UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4750/9750 — Mathematical Finance: Modelling and risk management

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This problem set consists of 10 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

A subordinator S is a one-dimensional Lévy process with non-decreasing sample paths, P-almost-surely. Subordinators can be used as a random model of time evolution or random clocks because they satisfy

$$S_t \geq 0, \qquad P\text{-a.s}, \quad \text{for each } t > 0,$$

and

$$S_{t_1} \leq S_{t_2}, \qquad P\text{-a.s}, \quad \text{whenever } t_1 \leq t_2.$$

The Lévy symbol of a subordinator takes the form

$$\eta(u) = \mathrm{i}\gamma_0 u + \int_0^\infty \left(e^{\mathrm{i}uy} - 1\right) \nu(dy),$$

where $\gamma_0 \geq 0$ and the Lévy measure ν satisfies the additional requirements

$$\nu((-\infty, 0)) = 0 \qquad \text{and} \qquad \int_0^\infty (y \wedge 1)\, \nu(dy) < \infty.$$

The measure $\nu(dy) = y^{-1/2}\mathbf{1}_{(0,+\infty)}(y)\, dy$ is not Lévy measure because

$$\int_1^{+\infty} y^{-1/2} dy = \left[\frac{y^{1/2}}{1/2}\right]_1^{+\infty} = +\infty,$$

(Continued on page 2.)

so ν cannot be the Lévy measure of a subordinator.

First we check that the measure $\mu(dy) = y^{-1}e^{-y}\mathbf{1}_{(0,+\infty)}(y)\,dy$ is a Lévy measure. We can write

$$\int_{\mathbb{R}_0} \left(y^2 \wedge 1\right) \mu(dy) = \int_0^1 y^2 y^{-1} e^{-y} dy + \int_1^{+\infty} y^{-1} e^{-y} dy.$$

and

$$\int_0^1 y e^{-y} dy < 1,$$
$$\int_1^{+\infty} y^{-1} e^{-y} dy < \int_1^{+\infty} e^{-y} dy < \int_0^{+\infty} e^{-y} dy = 1,$$

which yields

$$\int_{\mathbb{R}_0} \left(y^2 \wedge 1\right) \mu(dy) < 2 < +\infty.$$

Obviously

$$\mu((-\infty, 0)) = \int_{\mathbb{R}_0} \mathbf{1}_{(-\infty,0)} y^{-1} e^{-y} \mathbf{1}_{(0,+\infty)}(y)\,dy = \int_{\mathbb{R}_0} 0 dy = 0,$$

and

$$\begin{aligned}\int_0^{\infty} (y \wedge 1)\,\mu(dy) &= \int_0^1 y y^{-1} e^{-y} dy + \int_1^{+\infty} y^{-1} e^{-y} dy \\ &< \int_0^1 e^{-y} dy + \int_1^{+\infty} e^{-y} dy = \int_0^{+\infty} e^{-y} dy = 1 < +\infty.\end{aligned}$$

so μ is the Lévy measure of a subordinator.

b (weight 10p)

We consider the filtration of reference to be the minimal augmented filtration generated by S. First note that

$$\begin{aligned}\exp\left(-uS_t + t\psi(u)\right) &= \exp\left(-uS_t + t\left(-\frac{1}{t}\log\left(\mathbb{E}\left[e^{-uS_t}\right]\right)\right)\right) \\ &= \exp\left(-uS_t + t\left(-\frac{1}{t}\log\left(\mathbb{E}\left[e^{-uS_t}\right]\right)\right)\right) \\ &= \frac{e^{-uS_t}}{\mathbb{E}\left[e^{-uS_t}\right]}.\end{aligned}$$

Then, $Z_t(u)$ is clearly $\mathcal{F}_t$ measurable because $Z_t(u)) = f(S_t)$ for f Borel measurable. We have that

$$\mathbb{E}\left[|Z_t(u)|\right] = \mathbb{E}\left[\left|\frac{e^{-uS_t}}{\mathbb{E}\left[e^{-uS_t}\right]}\right|\right]$$

