3rd May, 2024

MAT4750/9750

Mandatory assignment 1 of 1

Submission deadline

Thursday 11th APRIL 2024, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have one attempt to pass the assignment. This means that there are no second attempts.

For courses on bachelor level, you can choose between scanning handwritten notes or using a typesetting software for mathematics (e.g. LaTeX). Scanned pages must be clearly legible. For courses on master level the assignment must be written with a typesetting software for mathematics. It is expected that you give a clear presentation with all necessary explanations. The assignment must be submitted as a single PDF file. Remember to include any relevant programming code and resulting plots and figures, in the PDF-file.

All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, you may be asked to give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline. Note that teaching staff cannot grant extensions.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

To pass the assignment you need a score of at least 50p. All questions have equal weight.

Problem 1. 1. (10p) Describe the main ingredients of an Itô financial market: risk free asset, risky assets, portfolio, wealth process,...

2. Suppose that the Itô financial market is given by

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \qquad S_0^0 = 1, \\ dS_t^1 &= \left(\mu - S_t^1\right) dt + \sigma dB_t, \qquad S_0^1 = s_1 > 0, \end{aligned}$$

where $r > 0, \mu > 0$, and $\sigma \neq 0$ are constants.

a) (10p) Find the price of the European T-claim $F = S_T^1$.

b) (10p) Find the replicating portfolio $\varphi = (\varphi_0, \varphi_1)$ for this claim.

Problem 2. 1. (10p) Explain what is an infinitely divisible distribution on $\mathbb{R}^d$. Let μ be the uniform probability distribution over the d-dimensional open ball with radius 1 and centered at the origin. That is the probability distribution with density $\mu(dx) = \frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{d/2}} \mathbf{1}_{\{|x|<1\}}(x)\, dx, A \in \mathcal{B}\left(\mathbb{R}^d\right)$, where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$. Is μ infinitely divisible? (**Hint**: The important point to notice is that this distribution has bounded support. The normalization constant does not matter)

2. (10p) Define what is a Lévy measure on $\mathbb{R}_0^d$. Let $d = 1$, is $\nu(dx) = \frac{1}{x^2} \mathbf{1}_{\{x \neq 0\}} dx$ a Lévy measure? And $\mu(dx) = \frac{1}{x^3} \mathbf{1}_{\{|x|>1\}} dx$?

3. (10p) State the Lévy-Kintchine formula. Justify that all the terms in the formula are well defined.

4. (10p) Show that if ν is a Lévy measure on $\mathbb{R}_0^d$, then for all $\varepsilon > 0$ one has that

$$\nu\left(\left\{y \in \mathbb{R}^d : |y| > \varepsilon\right\}\right) < \infty,$$

and conclude that ν is σ-finite.

Problem 3. Let μ be a probability measure on $\mathbb{R}^d$ and $\varphi_\mu(u)$ its characteristic function, i.e.,

$$\varphi_\mu(u) = \int_{\mathbb{R}^d} e^{i\langle u, x\rangle} \mu(dx).$$

We say that $\tilde{\mu}$ is the dual of μ if $\tilde{\mu}(A) = \mu(-A), A \in \mathcal{B}\left(\mathbb{R}^d\right)$ and we say that $\mu^\sharp$ is the symmetrization of μ if $\mu^\sharp = \mu * \tilde{\mu}$.

1. (10p) Prove that $\varphi_{\tilde{\mu}}(u) = \varphi_{\mu}(-u) = \overline{\varphi_{\mu}(u)}$, where $\overline{z} = \overline{(a + ib)} = a - ib$ is the complex conjugation. (**Hint**: use the image measure theorem)

2. (10p) Prove that $\varphi_{\mu^{\sharp}}(u) = |\varphi(u)|^2$.

3. (10p) Show that if μ is infinitely divisible divisible probability measure on $\mathbb{R}^d$ then $\varphi_{\mu}(u) \neq 0$, $u \in \mathbb{R}^d$. (**Hint:** consider the limit $\varphi(u) := \lim_{n\to\infty} \left|\varphi_{\mu^{1/n}}(u)\right|^2$ and use Lévy's continuity theorem)

Problem 4. Let $\{Z_n\}_{n\geq 1}$ be a sequence of independent, identically distributed d-dimensional random variables with common law P_Z and N be a Poisson process of intensity λ that is independent of $\{Z_n\}_{n\geq 1}$. Recall that the compound Poisson process is defined as

$$Y_t = Z_1 + \cdots + Z_{N_t}, \qquad t \geq 0,$$

and each $Y_t \sim \text{Poisson}(\lambda t, P_Z)$.

1. (10p) Prove that $Y = \{Y_t\}_{t\in\mathbb{R}_+}$ has stationary and independent increments.

2. (10p) Find $\Theta \subseteq \mathbb{R}^d$, which may depend on Z and N, such that if $\theta \in \Theta$ then $\mathbb{E}\left[e^{\langle\theta, Y_t\rangle}\right] < \infty$,$t \in \mathbb{R}_+$.

