Exercises Chap 1 : Review of Measure Theory

1 $\mathcal{E} = \{$ all finite subsets of E and its complements $\}$

Is $\mathcal{E}$ always a σ-algebra ?

If $\# \mathcal{E} < +\infty \Rightarrow \mathcal{E} = \mathcal{P}(E)$

If $\# \mathcal{E} = +\infty \Rightarrow$ No. Take $E = \mathbb{N}$.

Let $A_n = \{2n\} \in \mathcal{E}$. However,

$$\#\Big(\bigcup_{n \geq 1} A_n\Big) = +\infty \Rightarrow \bigcup_{n\geq 1} A_n \notin \mathcal{E} \Rightarrow \mathcal{E} \text{ is not a } \sigma\text{-algebra.}$$

2 $X : \hat{E} \to \cancel{\tilde{E}}\ E$ a mapping.

Show that if $\mathcal{E}$ is a σ-algebra on E, then $\hat{\mathcal{E}} = \{ X^{-1}(A) : A \in \mathcal{E} \}$ is a σ-algebra on $\hat{E}$.

- $\tilde{E} \in \tilde{\mathcal{E}}$? Yes, $\hat{E} = X^{-1}(E)$
- If $B \in \hat{\mathcal{E}} \Rightarrow B^c \in \hat{\mathcal{E}}$? Yes, if $B \in \hat{\mathcal{E}} \Rightarrow B = X^{-1}(A)$ for $A \in \mathcal{E}$.
  Note that $B^c = (X^{-1}(A))^c = X^{-1}(A^c) \in \hat{\mathcal{E}}$, because $A^c \in \mathcal{E}$.
- If $\{B_i\}_{i\geq 1} \subseteq \hat{\mathcal{E}}$, then $\bigcup_{i\geq 1} B_i \in \hat{\mathcal{E}}$? Yes.
  We have that $B_i = X^{-1}(A_i)$ for some $A_i \in \mathcal{E}$, $i \geq 1$.
  Note that
  $$\bigcup_{i\geq 1} B_i = \bigcup_{i\geq 1} X^{-1}(A_i) = X^{-1}\Big(\bigcup_{i\geq 1} A_i\Big) \in \hat{\mathcal{E}},$$ because $\bigcup_{i\geq 1} A_i \in \mathcal{E}$.

3 Let $\mathcal{E}$ be a σ-algebra on E and let $\cancel{A \in \mathcal{E}}$ $A \subset E$. Show that $\mathcal{E}_A = \{ A \cap B : B \in \mathcal{E} \}$ is a σ-algebra on $\cancel{E}$ A

Define $X : A \to E$ by $X(w) = w$, for all $w \in A$. Then $A \cap B = X^{-1}(B)$ for any $B \in \mathcal{E}$. This means that $\mathcal{E}_A = \{ X^{-1}(B) : B \in \mathcal{E} \}$, which is a σ-algebra by Exercise 2.

4 If $\{\mathcal{E}_i\}_{i\in I}$ is a family of σ-algebras on E, then $\bigcap_{i\in I} \mathcal{E}_i$ is also a σ-algebra on E.

- $E \in \bigcap_{i\in I} \mathcal{E}_i$? Yes.

  $E \in \mathcal{E}_i, \forall i \in I$ because $\mathcal{E}_i$ are σ-algebras $\Rightarrow E \in \bigcap_{i\in I} \mathcal{E}_i$

- $B \in \bigcap_{i\in I} \mathcal{E}_i \Rightarrow B^c \in \bigcap_{i\in I} \mathcal{E}_i$? Yes.

  $B \in \mathcal{E}_i, \forall i \in I \underset{\mathcal{E}_i - \sigma\text{-algebra}}{\Rightarrow} B^c \in \mathcal{E}_i, \forall i \in I \Rightarrow B^c \in \bigcap_{i\in I} \mathcal{E}_i$

- $\{A_j\}_{j\geq 1} \subseteq \bigcap_{i\in I} \mathcal{E}_i \Rightarrow \bigcup_{j\geq 1} A_j \in \bigcap_{i\in I} \mathcal{E}_i$? Yes

  $A_j \in \mathcal{E}_i \ \forall i \in I, j \geq 1. \underset{\mathcal{E}_i - \sigma\text{-algebra}}{\Rightarrow} \bigcup_{j\geq 1} A_j \in \mathcal{E}_i, \forall i \in I; \Rightarrow \bigcup_{j\geq 1} A_j \in \bigcap_{i\in I} \mathcal{E}_i$