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$$= \frac{\mathbb{E}\left[e^{-uS_t}\right]}{\mathbb{E}\left[e^{-uS_t}\right]} = 1 < \infty, \quad t \geq 0.$$

For the martingale property, since $Z_t(u) > 0$, it suffices to prove that

$$\mathbb{E}\left[\left.\frac{Z_t(u)}{Z_s(u)}\right| \mathcal{F}_s\right] = 1.$$

We have that

$$\begin{aligned}
\mathbb{E}\left[\left.\frac{Z_t(u)}{Z_s(u)}\right| \mathcal{F}_s\right] &= \mathbb{E}\left[\exp\left(-u\left(S_t - S_s\right) + (t-s)\psi(u)\right) \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\exp\left(-u\left(S_t - S_s\right)\right) \middle| \mathcal{F}_s\right] e^{(t-s)\psi(u)} \\
&= \mathbb{E}\left[\exp\left(-u\left(S_t - S_s\right)\right)\right] e^{(t-s)\psi(u)} \\
&= \mathbb{E}\left[\exp\left(-uS_{t-s}\right)\right] e^{(t-s)\psi(u)} \\
&= 1
\end{aligned}$$

Where in the second equality we have used that $e^{(t-s)\psi(u)}$ is deterministic (goes out of the conditional expectation), in the third inequality we have used that $(S_t - S_s)$ is independent of $\mathcal{F}_s$, in the fourth inequality we have used that S has stationary increments and in the last inequality we have used that

$$\mathbb{E}\left[\exp\left(-uS_{t-s}\right)\right] = e^{-(t-s)\psi(u)},$$

by the definition of $\psi(u)$.

c (weight 10p)

Define the functions

$$f_u(x) := \mathbb{E}\left[e^{\mathrm{i}uY_x}\right], \qquad x \geq 0,$$

and

$$g_u(x_1, x_2) := \mathbb{E}\left[e^{\mathrm{i}u\left(Y_{x_2} - Y_{x_1}\right)}\right], \qquad 0 \leq x_1 \leq x_2.$$

Since Y has stationary increments we have that $g_u(x_1, x_2) = f_u(x_2 - x_1)$. Now, using the law of total expectation , the substitution property, that $S_t \geq S_s$,P-a.s, and that S has stationary increments, we have

$$\begin{aligned}
\mathbb{E}\left[e^{\mathrm{i}u\left(Y_{S_t} - Y_{S_s}\right)}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{iu\left(Y_{S_t} - Y_{S_s}\right)} \middle| \sigma\left(S_t, S_s\right)\right]\right] \\
&= \mathbb{E}\left[g_u\left(S_s, S_t\right)\right] \\
&= \mathbb{E}\left[f_u\left(S_t - S_s\right)\right] \\
&= \mathbb{E}\left[f_u\left(S_{t-s}\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}u\left(Y_{S_{t-s}}\right)} \middle| \sigma\left(S_{t-s}\right)\right]\right] \\
&= \mathbb{E}\left[e^{\mathrm{i}uY_{S_{t-s}}}\right],
\end{aligned}$$

which proves that $\mathcal{L}(Y_{S_t} - Y_{S_s}) = \mathcal{L}\left(Y_{S_{t-s}}\right)$.

(Continued on page 4.)

d (weight 10p)

The solution is based on the following result:

Proposition 1. *Let $X = \{X_t\}_{t\in\mathbb{R}_+}$ be a Lévy process on $\mathbb{R}^d$ with generating triplet (γ, A, ν) and let U be an $n\times d$ matrix. Then $Y = \{Y_t = UX_t\}_{t\in\mathbb{R}_+}$ is a Lévy process on $\mathbb{R}^n$ with generating triplet (γ_U, A_U, ν_U) given by*

$$\begin{aligned} A_U &= UAU^T, \\ \nu_U &= \nu \circ U^{-1}, \\ \gamma_U &= U\gamma + \int_{\mathbb{R}_0^d} Ux\left(\mathbf{1}_{B_1^n(0)}(Ux) - \mathbf{1}_{B_1^d(0)}(x)\right)\nu(dx) \end{aligned}$$

where

$$\nu \circ U^{-1}(A) = \nu\left(\left\{x \in \mathbb{R}^d : Ux \in A\right\}\right), A \in \mathcal{B}(\mathbb{R}^n),$$

and $B_1^d(0) = \left\{x \in \mathbb{R}^d : |x| < 1\right\}, B_1^n(0) = \{y \in \mathbb{R}^n : |y| < 1\}$.

Let $X = \left\{\left(X_t^1, X_t^2\right)\right\}_{t\in\mathbb{R}_+}$ be a two dimensional Lévy process with Lévy generating triplet (γ, A, ν). Then $Y = UX$ with $U = (1, -1) \in \mathbb{R}^{1\times 2}$. Then,

$$\begin{aligned} A_Y &= UAU^T = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} A_{11} - A_{12} \\ A_{21} - A_{22} \end{pmatrix} \\ &= A_{11} - 2A_{12} + A_{22}, \\ \gamma_Y &= U\gamma + \int_{\mathbb{R}_0^2} Ux\left(\mathbf{1}_{B_1^1(0)}(Ux) - \mathbf{1}_{B_1^2(0)}(x)\right)\nu(dx) \\ &= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \\ &+ \int_{\mathbb{R}_0^2} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \left(\mathbf{1}_{B_1^1(0)}\left(\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - \mathbf{1}_{B_1^2(0)}(x_1, x_2)\right)\nu(dx_1, dx_2) \\ &= \gamma_1 - \gamma_2 + \int_{\mathbb{R}_0^2} (x_1 - x_2)\left(\mathbf{1}_{\{(x_1-x_2)^2<1\}} - \mathbf{1}_{\{x_1^2+x_2^2<1\}}\right)\nu(dx_1, dx_2). \end{aligned}$$

Finally, for all $B \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned} \nu_Y(B) = \nu \circ U^{-1}(B) &= \nu\left(\left\{x \in \mathbb{R}^2 : Ux \in B\right\}\right) \\ &= \nu\left(\left\{x \in \mathbb{R}^2 : (1, -1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in B\right\}\right) \\ &= \nu\left(\left\{x \in \mathbb{R}^2 : x_1 - x_2 \in B\right\}\right). \end{aligned}$$

Problem 2

a (weight 10p)

Definition. a financial market with $n + 1$ investment possibilities consists in

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1. A risk free asset, where the unit price S_t^0 at time t is given by
$$dS_t^0 = r(t) S_t^0 dt, \qquad t \in [0,T],$$
$$S_0^0 = 1.$$

2. n risky assets, where the unit price S_t^i at time t of the i-the risky asset is given by
$$dS_t^i = \mu^i(t) dt + \sum_{j=1}^{m} \sigma_j^i(t) dB_t^j \qquad t \in [0,T],$$
$$S_0^i = s_0^i \in \mathbb{R},$$
for $i = 1, ..., n$.

Here:

- $T > 0$ is the investment horizon,
- $r \geq 0$ is the interest rate, $r^{1/2} \in L^0_{a,T}$ and we will assume it to be bounded.
- $\mu = \left(\mu^1(t), ..., \mu^n(t)\right)^T$ is the vector of appreciation rates of the risky assets and $\left(\mu^i\right)^{1/2} \in L^0_{a,T}, i = 1, ..., n$.
- $\sigma = \left(\sigma_j^i(t)\right)_{i=1,...,n,j=1,...,m}$ is the volatility matrix of the risky assets and $\sigma_j^i \in L^0_{a,T}$, $i = 1, ..., n, j = 1, ..., m$. Note that σ^i denote the ith row of the $n \times m$ matrix σ, that is, $\sigma^i = \left(\sigma_1^i(t), ..., \sigma_m^i\right)$..