3. (10p) For $\theta \in \Theta$, consider the process $X(\theta) = \{X_t(\theta)\}_{t\in\mathbb{R}_+}$ where

$$X_t(\theta) = \exp\left(\langle\theta, Y_t\rangle - \gamma(\theta)\, t\right),$$

for some function $\gamma(\theta)$. Find $\gamma(\theta)$ such that X is a martingale.

Solution Problem 1

1. Let B be an m-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and let $\mathbb{F} := \mathbb{F}^B$ be the usual augmentation of the natural filtration generated by B.

   Definition 1. In the previous probabilistic setup, a financial market with $n+1$ investment possibilities consists in

   a) A risk free asset, where the unit price S_t^0 at time t is given by
   $$\begin{aligned} dS_t^0 &= r(t) S_t^0 dt, \qquad t \in [0,T], \\ S_0^0 &= 1. \end{aligned}$$

   b) n risky assets, where the unit price S_t^i at time t of the i-the risky asset is given by
   $$\begin{aligned} dS_t^i &= \mu^i(t)\, dt + \sum_{j=1}^m \sigma_j^i(t)\, dB_t^j \qquad t \in [0,T], \\ S_0^i &= s_0^i \in \mathbb{R}, \end{aligned}$$
   for $i = 1, ..., n$.
   Here:
   - $T > 0$ is the investment horizon,
   - $r \geq 0$ is the interest rate, $r^{1/2} \in L_{a,T}^0$ and we will assume it to be bounded.
   - $\mu = (\mu^1(t), ..., \mu^n(t))^T$ is the vector of appreciation rates of the risky assets and $(\mu^i)^{1/2} \in L_{a,T}^0$, $i = 1, ..., n$.
   - $\sigma = \left(\sigma_j^i(t)\right)_{i=1,...,n, j=1,...,m}$ is the volatility matrix of the risky assets and $\sigma_j^i \in L_{a,T}^0$, $i = 1, ..., n, j = 1, ..., m$. Note that σ^i denote the ith row of the $n \times m$ matrix σ, that is, $\sigma^i = (\sigma_1^i(t), ..., \sigma_m^i)$..
     The market is called **normalized** or **discounted** if $S_t^0 \equiv 1$.

   Remark 2. S^0 is called the safe investment because there is no Brownian part in its dynamics and since $r \geq 0$ the value of this investment cannot decrease in time. Note however that S^0 is in general a stochastic process. Since
   $$S_t^0 = \exp\left(\int_0^t r(s)\, ds\right) > 0, \qquad 0 \leq t \leq T,$$
   we can define the discount factor (or normalizing factor) $\xi(t)$ by
   $$\xi(t) := \left(S_t^0\right)^{-1} = \exp\left(-\int_0^t r(s)\, ds\right) > 0, \qquad 0 \leq t \leq T.$$

Note that $d\xi(t) = -r(t)\xi(t)\,dt$. Hence, we can always normalize the market by defining $\bar{S}_t^i = \xi(t) S_t^i, i = 1, ..., n$. The market

$$\bar{S}_t = \left(1, \bar{S}_t^1 ..., \bar{S}_t^n\right)^T,$$

is called the normalization of S. Normalization corresponds to consider the price S^0 of the safe investment as the unit price (the numeraire) and compute the other prices in terms of this unit.

Definition 3. A **portfolio** φ is an $(n+1)$-dimensional process

$$\varphi(t) = (\varphi_0(t), \varphi_1(t), ..., \varphi_n(t)),$$

with $\varphi_i \in L^0_{a,T}$ representing the number of units of the i-th asset held at time t. Note that this is a row vector.

Definition 4. The **wealth (or value) process** $V = V^\varphi$ associated to the portfolio φ is defined by

$$V_t^\varphi = \varphi(t) S_t = \sum_{i=0}^{n} \varphi_i(t) S_t^i.$$

Definition 5. A portfolio φ is called **self-financing** if $\int_0^t \varphi(s)\,dS_s$ exists for all $t > 0$ and

$$V_t^\varphi = V_0^\varphi + \int_0^t \varphi(s)\,dS_s,$$

or, in differential notation,

$$dV_t^\varphi = \varphi(t)\,dS_t.$$

Definition 6. The **normalized (or discounted) wealth process** $\bar{V}^\varphi$ is defined by

$$\bar{V}_t^\varphi = \xi(t) V_t^\varphi = \xi(t)\varphi(t) S_t = \varphi(t)\bar{S}_t.$$

Definition 7. A self-financing portfolio φ is called **admissible** if there exists a constant $K = K_\varphi > 0$ such that

$$V_t^\varphi \geq -K_\varphi, \qquad a.s. \text{ for all } t \in [0, T].$$

The set of all admissible portfolios is denoted by $\mathcal{A}$.

