Exercises

1. Suppose we have

$$dX_t = \alpha dt + \sigma dB_t + \int_{|x|<1} h(x)\,\tilde{N}(dt,dx) + \int_{|x|\geq 1} k(x)\,N(dt,dx),$$
$$X_0 = x_0,$$

where α, σ are constants and h and k are given functions from $\mathbb{R} \to \mathbb{R}$.

(a) Use Itô's formula to find dY_t when $Y_t = \exp(X_t)$.

(b) How do we choose α, σ, h and k if we want Y_t to solve the SDE

$$dY_t = Y_{t-}\left(\beta dt + \theta dB_t + \lambda \int_{|x|<1} x\tilde{N}(dt,dx) + \rho \int_{|x|\geq 1} xN(dt,dx)\right),$$

for given constants β, θ, λ and ρ.

2. Solve the following Lévy SDEs:

(a) Let m, σ, λ be constants and

$$dX_t = (m - X_t)\,dt + \sigma dB_t + \lambda \int_{|x|<1} x\tilde{N}(dt,dx) + \rho \int_{|x|\geq 1} xN(dt,dx),$$
$$X_0 = x_0.$$

(b) Let

$$dX_t = \alpha dt + \gamma X_{t-}\left(\int_{|x|<1} x\tilde{N}(dt,dx) + \int_{|x|\geq 1} xN(dt,dx)\right),$$
$$X_0 = x_0.$$

3. Let $h \in L^2(\mathbb{R})$ be deterministic and define

$$Y_t = \exp\left\{\int_0^t \int_{\mathbb{R}_0} h(s)\,x\tilde{N}(ds,dx) - \int_0^t \int_{\mathbb{R}_0} \left(e^{h(s)x} - 1 - h(s)\,x\right)\nu(dx)\,ds\right\}.$$

Show that

$$dY_t = Y_{t-}\int_{\mathbb{R}_0} \left(e^{h(t)x} - 1\right)\tilde{N}(dt,dx).$$

4. Show that, under some conditions on $\gamma(s,x)$ (assumed to be deterministic) we have

$$\mathbb{E}\left[\exp\left(\int_0^t \int_{\mathbb{R}_0} \gamma(s,x)\,\tilde{N}(ds,dx)\right)\right] = \exp\left(\int_0^t \int_{\mathbb{R}_0} \left\{e^{\gamma(s,x)} - 1 - \gamma(s,x)\right\}\nu(dx)\,ds\right).$$

5. Let

$$dX_t^i = \int_{\mathbb{R}_0} \gamma_i(t,x)\,\tilde{N}(dt,dx), \qquad i = 1,2,$$

be two one-dimensional Itô-Lévy processes. Prove that

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_{s-}^1 dX_s^2 + \int_0^t X_{s-}^2 dX_s^1 + \int_0^t \int_{\mathbb{R}_0} \gamma_1(s,x)\,\gamma_2(s,x)\,N(ds,dx).$$

6. Define, with suitable conditions on $\theta(s,x)$,

$$Z_t(\theta) = \exp\left(\int_0^t \int_{\mathbb{R}_0} \log(1-\theta(s,x))\,\tilde{N}(ds,dx) + \int_0^t \int_{\mathbb{R}_0} \{\log(1-\theta(s,x)) + \theta(s,x)\}\,\nu(dx)\,ds\right).$$

Show that

$$dZ_t(\theta) = -Z_{t-}(\theta)\int_{\mathbb{R}_0} \theta(t,x)\,\tilde{N}(dt,dx).$$

7. Let $N = \{N_t\}_{t\geq 0}$ be a Poisson process with intensity parameter λ. Compute $\int_0^t N_{s-}dN_s$ and $\int_0^t N_s dN_s$.

8. Decide whether or not the following markets have arbitrages. If the market has an arbitrage find one.

   (a) Consider

$$\begin{aligned} dS_t^0 &= 0, \quad S_0^0 = 1, \\ dS_t^1 &= S_{t-}^1\left(\alpha dt + \int_{\mathbb{R}_0} x\tilde{N}(dt,dx)\right), \quad S_0^1 > 0, \end{aligned}$$

   where ν is supported in $(-1,+\infty)$, $\alpha \in \mathbb{R}$ is a constant and $\int_{\mathbb{R}} |z|\,\nu(dz) > |\alpha|$.

   (b) Consider

$$\begin{aligned} dS_t^0 &= 0, \quad S_0^0 = 1, \\ dS_t^1 &= S_{t-}^1\left(-dt - 1dB_t + 3\int_{\mathbb{R}_0} x\tilde{N}(dt,dx)\right), \quad S_0^1 > 0, \\ dS_t^2 &= S_{t-}^2\left(4dt + 2dB_t - 6\int_{\mathbb{R}_0} x\tilde{N}(dt,dx)\right), \quad S_0^2 > 0. \end{aligned}$$

9. Consider the following market

$$\begin{aligned} dS_t^0 &= 0, \quad S_0^0 = 1, \\ dS_t^1 &= S_{t-}^1\left(\mu_1 dt + \gamma_1^1 d\eta_t^1 + \gamma_2^1 d\eta_t^2\right), \quad S_0^1 > 0, \\ dS_t^2 &= S_{t-}^2\left(\mu_2 dt + \gamma_1^2 d\eta_t^1 + \gamma_2^2 d\eta_t^2\right), \quad S_0^2 > 0, \end{aligned}$$

where μ_i and $\gamma_j^i, i,j = 1,2$, are constants and η^1 and η^2 are independent Lévy martingales of the form

$$d\eta_t^i = \int_{\mathbb{R}_0} x\tilde{N}^i(dt,dx), \qquad i = 1,2.$$

Assume that the matrix $\gamma = \left(\gamma_j^i\right)_{1\leq i,j\leq 2}$ is invertible, with inverse $\gamma^{-1} = \lambda = \left(\lambda_j^i\right)_{1\leq i,j\leq 2}$ and assume that

$$\int_{\mathbb{R}_0} |x|\,\nu^i(dx) > \left|\lambda_1^i \mu_1 + \lambda_2^i \mu_2\right|, \qquad i = 1,2.$$

Show that this market is arbitrage free.