Definition. A **portfolio** φ is an $(n+1)$-dimensional process
$$\varphi(t) = (\varphi_0(t), \varphi_1(t), ..., \varphi_n(t)),$$
with $\varphi_i \in L^0_{a,T}$ representing the number of units of the i-th asset held at time t. Note that this is a row vector.

Definition. The **wealth (or value) process** $V = V^\varphi$ associated to the portfolio φ is defined by
$$V_t^\varphi = \varphi(t) S_t = \sum_{i=0}^{n} \varphi_i(t) S_t^i.$$

Definition. A portfolio φ is called **self-financing** if $\int_0^t \varphi(s) dS_s$ exists for all $t > 0$ and
$$V_t^\varphi = V_0^\varphi + \int_0^t \varphi(s) dS_s,$$
or, in differential notation,
$$dV_t^\varphi = \varphi(t) dS_t.$$

Definition. A self-financing portfolio φ is called **admissible** if there exists a constant $K = K_\varphi > 0$ such that
$$V_t^\varphi \geq -K_\varphi, \qquad a.s. \text{ for all } t \in [0,T].$$
The set of all admissible portfolios is denoted by $\mathcal{A}$.

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b (weight 10p)

Definition. A portfolio $\varphi \in \mathcal{A}$ is called an arbitrage if

$$V_0^{\varphi} = 0, \qquad V_T^{\varphi} \geq 0, \quad P\text{-}a.s. \quad \text{and} \quad P\left(V_T^{\varphi} > 0\right) > 0.$$

Definition. A probability measure Q on $\mathcal{F}_T$ is called an equivalent (local) martingale measure E(L)MM if $Q \sim P$ (that is, $Q \ll P$ and $P \ll Q$) and the normalized price process $\bar{S}_t := \exp\left(-\int_0^t r(s)\,ds\right) S_t$ is a (local) Q-martingale.

Theorem. *Suppose there exists an* ELMM. *Then the market has no arbitrage opportunities (*NA*).*

Proof. Let Q be an ELMM. Suppose there exists an arbitrage φ. Then

$$V_0^{\varphi} = 0, \qquad V_T^{\varphi} \geq 0, \quad P\text{-a.s.} \quad \text{and} \quad P\left(V_T^{\varphi} > 0\right) > 0.$$

Since $Q \sim P$ we deduce that

$$\bar{V}_T^{\varphi} \geq 0, \quad Q\text{-a.s.} \quad \text{and} \quad Q\left(\bar{V}_T^{\varphi} > 0\right) > 0. \tag{1}$$

Since $\bar{S}$ is a local Q-martingale, we also have that

$$\bar{V}_t^{\varphi} = V_0^{\varphi} + \int_0^t \varphi(u)\,d\bar{S}_u,$$

is local Q-martingale. Since φ is admissible then $\bar{V}^{\varphi}$ is lower bounded Q-a.s. Therefore, $\bar{V}^{\varphi}$ is a Q-supermartingale, which implies that

$$0 = \bar{V}_0^{\varphi} \geq \mathbb{E}_Q\left[\bar{V}_T^{\varphi}\right].$$

This contradicts (1) and shows that an arbitrage cannot exist. □

c (weight 10p)

1. In this market consider $\theta \in L^2_{a,T}$ such that

$$\sigma\theta_t = \mu - S_t^1 - rS_t^1, \qquad \lambda \otimes P\text{-}a.e.,$$

that is $\theta_t = \frac{\mu-(1-r)S_t^1}{\sigma}$. We need to consider the the change of measure

$$\frac{dQ}{dP} = Z_t(\theta) = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right).$$

To show that $Z(\theta)$ is a martingale is not straightforward (it does not follow by a direct application of Novikov's theorem, but it is true) so we will assume it without proof. By Girsanov's theorem we have that

$$dB_t^Q := \theta_t dt + dB_t,$$

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is a Q-Brownian motion and the dynamics of S^1 can be rewriten in terms of B^Q as

$$\begin{aligned} dS_t^1 &= \left(\mu - S_t^1\right) dt + \sigma dB_t = \left(\mu - S_t^1\right) dt + \sigma(dB_t^Q - \theta_t dt) \\ &= rS_t^1 dt + \sigma dB_t^Q. \end{aligned}$$