Definition 8. A portfolio $\varphi \in \mathcal{A}$ is called an arbitrage if

$$V_0^\varphi = 0, \qquad V_T^\varphi \geq 0, \quad P\text{-}a.s. \quad \text{and} \quad P(V_T^\varphi > 0) > 0.$$

2.

a) In this market consider $\theta \in L^2_{a,T}$ such that

$$\sigma\theta_t = \mu - S_t^1 - rS_t^1, \qquad \lambda \otimes P\text{-}a.e.,$$

that is $\theta_t = \frac{\mu-(1-r)S_t^1}{\sigma}$. We need to consider the the change of measure

$$\frac{dQ}{dP} = Z_t(\theta) = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right).$$

To show that $Z(\theta)$ is a martingale is not straightforward (it does not follow by a direct application of Novikov's theorem, but it is true) so we will assume it without proof. By Girsanov's theorem we have that

$$dB_t^Q := \theta_t dt + dB_t,$$

is a Q-Brownian motion and the dynamics of S^1 can be rewriten in terms of B^Q as

$$\begin{aligned} dS_t^1 &= \left(\mu - S_t^1\right) dt + \sigma dB_t = \left(\mu - S_t^1\right) dt + \sigma(dB_t^Q - \theta_t dt) \\ &= rS_t^1 dt + \sigma dB_t^Q. \end{aligned}$$

Moreover, using Ito's product rule with $e^{-rt}S_t^1$ and taking into account that $d\left[e^{-r\cdot}, S^1\right]_t = 0$ we get that

$$d\left(e^{-rt}S_t^1\right) = -rS_t^1 dt + e^{-rt}dS^1 = \sigma dB_t^Q,$$

which yields

$$S_t^1 = e^{rt}S_0^1 + \int_0^t e^{r(t-s)}\sigma dB_s^Q. \tag{1}$$

By Theorem 5.16, we have that the market is arbitrage free. By Corollary 5.27 (1), since $n = m$ and σ is invertible we have that the market is complete. Then, using Theorem 5.31 we know that the upper and lower price of F coincide and are equal to

$$p(F) = \mathbb{E}_Q\left[e^{-rT}S_T^1\right] = \mathbb{E}_Q\left[e^{-rT}\left(e^{rT}S_0^1 + \int_0^T e^{r(T-s)}\sigma dB_s^Q\right)\right] = S_0^1,$$

because the (Q-)expectation of an Itô integral with respect to a Q-Brownian motion is zero.

b) By Theorem 5.36, the replicating portfolio will satisfy

$$e^{-rt}\varphi_1(t)\,\sigma=\psi_t,$$

where ψ is such that

$$e^{-rT}F=e^{-rT}S_T^1=\mathbb{E}_Q\left[e^{-rT}S_T^1\right]+\int_0^T\psi_t dB_t^Q=S_0^1+\int_0^T\psi_t dB_t^Q.$$

Looking at equation (1) we deduce that $\psi_t=\sigma e^{-rt}$ and then $\varphi_1(t)\equiv 1$. Moreover, using the formulas in Lemma 5.8, we find that

$$\begin{aligned}\varphi_0(t)&=\varphi_0(0)+\int_0^t e^{-rs}d\left(\int_0^s\varphi_1(u)\,dS_u^1-\varphi_1(s)\,S_s^1\right)\\&=\varphi_0(0)+\int_0^t e^{-rs}d\left(S_s^1-S_0^1-S_s^1\right)=\varphi_0(0),\end{aligned}$$

and, choosing $V_0^\varphi:=\mathbb{E}_Q\left[e^{-rT}S_T^1\right]=S_0^1$, we have that $\varphi_0(0)$ must satisfy

$$S_0^1=\varphi_0(0)\,S_0^0+\varphi_1(0)\,S_0^1=\varphi_0(0)+S_0^1,$$

which yields $\varphi_0(0)=0$. Choosing $V_0^\varphi:=0$, would yield $\varphi_0(0)=-S_0^1$. In any case, the interpretation of the hedging strategy is the buy and hold strategy.

Remark 9. In this problem, since it is clear that the hedging strategy is the buy and hold strategy, one can make simpler reasonings to compute the price.

Solution Problem 2

1. A probability distribution on $\mathbb{R}^d$,which can be associated to the law of a random vector, is infinitely divisible if it can be written as the law of the sum of an *arbitrary* number of independent and identically distributed (i.i.d.) random vectors. The fact that the number of summands can be arbitrarily large and they are identically distributed ensures that the contribution of each term in the decomposition can be made aritrarily/infinitely small. A precise definition is as follows: let X be a random variable in $\mathbb{R}^d$ with law P_X. We say that X is *infinitely divisible* if, for all $n\in\mathbb{N}$, there exists $Y_{n,1},\ldots,Y_{n,n}$, i.i.d. random variables in $\mathbb{R}^d$, such that

$$\mathcal{L}(X)=\mathcal{L}(Y_{n,1}+\cdots+Y_{n,n}).$$

The previous property can be expressed in term of convolution products of probability distributions, because the law of the sum of independent random

vectors correspond to the convolution product of the laws of its summands. This yields a criteria for infinite divisibility in terms of convolution n-th roots of probability distributions. Similarly, the characteristic function of the sum of independent random vectors correspond to the product of the laws of its summands. This yields a criteria for infinite divisibility in terms of n-th roots of characteristic functions.
The probability distribution is the uniform distribution in