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5 $(E, \mathcal{E})$ measurable space. Let $\{A_n\}_{n\geq 1} \subset E$. Define

$$\overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \quad \text{and} \quad \underline{\lim_{n\to\infty}} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

- Show that if $\{A_n\}_{n\geq 1} \subset \mathcal{E}$, then $\overline{\lim_{n\to\infty}} A_n$ and $\underline{\lim_{n\to\infty}} A_n$ belong to $\mathcal{E}$.
- Show that $\overline{\lim_{n\to\infty}} A_n = \{A_n \text{ occurs for infinitely many } n\}$

  $\underline{\lim_{n\to\infty}} A_n = \{A_n \text{ occurs for all but finitely many } n\}$

  and $\underline{\lim_{n\to\infty}} A_n \subseteq \overline{\lim_{n\to\infty}} A_n$.

First we prove an auxiliary claim.

(*) Let $\{B_n\}_{n\geq 1} \subset \mathcal{E}$, then $\bigcap_{n\geq 1} B_n \in \mathcal{E}$. This follows from

$$\bigcap_{n\geq 1} B_n = \Big(\bigcup_{n\geq 1} B_n^c\Big)^c$$

Note that $\bigcup_{n\geq 1} B_n^c \in \mathcal{E}$

- $\overline{\lim} A_n \in \mathcal{E}$? ✓ $\bigcup_{k=n}^{\infty} A_k \in \mathcal{E} \overset{(*)}{\Rightarrow} \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k \in \mathcal{E}$

  Since $\{A_n\}_{n\geq 1} \in \mathcal{E} \Rightarrow$

- $\underline{\lim} A_n \in \mathcal{E}$? ✓

Since $\{A_n\}_{n\geq 1} \subset \mathcal{E} \overset{(*)}{\Longrightarrow} \bigcap_{k=n}^{\infty} A_k \in \mathcal{E} \longrightarrow \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \in \mathcal{E}$.

- $w \in \overline{\lim} A_n \iff w \in \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k$

$\iff \forall n \geq 1 . \exists k \geq n$ such that $w \in A_k$

$\iff w \in A_n$ for infinitely many n.

- $w \in \underline{\lim} A_n \iff w \in \bigcup_{n\geq 1} \bigcap_{k\geq n} A_k$

$\iff \exists n \geq 1$ such that $\forall k \geq n$, $w \in A_k$.

$\iff w \in A_n$ for all but finitely many n.

- If $w \in A_n$ for all but finitely many n then $w \in A_n$ for infinitely many n.

$\boxed{6}$ $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, define $P(\{i\}) = \alpha_i$, $i \geq 1$.

Extend P to a probability measure on $\mathcal{P}(\mathbb{N})$.

Conditions on α_i. Can (α_i) be chosen the same?

Since P must be non-negative $\Rightarrow \alpha_i \geq 0$.

To have $P(\mathbb{N}) = 1 \Rightarrow \sum_{i\geq 1} \alpha_i = 1$

The convergence of the series implies that $\alpha_i \to 0 \Rightarrow \alpha_i \neq \alpha_j, i \neq j$

For any $A \subset \mathbb{N}$, we set

$$P(A) = \sum_{i\in A} P(\{i\}) = \sum_{i \in A} \alpha_i \quad \text{and} \quad P(\emptyset) = 0.$$

Let's prove the σ-additivity. Let $\{A_i\}_{i\geq 1} \subset \mathcal{P}(\mathbb{N})$ be a sequence of pairwise disjoint sets.

$$P\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{i \in \bigcup_{k\geq 1} A_k} \alpha_i \underset{\uparrow}{=} \sum_{k=1}^{\infty} \sum_{i\in A_k} \alpha_i = \sum_{k=1}^{\infty} P(A_k)$$

Absolutely conv. series $\Rightarrow$ We can reorder the terms.

7 $(E, \mathcal{E}, \mu)$ meas. space.

Show that μ is finitely subadditive, i.e., for any $\{A_n\}_{n=1,\dots,N} \subset \mathcal{E}$ where $N \in \mathbb{N}$ one has

$$\mu\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} \mu(A_n)$$

Preliminary result.