Moreover, using Ito's product rule with $e^{-rt}S_t^1$ and taking into account that $d\left[e^{-r\cdot}, S^1\right]_t = 0$ we get that

$$d\left(e^{-rt}S_t^1\right) = -rS_t^1 dt + e^{-rt}dS^1 = \sigma dB_t^Q,$$

which yields

$$S_t^1 = e^{rt}S_0^1 + \int_0^t e^{r(t-s)}\sigma dB_s^Q. \tag{2}$$

By Theorem 5.16, we have that the market is arbitrage free. By Corollary 5.27 (1), since $n = m$ and σ is invertible we have that the market is complete. Then, using Theorem 5.31 we know that the upper and lower price of F coincide and are equal to

$$\begin{aligned} p(F) &= \mathbb{E}_Q\left[e^{-rT}\left(S_T^1\right)^2\right] = \mathbb{E}_Q\left[e^{-rT}\left(e^{rT}S_0^1 + \int_0^T e^{r(T-s)}\sigma dB_s^Q\right)^2\right] \\ &= \mathbb{E}_Q\left[e^{-rT}\left(e^{rT}S_0^1\right)^2\right] + \mathbb{E}_Q\left[2S_0^1\int_0^T e^{r(T-s)}\sigma dB_s^Q\right] + \mathbb{E}_Q\left[e^{-rT}\left(\int_0^T e^{r(T-s)}\sigma dB_s^Q\right)^2\right] \\ &= e^{rT}\left(S_0^1\right)^2 + e^{rT}\int_0^T e^{-2rs}\sigma^2 ds = e^{rT}\left(S_0^1\right)^2 - \frac{\sigma^2}{2r}e^{rT}\left\{e^{-2rT} - 1\right\} \\ &= e^{rT}\left(S_0^1\right)^2 + \frac{\sigma^2}{2r}\left\{e^{rT} - e^{-rT}\right\}. \end{aligned}$$

because the (Q-)expectation of an Itô integral with respect to a Q-Brownian motion is zero and by the Itô isometry.

2. By Theorem 5.36, the replicating portfolio will satisfy

$$e^{-rt}\varphi_1(t)\sigma = \psi_t,$$

where ψ is such that

$$e^{-rT}F = e^{-rT}\left(S_T^1\right)^2 = \mathbb{E}_Q\left[e^{-rT}\left(S_T^1\right)^2\right] + \int_0^T \psi_t dB_t^Q.$$

To find ψ we can use Corollary 5.35, which yields that

$$\psi_t = e^{-rT}\frac{\partial}{\partial s}\mathbb{E}_Q^s\left[\left(S_{T-t}^1\right)^2\right]\Big|_{s=S_t}\sigma.$$

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By similar reasonings as before we have that

$$\begin{aligned}\mathbb{E}_Q^s\left[\left(S_{T-t}^1\right)^2\right] &= \mathbb{E}_Q^s\left[\left(e^{r(T-t)}s + \int_0^{T-t} e^{r(T-t-s)}\sigma dB_s^Q\right)^2\right] \\ &= e^{2r(T-t)}s^2 + \int_0^{T-t} e^{2r(T-t-s)}\sigma^2 ds \\ &= e^{2r(T-t)}s^2 - \sigma^2\frac{e^{2r(T-t)}}{2r}\left\{e^{-2r(T-t)} - 1\right\} \\ &= e^{2r(T-t)}s^2 + \frac{\sigma^2}{2r}\left\{e^{2r(T-t)} - 1\right\},\end{aligned}$$

and

$$\frac{\partial}{\partial s}\mathbb{E}_Q^s\left[\left(S_{T-t}^1\right)^2\right] = 2e^{2r(T-t)}s.$$