$$B_1(0) = \left\{x \in \mathbb{R}^d : |x| < 1\right\}.$$

Let $Y \sim \mu$. Note that

$$P(|Y| < 1) = 1. \tag{2}$$

Suppose that Y is infinitely divisible, then for every $n \in \mathbb{N}$ there exists $\{Y_{n,i}\}_{i=1,\dots,n}$ i.i.d. such that

$$\mathcal{L}(Y) = \mathcal{L}(Y_{n,1} + \cdots + Y_{n,n}).$$

Fix $n \in \mathbb{N}$, then equation (2) and the fact that $\{Y_{n,i}\}_{i=1,\dots,n}$ are identically distributed implies that

$$P\left(|Y_{n,i}| < \frac{1}{n}\right) = 1, \qquad i = 1, \dots, n.$$

Moreover,

$$\mathrm{Var}[Y_{n,i}] = \mathbb{E}\left[|Y_{n,i}|^2\right] - (E[|Y_{n,1}|])^2 \leq \mathbb{E}\left[|Y_{n,i}|^2\right] \leq \frac{1}{n^2}, \qquad i = 1, \dots, n.$$

Since $\{Y_{n,i}\}_{i=1,\dots,n}$ are independent, we get

$$\mathrm{Var}[Y] = \sum_{i=1}^{n} \mathrm{Var}[Y_{n,i}] \leq \frac{n}{n^2} = \frac{1}{n}.$$

However, as n can be arbitrarily large, the last inequality implies that the variance of Y must be arbitrarily small. This contradicts the assumption $Y \sim \mu$. Indeed, we have proved that the unique infinitely divisible distribution with bounded support is the Dirac distribution at the origin, that is, $Y \equiv 0$,P-a.s.

2. Let ν be a Borel measure defined on $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$. We say that ν is a Lévy measure if

$$\int_{\mathbb{R}_0^d} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty. \tag{3}$$

If we take $d = 1$ and $\nu(dx) = \frac{1}{x^2}\mathbf{1}_{\{x\neq 0\}}dx$ we have that ν is a Borel measure and

$$\begin{aligned}
\int_{\mathbb{R}_0^d}\left(|x|^2 \wedge 1\right)\nu(dx) &= \int_{\mathbb{R}_0^d}\left(|x|^2 \wedge 1\right)\frac{1}{x^2}dx \\
&= \int_{0<|x|<1}\frac{|x|^2}{x^2}dx + \int_{|x|\geq 1}\frac{1}{x^2}dx \\
&= \int_{-1}^{1}dx + 2\int_{1}^{+\infty}\frac{1}{x^2}dx \\
&= 2 - 2\left[x^{-1}\right]_1^{+\infty} = 2 + 2 = 4 < \infty,
\end{aligned}$$

and (3) is satisfied. If we take $d = 1$ and $\mu(dx) = \frac{1}{x^3}\mathbf{1}_{\{|x|>1\}}dx$, note that μ is not a Borel measure as it is negative for $x < 0$. Hence, μ is not a Lévy measure. (The problem is not the integrability because we are integrating over $\{|x| > 1\}$).

3. The Lévy-Kintchine theorem is as follows: $\mu \in \mathcal{M}_1\left(\mathbb{R}^d\right)$ is infinitely divisible if there exists a vector $\gamma \in \mathbb{R}^d$, a positive definite symmetric $d \times d$ matrix A and a Lévy measure ν on $\mathbb{R}_0^d$ such that, for all $u \in \mathbb{R}^d$,

$$\varphi_\mu(u) = \exp\left(\mathrm{i}\langle\gamma, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}_0^d}\left(e^{\mathrm{i}\langle u,y\rangle} - 1 - \mathrm{i}\langle u, y\rangle \mathbf{1}_{B_1(0)}(y)\right)\nu(dy)\right), \tag{4}$$

where $B_1(0) := \left\{x \in \mathbb{R}^d : |x| < 1\right\}$. Conversely, any mapping of the form (4) is the characteristic function of an infinitely divisible probability measure on $\mathbb{R}^d$.
Note that the formula (4) is well defined if

$$\int_{\mathbb{R}_0^d}\left(e^{\mathrm{i}\langle u,y\rangle} - 1 - \mathrm{i}\langle u, y\rangle \mathbf{1}_{B_1(0)}(y)\right)\nu(dy) < \infty. \tag{5}$$

Using Taylor's theorem and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left|e^{\mathrm{i}\langle u,y\rangle} - 1 - \mathrm{i}\langle u, y\rangle \mathbf{1}_{B_1(0)}(y)\right| &\leq \left|\mathbf{1}_{B_1(0)}(y)\left(e^{\mathrm{i}\langle u,y\rangle} - 1 - \mathrm{i}\langle u, y\rangle\right)\right| + \left|\mathbf{1}_{B_1^c(0)}(y)\left(e^{\mathrm{i}\langle u,y\rangle} - 1\right)\right| \\
&\leq \frac{1}{2}\left|\langle u, y\rangle\right|^2 \mathbf{1}_{B_1(0)}(y) + 2\mathbf{1}_{B_1^c(0)}(y) \\
&\leq \frac{1}{2}|u|^2|y|^2 \mathbf{1}_{B_1(0)}(y) + 2\mathbf{1}_{B_1^c(0)}(y).
\end{aligned}$$

