a Hilbert space. To work with general martingales as integrators, the stochastic integrands are continuous adapted processes $\varphi \in L^2(P \times d < M >)$, and this is going to called the space $M^2[0, T]$ in correspondence with what we did in class for Brownian motion. It is possible to prove the following statement:
Any element $\varphi \in M^2[0, T]$ admits an approximating sequence $(\varphi_n)_{n=1,2,...}$ in $L^2(P \times d < M >)$ of simple integrands of the type:

$$\varphi_n(\omega, t) = \sum_{k=1}^{K_n} \phi_{n,k}(\omega) 1_{[t^n_{k-1}, t^n_k)}(t), \qquad (\omega, t) \in \Omega \times [0, T],$$

for the partitions $0 = t^n_0 < ... < t^n_{K_n} = T$ with mesh vanishing and where, for all k, $\phi_{n,k}$ is a bounded and $\mathcal{F}_{t^n_{k-1}}$-measurable random variable.

Mimicking the arguments we have seen for Brownian motion,

1. prove the Itô isometry for simple integrands in this framework. Provide full detail.
2. Define the Itô integral by an appropriate limiting argument. Provide full detail.
3. Also, prove that in this framework the following holds true for the simple integrands φ and all s, t with $s < t$:

$$E\Big[\int_s^t \varphi(u) dM(u) | \mathcal{F}_s \Big] = 0.$$

Exercise 2.
Let $B = B_t$, $t \geq 0$, be a Brownian motion. Provide full detail when replying the following questions.

1. Study the process $Y_t := e^{B_t}$, $t \in [0,T]$, and establish whether it is a martingale, sub-martingale, or super-martingale.
2. Let $f \in L^2[0,T]$ be a deterministic function. Study the process
$$Z_t := e^{\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds}, \qquad t \in [0,T],$$
and establish whether it is a martingale, sub-martingale, or super-martingale.
3. Use the properties of Gaussian distributions to compute the first and second moment of Y_t and Z_t for a given t
4. By means of the Itô formula show that the processes Y and Z above are Itô processes.

Exercise 3.
By application of the Itô formula show that the following processes are Itô processes:

1. $X_t = xe^{\lambda t} + \sigma e^{\lambda t}\int_0^t e^{-\lambda s}dB_s, \qquad t \geq 0 \qquad (\lambda, \sigma > 0, x \in \mathbb{R})$
2. $Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}dB_s, \qquad t \in [0,1) \qquad (a, b \in \mathbb{R})$

Exercise 4.
On the finite probability space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with the σ-algebra of all the subset of Ω and with probability
$$P(\omega_k) = \begin{cases} \frac{1}{9}, & k=1 \\ \frac{2}{9}, & k=2 \\ \frac{2}{15}, & k=3 \\ \frac{8}{15}, & k=4, \end{cases}$$
we consider the following stochastic process
$$M(0) = 3; \qquad M(1,\omega_k) = \begin{cases} 5, & k=1,2 \\ 2, & k=3,4 \end{cases}; \qquad M(2,\omega_k) = \begin{cases} 7, & k=1 \\ 4, & k=2, \\ 6, & k=3, \\ 1, & k=4 \end{cases}$$
.

1. Show that it is a martingale with respect to its natural filtration $\mathbb{F} := \{\mathcal{F}_t, t = 0,1,2\}$.

2. Find an $\mathbb{F}$-predictable increasing stochastic process $A(t)$, $t = 0, 1, 2$ with $A(0) = 0$ such that the stochastic process

$$X(t) := M^2(t) - A(t), \qquad t = 0, 1, 2,$$

is a martingale.

3. Find an $\mathbb{F}$-predictable stochastic process $B(t)$, $t = 0, 1, 2$ with $B(0) = 1$ such that the stochastic process

$$Y(t) := B(t)M(t), \qquad t = 0, 1, 2,$$

is a martingale.

4. Find an $\mathbb{F}$-predictable stochastic process $C(t)$, $t = 0, 1, 2$ with $C(0) = 1$ such that the stochastic process

$$Z(t) := C(t)M^3(t), \qquad t = 0, 1, 2,$$

is a martingale.