Hence, $\psi_t = 2\sigma e^{rT}e^{-2rt}S_t^1$ and $\varphi_1(t) = 2e^{r(T-t)}S_t^1$. We choose $\varphi_0(t)$ such that $\varphi = (\varphi_0, \varphi_1)$ is self-financing. That is, let

$$V_0^\varphi = p(F),$$

$$A_t := \int_0^t \varphi_1(u)\, dS_u^1 - \varphi_1(t)\, S_t^1 = \int_0^t 2e^{r(T-u)}S_u^1 dS_u^1 - 2e^{r(T-t)}\left(S_t^1\right)^2$$

$$\varphi_0(t) = p(F)\, e^{-rt}A_t + r\int_0^t A_s e^{-rs} ds.$$

Problem 3

a (weight 10p)

Let $Y = \{Y_t\}_{t\in[0,T]}$ be an $\mathbb{R}$-valued stochastic process with stochastic differential

$$dY_t = G(t)\, dt + F(t)\, dB_t + \int_{|x|<1} H(t,x)\, \tilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\, N(dt,dx), \quad (3)$$

where, $|G|^{1/2}$, $F \in \mathcal{P}_2(T)$, $H \in \mathcal{P}_2(T, \hat{B}_1(0))$, and K is predictable and .

We will use the notation

$$dY_t^c = G(t)\, dt + F(t)\, dB_t.$$

Theorem. *If Y is a Lévy-type stochastic integral of the form* (3)*, such that*

$$\sup_{0\leq t\leq T}\sup_{x\in\hat{B}_1(0)} |H(t,x)| < \infty, \qquad P\text{-a.s.}$$

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then for each $f \in C^2(\mathbb{R})$ *we have*

$$\begin{aligned} f(Y_t) = f(Y_0) &+ \int_0^t \frac{\partial f}{\partial x}(Y_{s-})\, dY_s^c + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(Y_{s-})\, d[Y^c, Y^c]_s \\ &+ \int_0^t \int_{|x|\geq 1} \{f(Y_{s-} + K(s,x)) - f(Y_{s-})\}\, N(ds, dx) \\ &+ \int_0^t \int_{|x|<1} \{f(Y_{s-} + H(s,x)) - f(Y_{s-})\}\, \tilde{N}(ds, dx) \\ &+ \int_0^t \int_{|x|<1} \left\{ f(Y_{s-} + H(s,x)) - f(Y_{s-}) - H(s,x)\frac{\partial f}{\partial x_i}(Y_{s-}) \right\} \nu(dx)\, ds, \end{aligned}$$

for $0 \leq t \leq T$, P*-a.s.*

b (weight 10p)

First we compute the derivatives $f(y) = y^2$, $\frac{\partial f}{\partial x}(y) = 2y$ and $\frac{\partial^2 f}{\partial x^2}(y) = 2$. Then, note that

$$d[Y^c, Y^c]_t = F^2(t)\, dt$$

Applying Itô's formula we have

$$\begin{aligned} d(Y_t^2) &= 2Y_{t-}G(t)\, dt + 2Y_{t-}F(t)\, dB_t + F^2(t)\, dt \\ &+ \int_{|x|\geq 1} \left\{ (Y_{t-} + K(t,x))^2 - (Y_{t-})^2 \right\} N(dt, dx) \\ &+ \int_{|x|<1} \left\{ (Y_{t-} + H(t,x))^2 - (Y_{t-})^2 \right\} \tilde{N}(dt, dx) \\ &+ \int_{|x|<1} \left\{ (Y_{t-} + H(t,x))^2 - (Y_{t-})^2 - 2H(t,x)Y_{t-} \right\} \nu(dx)\, dt \\ &= 2Y_{t-}G(t)\, dt + 2Y_{t-}F(t)\, dB_t + F^2(t)\, dt \\ &+ \int_{|x|\geq 1} \left\{ K^2(t,x) + 2K(t,x)Y_{t-} \right\} N(dt, dx) \\ &+ \int_{|x|<1} \left\{ H^2(t,x) + 2H(t,x)Y_{t-} \right\} \tilde{N}(dt, dx) \\ &+ \int_{|x|<1} H^2(t,x)\, \nu(dx)\, dt \end{aligned}$$

c (weight 10p)