Combining the previous inequality with equation (3) we obtain (5), that is,

$$e^{\mathrm{i}\langle u,y\rangle} - 1 - \mathrm{i}\langle u, y\rangle \mathbf{1}_{B_1(0)}(y) \in L^1(v).$$

4. Since ν is a Lévy measure we know that it satisfies (3). Note that (3) is equivalent to the following two conditions

$$\int_{0<|y|<1} |y|^2 \nu(dy) < \infty \tag{6}$$

and

$$\int_{|y|\geq 1} \nu(dy) < \infty. \tag{7}$$

If $\varepsilon \geq 1$, then (7) implies the result. If $\varepsilon < 1$, then

$$\nu\left(\left\{y \in \mathbb{R}^d : |y| > \varepsilon\right\}\right) = \int_{\varepsilon<|y|<1} \nu(dy) + \int_{|y|\geq 1} \nu(dy).$$

The second integral is finite by (7). For the first integral we have

$$\begin{aligned}\int_{\varepsilon<|y|<1} \nu(dy) &= \frac{1}{\varepsilon^2}\int_{\varepsilon<|y|<1} \varepsilon^2 \nu(dy) < \frac{1}{\varepsilon^2}\int_{\varepsilon<|y|<1} |y|^2 \nu(dy) \\ &< \frac{1}{\varepsilon^2}\int_{0<|y|<1} |y|^2 \nu(dy) < \infty,\end{aligned}$$

where in the second inequality we have used that the monotonicity of the integral and

$$\mathbf{1}_{\{\varepsilon<|y|<1\}} |y|^2 < \mathbf{1}_{\{0<|y|<1\}} |y|^2$$

and in the third inequality we have used (6). To show that ν is σ-finite, we consider the decomposition

$$\mathbb{R}_0^d = \biguplus_{n\geq 1} A_n,$$

where $A_1 = \{|y| \geq 1\}$ and

$$A_n = \left\{\frac{1}{n} \leq |y| < \frac{1}{n-1}\right\}, \qquad n \geq 2.$$

Clearly, $\nu(A_1) < \infty$ by (6) and, for $n \geq 2$,

$$\nu(A_n) \leq \nu\left(\left\{|y| > \frac{1}{n+1}\right\}\right) < \infty,$$

where we have used that $A_n \subset \left\{|y| > \frac{1}{n+1}\right\}$, the monotonicity of measures and (7).

Solution Problem 3

1. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be the (bijective, $T\left(\mathbb{R}^d\right) = \mathbb{R}^d$) mapping given by $y = T(x) = -x$. Note that, by construction $\tilde{\mu} = \mu \circ T^{-1} = \mu_T$. Then, by the image measure theorem we have that

$$\begin{aligned}\varphi_{\tilde{\mu}}(u) &= \int_{\mathbb{R}^d} e^{\mathrm{i}\langle u,y\rangle}\tilde{\mu}(dy) = \int_{T\left(\mathbb{R}^d\right)} e^{\mathrm{i}\langle u,y\rangle}\mu_T(dy) \\ &= \int_{\mathbb{R}^d} e^{\mathrm{i}\langle u,T(x)\rangle}\mu(dx) = \int_{\mathbb{R}^d} e^{\mathrm{i}\langle u,-x\rangle}\mu(dx) \\ &= \int_{\mathbb{R}^d} e^{\mathrm{i}\langle -u,x\rangle}\mu(dx) = \varphi_\mu(-u).\end{aligned}$$

Note that, by Euler's formula and the basic properties of trigonometric functions,

$$\begin{aligned}e^{\mathrm{i}\langle -u,x\rangle} &= \cos(\langle -u,x\rangle) + \mathrm{i}\sin(\langle -u,x\rangle) \\ &= \cos(\langle u,x\rangle) - \mathrm{i}\sin(\langle u,x\rangle).\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi_\mu(-u) &= \int_{\mathbb{R}^d} e^{\mathrm{i}\langle -u,x\rangle}\mu(dx) \\ &= \int_{\mathbb{R}^d}\cos(\langle u,x\rangle)\mu(dx) - \mathrm{i}\int_{\mathbb{R}^d}\sin(\langle u,x\rangle)\mu(dx) \\ &= \overline{\int_{\mathbb{R}^d}\cos(\langle u,x\rangle)\mu(dx) + \mathrm{i}\int_{\mathbb{R}^d}\sin(\langle u,x\rangle)\mu(dx)} \\ &= \overline{\int_{\mathbb{R}^d} e^{\mathrm{i}\langle u,x\rangle}\mu(dx)} = \overline{\varphi_\mu(u)}.\end{aligned}$$

