22$^{\text{nd}}$ March, 2024

MAT4750/9750

Mandatory assignment 1 of 1

Submission deadline

Thursday 11$^{\text{th}}$ APRIL 2024, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have one attempt to pass the assignment. This means that there are no second attempts.

For courses on bachelor level, you can choose between scanning handwritten notes or using a typesetting software for mathematics (e.g. LaTeX). Scanned pages must be clearly legible. For courses on master level the assignment must be written with a typesetting software for mathematics. It is expected that you give a clear presentation with all necessary explanations. The assignment must be submitted as a single PDF file. Remember to include any relevant programming code and resulting plots and figures, in the PDF-file.

All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, you may be asked to give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline. Note that teaching staff cannot grant extensions.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

To pass the assignment you need a score of at least 50p. All questions have equal weight.

Problem 1. 1. (10p) Describe the main ingredients of an Itô financial market: risk free asset, risky assets, portfolio, wealth process,...

2. Suppose that the Itô financial market is given by

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \qquad S_0^0 = 1, \\ dS_t^1 &= \left(\mu - S_t^1\right) dt + \sigma dB_t, \qquad S_0^1 = s_1 > 0, \end{aligned}$$

where $r > 0, \mu > 0$, and $\sigma \neq 0$ are constants.

a) (10p) Find the price of the European T-claim $F = S_T^1$.

b) (10p) Find the replicating portfolio $\varphi = (\varphi_0, \varphi_1)$ for this claim.

Problem 2. 1. (10p) Explain what is an infinitely divisible distribution on $\mathbb{R}^d$. Let μ be the uniform probability distribution over the d-dimensional open ball with radius 1 and centered at the origin. That is the probability distribution with density $\mu(dx) = \frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{d/2}} \mathbf{1}_{\{|x|<1\}}(x)\, dx$,$A \in \mathcal{B}\left(\mathbb{R}^d\right)$, where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$. Is μ infinitely divisible? (**Hint**: The important point to notice is that this distribution has bounded support. The normalization constant does not matter)

2. (10p) Define what is a Lévy measure on $\mathbb{R}_0^d$. Let $d = 1$, is $\nu(dx) = \frac{1}{x^2} \mathbf{1}_{\{x \neq 0\}} dx$ a Lévy measure? And $\mu(dx) = \frac{1}{x^3} \mathbf{1}_{\{|x|>1\}} dx$?

3. (10p) State the Lévy-Kintchine formula. Justify that all the terms in the formula are well defined.

4. (10p) Show that if ν is a Lévy measure on $\mathbb{R}_0^d$, then for all $\varepsilon > 0$ one has that

$$\nu\left(\left\{y \in \mathbb{R}^d : |y| > \varepsilon\right\}\right) < \infty,$$

and conclude that ν is σ-finite.

Problem 3. Let μ be a probability measure on $\mathbb{R}^d$ and $\varphi_\mu(u)$ its characteristic function, i.e.,

$$\varphi_\mu(u) = \int_{\mathbb{R}^d} e^{i\langle u, x\rangle} \mu(dx).$$

We say that $\tilde{\mu}$ is the dual of μ if $\tilde{\mu}(A) = \mu(-A)$,$A \in \mathcal{B}\left(\mathbb{R}^d\right)$ and we say that $\mu^\sharp$ is the symmetrization of μ if $\mu^\sharp = \mu * \tilde{\mu}$.

1. (10p) Prove that $\varphi_{\tilde{\mu}}(u) = \varphi_{\mu}(-u) = \overline{\varphi_{\mu}(u)}$, where $\overline{z} = \overline{(a + \mathrm{i}b)} = a - \mathrm{i}b$ is the complex conjugation. (**Hint**: use the image measure theorem)

2. (10p) Prove that $\varphi_{\mu^{\sharp}}(u) = |\varphi(u)|^2$.

3. (10p) Show that if μ is infinitely divisible divisible probability measure on $\mathbb{R}^d$ then $\varphi_{\mu}(u) \neq 0$, $u \in \mathbb{R}^d$. (**Hint:** consider the limit $\varphi(u) := \lim_{n\to\infty} \left|\varphi_{\mu^{1/n}}(u)\right|^2$ and use Lévy's continuity theorem)

Problem 4. Let $\{Z_n\}_{n\geq 1}$ be a sequence of independent, identically distributed d-dimensional random variables with common law P_Z and N be a Poisson process of intensity λ that is independent of $\{Z_n\}_{n\geq 1}$. Recall that the compound Poisson process is defined as

$$Y_t = Z_1 + \cdots + Z_{N_t}, \qquad t \geq 0,$$

and each $Y_t \sim \text{Poisson}(\lambda t, P_Z)$.

1. (10p) Prove that $Y = \{Y_t\}_{t\in\mathbb{R}_+}$ has stationary and independent increments.

2. (10p) Find $\Theta \subseteq \mathbb{R}^d$, which may depend on Z and N, such that if $\theta \in \Theta$ then $\mathbb{E}\left[e^{\langle\theta, Y_t\rangle}\right] < \infty$,$t \in \mathbb{R}_+$.

3. (10p) For $\theta \in \Theta$, consider the process $X(\theta) = \{X_t(\theta)\}_{t\in\mathbb{R}_+}$ where

$$X_t(\theta) = \exp\left(\langle\theta, Y_t\rangle - \gamma(\theta)\, t\right),$$

for some function $\gamma(\theta)$. Find $\gamma(\theta)$ such that X is a martingale.