For a process Y of the form (3), under the assumptions

- (LM1) $\mathbb{E}\left[\int_0^T \int_{|x|\geq 1} \left|e^{K(s,x)} - 1\right| \nu(dx)\, ds\right] < \infty$,
- (LM2) $|G|^{1/2} \in \mathcal{H}_2(T)$,

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we have that e^Y is a local martingale if and only if

$$G(s) + \frac{1}{2}F(s)^2 + \int_{|x|<1} \left(e^{H(s,x)} - 1 - H(s,x)\right)\nu(dx) + \int_{|x|\geq 1}\left(e^{K(s,x)} - 1\right)\nu(dx) = 0, \tag{4}$$

P-a.s, for Lebesgue almost all $s \in [0,T]$. Moreover, we can write

$$\begin{aligned} e^{Y_t} &= 1 + \int_0^t e^{Y_{s-}} F(s)\, dBs + \int_0^t \int_{|x|<1} e^{Y_{s-}} \left(e^{H(s,x)} - 1\right)\tilde{N}(ds, dx) \\ &+ \int_0^t \int_{|x|\geq 1} e^{Y_{s-}} \left(e^{K(s,x)} - 1\right)\tilde{N}(ds, dx), \qquad 0 \leq t \leq T. \end{aligned}$$

d (weight 10p)

In our case, $H(t,x) = K(t,x) = x$, $F(t) \equiv 0$ and $\nu(dx) = xe^{-\lambda x}\mathbf{1}_{(0,+\infty)}(x)\,dx$. Hence, (LM1) reduces to

$$\int_1^{+\infty} |e^x - 1|\, xe^{-\lambda x}dx < \infty.$$

But note that

$$\begin{aligned} \int_1^{+\infty} |e^x - 1|\, xe^{-\lambda x}dx &\leq \int_1^{+\infty} (e^x + 1)\, xe^{-\lambda x}dx \\ &\leq \int_0^{+\infty} xe^{-(\lambda-1)x}dx + \int_0^{+\infty} xe^{-\lambda x}dx \\ &\leq \frac{1}{(\lambda-1)^2} + \frac{1}{\lambda^2} < \infty, \end{aligned}$$

if $\lambda > 1$. In the first inequality we have used that $|e^x - 1| \leq e^x + 1$. In the second inequality we have used the monotonicity of the integral of a positive function with respect to domain of integration. Finally, in the third inequality we have used that the integrals can be rewritten in terms of the first moment of exponential distributions. If $\lambda \leq 1$ then the integral in (LM1) diverges. So (LM1) holds if and only if $\lambda > 1$. Condition (4) reduces to

$$\begin{aligned} G(s) &= -\int_0^1 (e^x - 1 - x)\, xe^{-\lambda x}dx - \int_1^{+\infty} (e^x - 1)\, xe^{-\lambda x}dx \\ &= -\int_0^{+\infty} (e^x - 1)\, xe^{-\lambda x}dx + \int_0^1 x^2 e^{-\lambda x}dx \\ &= -\int_0^{+\infty} xe^{-(\lambda-1)x}dx + \int_0^{+\infty} xe^{-\lambda x}dx + \int_0^1 x^2 e^{-\lambda x}dx \\ &= -\frac{1}{(\lambda-1)^2} + \frac{1}{\lambda^2} + \frac{2 - e^{-\lambda}\left(\lambda^2 + 2\lambda + 2\right)}{\lambda^3}. \end{aligned}$$

where we have used the integration by parts formula twice to compute the integral $\int_0^1 x^2 e^{-\lambda x}dx$. Note that, since G is constant, $|G|^{1/2}$ belongs to $\mathcal{H}_2(T)$.