2. Using the formula that relates integrals with respect to the convolution product of two measures and integrals with respect to the product measure of these two measures, combined with Fubini's theorem, we obtain

$$\begin{aligned}\varphi_{\mu^\sharp}(u) &= \varphi_{\mu*\tilde{\mu}}(u) = \int_{\mathbb{R}^d} e^{\mathrm{i}\langle u,x\rangle}(\mu*\tilde{\mu})(dx) \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{\mathrm{i}\langle u,x+y\rangle}\mu\otimes\tilde{\mu}(dx,dy) \\ &= \int_{\mathbb{R}^d} e^{\mathrm{i}\langle u,x\rangle}\mu(dx)\int_{\mathbb{R}^d} e^{\mathrm{i}\langle u,y\rangle}\tilde{\mu}(dy) \\ &= \varphi_\mu(u)\,\varphi_{\tilde{\mu}}(u) = \varphi_\mu(u)\,\varphi_\mu(-u) \\ &= \varphi_\mu(u)\,\overline{\varphi_\mu(u)} = |\varphi_\mu(u)|^2.\end{aligned}$$

3. Let μ be an infinitely divisible divisible probability measure on $\mathbb{R}^d$. Then, by Proposition 6.8 in the lecture notes, we know that, for each $n \in \mathbb{N}$, there exists a probability distribution $\mu^{1/n}$, with characteristic function $\varphi_{\mu^{1/n}}$ such that
$$\varphi_\mu(u) = \left(\varphi_{\mu^{1/n}}(u)\right)^n.$$
By the previous step we have that $|\varphi_\mu(u)|^2$ and $\left|\varphi_{\mu^{1/n}}(u)\right|^2$ are (real-valued) characteristic function. Moreover, note that
$$|\varphi_\mu(u)|^2 = \varphi_\mu(u)\overline{\varphi_\mu(u)} = \left(\varphi_{\mu^{1/n}}(u)\right)^n \overline{\left(\varphi_{\mu^{1/n}}(u)\right)^n} = \left|\varphi_{\mu^{1/n}}(u)\right|^{2n}$$
and we get that
$$\left|\varphi_{\mu^{1/n}}(u)\right|^2 = |\varphi_\mu(u)|^{2/n}.$$
Hence, we can define $\varphi(u)$ by the following limit of real-valued functions
$$\varphi(u) := \lim_{n\to\infty}\left|\varphi_{\mu^{1/n}}(u)\right|^2 = \lim_{n\to\infty}|\varphi_\mu(u)|^{2/n} = \left\{\begin{array}{lll} 1 & \text{if} & \varphi_\mu(u) \neq 0 \\ 0 & \text{if} & \varphi_\mu(u) = 0 \end{array}\right. .$$
Since $\varphi_\mu(0) = 1$ and φ_μ is continuous, there exists a neighbourhood U_0 of 0 such that $\varphi_\mu(u) \neq 0, u \in U_0$ and, therefore, $\varphi(u) = 1, u \in U_0$ and φ is also continuous in U_0. By Lévy's continuity theorem φ is the characteristic function of a probability distribution and, hence, it is continuous for all $u \in \mathbb{R}^d$. But this can only happen if φ is identically equal to 1, that is, if $\varphi_\mu(u) \neq 0, u \in \mathbb{R}^d$.

Solution Problem 4

1. First we prove that Y has stationary increments. As $Y_0 = 0$, we have to prove that
$$\mathbb{E}\left[e^{\mathrm{i}\langle u, Y_t - Y_s\rangle}\right] = \mathbb{E}\left[e^{\mathrm{i}\langle u, Y_{t-s}\rangle}\right], \qquad u \in \mathbb{R}^d, 0 \leq s < t.$$
In the following computations we will be using the basic properties of conditional expectation such as conservation of expectation and the substitution property. We will also make use of the fact that $\{Z_i\}_{i\geq 1}$ are i.i.d. random variables and that N has stationary increments. We can write
$$\begin{aligned}\mathbb{E}\left[e^{\mathrm{i}\langle u, Y_t - Y_s\rangle}\right] &= \mathbb{E}\left[e^{\mathrm{i}\left\langle u, \sum_{j=N_s+1}^{N_t} Z_j\right\rangle}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\left\langle u, \sum_{j=N_s+1}^{N_t} Z_j\right\rangle}\middle|\sigma(N_t, N_s)\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\left\langle u, \sum_{j=n_s+1}^{n_t} Z_j\right\rangle}\right]\Bigg|_{n_s = N_s, n_t = N_t}\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\left\langle u,\sum_{j=1}^{n_t-n_s} Z_j\right\rangle}\right]\Big|_{n_s=N_s,n_t=N_t}\right]\\
&= \mathbb{E}\left[\left(\prod_{j=1}^{n_t-n_s}\mathbb{E}\left[e^{\mathrm{i}\langle u,Z_j\rangle}\right]\right)\Bigg|_{n_s=N_s,n_t=N_t}\right]\\
&= \mathbb{E}\left[\left(\prod_{j=1}^{N_t-N_s}\mathbb{E}\left[e^{\mathrm{i}\langle u,Z_j\rangle}\right]\right)\right]\\
&= \mathbb{E}\left[\left(\prod_{j=1}^{N_{t-s}}\mathbb{E}\left[e^{\mathrm{i}\langle u,Z_j\rangle}\right]\right)\right]\\
&= \mathbb{E}\left[\left(\prod_{j=1}^{n}\mathbb{E}\left[e^{\mathrm{i}\langle u,Z_j\rangle}\right]\right)\Bigg|_{n=N_{t-s}}\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\left\langle u,\sum_{j=1}^{n} Z_j\right\rangle}\right]\Big|_{n=N_{t-s}}\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\left\langle u,\sum_{j=1}^{N_{t-s}} Z_j\right\rangle}\,\middle|\,\sigma\left(N_{t-s}\right)\right]\right] = \mathbb{E}\left[e^{\mathrm{i}\langle u,Y_{t-s}\rangle}\right].
\end{aligned}
$$

