UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4750/9750 — Mathematical Finance: Modelling and risk management

Day of examination: Wednesday 29, May 2024

Examination hours: 13:00 PM – 17:00 PM

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Define what is a subordinator and give an expression for its Lévy symbol. Why is interesting for stochastic modelling? Can the measure $\nu(dy) = y^{-1/2}\mathbf{1}_{(0,+\infty)}(y)\,dy$ be the Lévy measure of a subordinator? And the measure $\mu(dy) = y^{-1}e^{-y}\mathbf{1}_{(0,+\infty)}(y)\,dy$?

b (weight 10p)

Let $S = \{S_t\}_{t\in\mathbb{R}_+}$ be a subordinator. The Laplace exponent of S is defined by

$$\psi(u) = -\frac{1}{t}\log\left(\mathbb{E}\left[e^{-uS_t}\right]\right), \qquad u \geq 0.$$

Prove that $Z(u) = \{Z_t(u)\}_{t\in\mathbb{R}_+}, u \geq 0$ given by

$$Z_t(u) = \exp\left(-uS_t + t\psi(u)\right), \qquad t \geq 0,$$

is a martingale.

c (weight 10p)

Let $Y = \{Y_t\}_{t\in\mathbb{R}_+}$ be a one dimensional Lévy process and $S = \{S_t\}_{t\in\mathbb{R}_+}$ be a subordinator, independent of Y. Consider the subordinated process $Z = \{Z_t = Y_{S_t}\}_{t\in\mathbb{R}}$. Prove that Z has stationary increments, i.e., prove that $\mathcal{L}(Z_t - Z_s) = \mathcal{L}(Z_{t-s})$.

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d (weight 10p)

Let $X = \{(X_t^1, X_t^2)\}_{t\in\mathbb{R}_+}$ be a two dimensional Lévy process with Lévy generating triplet (γ, A, ν). Prove that $Y := X^1 - X^2$ is a one dimensional Lévy process and give its generating triplet (be as explicit as possible).

Hint: You can use (without having to prove it) a general result stated in class on linear transformations of Lévy processes. Note that $X^1$and X^2 are NOT necessarily independent. If you are not able to recall this general result, you can still obtain some points by proving the result under the assumption that X^1 and X^2 are two independent Lévy processes.

Problem 2

Let B be an m-d imensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and let $\mathbb{F} := \mathbb{F}^B$ be the usual augmentation of the natural filtration generated by B. **In this setup**:

a (weight 10p)

Describe what is a financial market. Define portfolio, wealth process associated to a portfolio, self-financing portfolio and admissible porfolio.

b (weight 10p)

Define arbitrage opportunity (or portfolio) and equivalent local martingale measure (ELMM). Prove that if there exists an ELMM then there are no arbitrage opportunities in the market.

c (weight 20p)

Suppose that the Itô financial market is given by

$$dS_t^0 = rS_t^0 dt, \qquad S_0^0 = 1,$$
$$dS_t^1 = \left(\mu - S_t^1\right) dt + \sigma dB_t, \qquad S_0^1 = s_1 > 0,$$

where $r > 0, \mu > 0$, and $\sigma \neq 0$ are constants.

1. (10p) Find the price of the European T-claim $F = \left(S_T^1\right)^2$.
2. (10p) Find the replicating portfolio $\varphi = (\varphi_0, \varphi_1)$ for this claim.

Problem 3

Let B be a one dimensional standard Browian motion and N an independent Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0$ with intensity measure $dt \otimes \nu$, where ν is a Lévy measure. Let $\tilde{N}$ be the compensated Poisson random measure associated to N.

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Let $Y = \{Y_t\}_{t\in[0,T]}$ be an $\mathbb{R}$-valued stochastic process with stochastic differential

$$dY_t = G(t)\,dt + F(t)\,dB_t + \int_{|x|<1} H(t,x)\,\tilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx), \quad (1)$$

where, $|G|^{1/2}, F \in \mathcal{P}_2(T)$,$H \in \mathcal{P}_2(T, \hat{B}_1(0))$, and K is predictable.

a (weight 10p)

State Itô's formula for the process Y.

b (weight 10p)

Find the differential of $f(Y_t)$ with $f(y) = y^2$. Simplify as much as possible.

c (weight 10p)

State general conditions on G, F, H and K such that e^Y is a local martingale and find an expression for $d\left(e^{Y_t}\right)$ in such case.

d (weight 10p)

Take $F = 0, H(t,x) = K(t,x) = x$ and $\nu(dx) = xe^{-\lambda x}\mathbf{1}_{(0,+\infty)}dx$ for some $\lambda > 0$. Find G such that e^Y is a local martingale.