Secondly we prove that Y has independent increments. We have to show that for any $0 \le t_0 < t_1 < \cdots < t_n$ and $\{u_j\}_{j=1,\dots,n} \subseteq \mathbb{R}^d$ we have that

$$
\mathbb{E}\left[e^{\mathrm{i}\sum_{j=1}^{n}\left\langle u_j,Y_{t_j}-Y_{t_{j-1}}\right\rangle}\right] = \prod_{j=1}^{n}\mathbb{E}\left[e^{\mathrm{i}\left\langle u_j,Y_{t_j}-Y_{t_{j-1}}\right\rangle}\right].
$$

By similar arguments as before,

$$
\begin{aligned}
\mathbb{E}\left[e^{\mathrm{i}\sum_{j=1}^{n}\left\langle u_j,Y_{t_j}-Y_{t_{j-1}}\right\rangle}\right] &= \mathbb{E}\left[e^{\mathrm{i}\sum_{j=1}^{n}\left\langle u_j,\sum_{i=N_{t_{j-1}}+1}^{N_{t_j}} Z_i\right\rangle}\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\sum_{j=1}^{n}\left\langle u_j,\sum_{i=N_{t_{j-1}}+1}^{N_{t_j}} Z_i\right\rangle}\,\middle|\,\sigma\left(N_{t_0},\dots,N_{t_n}\right)\right]\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\sum_{j=1}^{n}\left\langle u_j,\sum_{i=n_{j-1}+1}^{n_j} Z_i\right\rangle}\right]\Big|_{n_0=N_{t_0},\dots,n_n=N_{t_n}}\right]\\
&= \mathbb{E}\left[\left(\prod_{j=1}^{n}\mathbb{E}\left[e^{\mathrm{i}\left\langle u_j,\sum_{i=n_{j-1}+1}^{n_j} Z_i\right\rangle}\right]\right)\Bigg|_{n_0=N_{t_0},\dots,n_n=N_{t_n}}\right]\\
&= \mathbb{E}\left[\left(\prod_{j=1}^{n}\varphi_Z\left(u_j\right)^{n_j-n_{j-1}}\right)\Bigg|_{n_0=N_{t_0},\dots,n_n=N_{t_n}}\right]
\end{aligned}
$$

$$= \mathbb{E}\left[\left(\prod_{j=1}^{n} \varphi_Z\left(u_j\right)^{N_{t_j}-N_{t_{j-1}}}\right)\right]$$
$$= \prod_{j=1}^{n} \mathbb{E}\left[\varphi_Z\left(u_j\right)^{N_{t_j}-N_{t_{j-1}}}\right]$$

On the other hand

$$\begin{aligned}
\mathbb{E}\left[e^{\mathrm{i}\left\langle u_j, Y_{t_j}-Y_{t_{j-1}}\right\rangle}\right] &= \mathbb{E}\left[e^{\mathrm{i}\left\langle u_j, \sum_{i=N_{t_{j-1}}+1}^{N_{t_j}} Z_i\right\rangle}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\left\langle u_j, \sum_{i=N_{t_{j-1}}+1}^{N_{t_j}} Z_i\right\rangle}\middle| \sigma\left(N_{t_j}, N_{t_{j-1}}\right)\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\left\langle u_j, \sum_{i=n_{j-1}+1}^{n_j} Z_i\right\rangle}\right]\Bigg|_{n_j=N_{t_j}, n_{j-1}=N_{t_{j-1}}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}\left\langle u_j, \sum_{i=1}^{n_j-n_{j-1}} Z_i\right\rangle}\right]\Bigg|_{n_j=N_{t_j}, n_{j-1}=N_{t_{j-1}}}\right] \\
&= \mathbb{E}\left[\left(\prod_{i=1}^{n_j-n_{j-1}} \mathbb{E}\left[e^{\mathrm{i}\langle u_j, Z_i\rangle}\right]\right)\Bigg|_{n_j=N_{t_j}, n_{j-1}=N_{t_{j-1}}}\right] \\
&= \mathbb{E}\left[\left(\varphi_Z\left(u_j\right)^{n_j-n_{j-1}}\right)\Big|_{n_j=N_{t_j}, n_{j-1}=N_{t_{j-1}}}\right] \\
&= \mathbb{E}\left[\varphi_Z\left(u_j\right)^{N_{t_j}-N_{t_{j-1}}}\right],
\end{aligned}$$

and we can conclude.

2. Let $\psi_Z(\theta)$ denote the moment generating function of Z, that is,

$$\mathbb{E}\left[e^{\langle\theta, Z\rangle}\right] = \int_{\mathbb{R}^d} e^{\langle\theta, z\rangle} P_Z(dz),$$

and assume that $\theta \in \Theta_Z$, where

$$\Theta_Z = \left\{\theta \in \mathbb{R}^d : \int_{\mathbb{R}^d} e^{\langle\theta, z\rangle} P_Z(dz) < \infty\right\}.$$

Then, if $\theta \in \Theta_Z$, we have that

$$\begin{aligned}
\mathbb{E}\left[e^{\langle\theta, Y_t\rangle}\right] &= \mathbb{E}\left[e^{\left\langle\theta, \sum_{i=1}^{N_t} Z_i\right\rangle}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{\left\langle\theta, \sum_{i=1}^{N_t} Z_i\right\rangle}\middle|\sigma\left(N_t\right)\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\left\langle\theta, \sum_{i=1}^{n} Z_i\right\rangle}\right]\Bigg|_{n=N_t}\right] = \mathbb{E}\left[\left(\prod_{i=1}^{n} \mathbb{E}\left[e^{\langle\theta, Z_i\rangle}\right]\right)\Bigg|_{n=N_t}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\psi_Z(\theta)^{N_t}\right] = \sum_{k=0}^{\infty} \psi_Z(\theta)^k e^{-\lambda t} \frac{(\lambda t)^k}{k!} \\
&= e^{-\lambda t} e^{\lambda t \psi_Z(\theta)} = e^{\lambda t(\psi_Z(\theta)-1)} < \infty.
\end{aligned}
$$

Therefore, we can conclude that the fact that $\mathbb{E}\left[e^{\langle\theta, Y_t\rangle}\right] < \infty$ does not depend on t or λ, but only on the fact that Z has finite exponential moments, that is, $\Theta_Z \neq \emptyset$. The desired set is $\Theta = \Theta_Z$.

3. Let $X(\theta)$ be the process

$$
X_t(\theta) = \exp\left(\langle\theta, Y_t\rangle - \gamma(\theta) t\right), \qquad t \in \mathbb{R}_+,
$$

for some function $\gamma(\theta), \theta \in \Theta$. The integrability requirement is satisfied

$$
\begin{aligned}
\mathbb{E}\left[|X_t(\theta)|\right] &= \mathbb{E}\left[|\exp\left(\langle\theta, Y_t\rangle - \gamma(\theta) t\right)|\right] \\
&= \mathbb{E}\left[\exp\left(\langle\theta, Y_t\rangle - \gamma(\theta) t\right)\right] \\
&= \mathbb{E}\left[e^{\langle\theta, Y_t\rangle}\right] e^{-\gamma(\theta) t} \\
&= e^{\lambda t(\psi_Z(\theta)-1) - \gamma(\theta) t} < \infty.
\end{aligned}
$$

We consider the filtration $\mathbb{F}^Y = \left\{\mathcal{F}_t^Y = \sigma\left(\{Y_s\}_{s \leq t}\right)\right\}_{t \in \mathbb{R}_+}$. $X_t(\theta)$ is $\sigma(Y_t)$-measurable because it is the composition of a Borel measurable function and Y_t. Hence, $X_t(\theta)$ is $\mathcal{F}_t^Y$-measurable for all $t \in \mathbb{R}_+$ and $X(\theta)$ is $\mathbb{F}^Y$-adapted. For the martingale property, it suffices to check that

$$
\mathbb{E}\left[\left.\frac{X_t(\theta)}{X_s(\theta)}\right| \mathcal{F}_s^Y\right] = 1.
$$

We have that

$$
\begin{aligned}
\mathbb{E}\left[\left.\frac{X_t(\theta)}{X_s(\theta)}\right| \mathcal{F}_s^Y\right] &= \mathbb{E}\left[\exp\left(\langle\theta, Y_t - Y_s\rangle - \gamma(\theta)(t-s)\right) \middle| \mathcal{F}_s^Y\right] \\
&= \mathbb{E}\left[\exp\left(\langle\theta, Y_t - Y_s\rangle\right) \middle| \mathcal{F}_s^Y\right] e^{-\gamma(\theta)(t-s)} \\
&= \mathbb{E}\left[\exp\left(\langle\theta, Y_t - Y_s\rangle\right)\right] e^{-\gamma(\theta)(t-s)} \\
&= \mathbb{E}\left[\exp\left(\langle\theta, Y_{t-s}\rangle\right)\right] e^{-\gamma(\theta)(t-s)} \\
&= e^{\lambda(t-s)(\psi_Z(\theta)-1)} e^{-\gamma(\theta)(t-s)} \\
&= e^{(t-s)\{\lambda(\psi_Z(\theta)-1)-\gamma(\theta)\}},
\end{aligned}
$$

where in the third equality we have used that $Y_t - Y_s$ is independent from $\mathcal{F}_s^Y$ and in the fourth equality we have used that Y has stationary increments. Therefore, the martingale property is satisfied if and only if

$$
\gamma(\theta) = \lambda\left(\psi_Z(\theta) - 1\right).
$$