If $A, B \in \mathcal{E}$, $B \subset A \Rightarrow \mu(B) \leq \mu(A)$ $(*)$

We have that $A = (A \cap B) \uplus (A \cap B^c)$

$$\underset{\substack{\mu \text{ is a} \\ \text{measure}}}{\Rightarrow} \quad \mu(A) = \mu(A \cap B) + \mu(A \cap B^c)$$
$$= \mu(B) + \mu(A \cap B^c)$$
$$\geq \mu(B) \qquad (B \subset A \Rightarrow A \cap B = B \text{ and } \mu(A \cap B^c) \geq 0)$$

We will prove the result by induction on N

<u>Case $K=1$</u> $\quad P(A_1) \leq P(A_1)$

<u>Case $K=N$</u>

Define the sets $B_1 = A_1$, $B_K = A_K \setminus \left(\bigcup_{i=1}^{K-1} A_i\right)$, $K = 2, \dots, N$.

Note that $B_K \subset A_K$, $K = 1, \dots, N$ and $\bigcup_{i=1}^{m} A_i = \biguplus_{i=1}^{m} B_i$, $1 \leq m \leq N$

$$\mu\left(\bigcup_{n=1}^{N} A_n\right) = \mu\left(\biguplus_{n=1}^{N} B_n\right) = \mu\left(\left(\biguplus_{n=1}^{N-1} B_n\right) \uplus B_N\right)$$

$$(\sigma\text{-additivity}) \quad = \mu\left(\biguplus_{n=1}^{N-1} B_n\right) + \mu(B_N)$$

$$(*) \quad \leq \mu\left(\bigcup_{n=1}^{N-1} A_n\right) + \mu(A_N) \qquad (B_N \subseteq A_N)$$

$$\leq \sum_{n=1}^{N} \mu(A_n) \qquad (\text{induction hypothesis})$$

8 $(E, \mathcal{E}, \mu)$ measure space.

$f : E \to \mathbb{R}$ be a meas. function such that $f \geq 0$, μ-a.e.

Prove that if $\int_E f \, d\mu = 0 \Rightarrow f = 0$ μ-a.e.

$A = \{x \in E : f(x) > 0\} \in \mathcal{E}$ and $A_n \{x \in E : f \geq 1/n\} \in \mathcal{E}$

$A_n \nearrow A$, $\{A_n\}$ is an increasing sequence and $A = \bigcup_{n \geq 1}^{\infty} A_n$

Since $f \geq f \, \mathbb{1}_{A_n} \geq \frac{1}{n} \mathbb{1}_{A_n}$, by the monotonicity of the integral we get

$$\underset{\substack{\uparrow \\ \text{By assumption}}}{0} = \int_E f \, d\mu \geq \int_E \frac{1}{n} \mathbb{1}_{A_n} \, d\mu = \frac{\mu(A_n)}{n} \Rightarrow \mu(A_n) = 0 \quad \forall n \geq 1$$

Moreover, as $A_n \subseteq A_{n+1} \Rightarrow 0 \leq \mathbb{1}_{A_n} \leq \mathbb{1}_{A_{n+1}}$

and $A = \bigcup_{n \geq 1} A_n \Rightarrow \mathbb{1}_{A_n} \xrightarrow[n \to \infty]{\text{a.s.}} \mathbb{1}_A$.

Hence applying Monotone Convergence we get

$$\mu(A) = \int_E \mathbb{1}_A \, d\mu = \lim_{n \to \infty} \int_E \mathbb{1}_{A_n} \, d\mu = \lim_{n \to \infty} \mu(A_n) = 0$$

Finally,

$$\{f = 0\}^c = \{f \neq 0\} = \{f < 0\} \uplus \{f > 0\}$$

$$\mu(\{f = 0\}^c) = \mu(\{f < 0\}) + \mu\{f > 0\} \underset{\substack{\uparrow \\ f \geq 0 \ \mu\text{-a.s.} \Rightarrow \mu(\{f < 0\}) = 0}}{=} 0 + \mu(A) = 0$$

$\boxed{9}$ $(E, \mathcal{E}, \mu)$ - measure space. $f \in L^1$

Prove that if $\int_E f \mathbb{1}_A d\mu = 0, \forall A \in \mathcal{E} \Rightarrow f \equiv 0$ μ-a.e.

Let $A_+ = \{x \in E : f(x) > 0\} \in \mathcal{E}$. (because f is measurable)

$$\left.\begin{array}{l} \text{Note that} \quad f \mathbb{1}_{A_+} \geq 0 \quad \mu\text{-a.e.} \\ \text{By hypothesis} \quad \int_E f \mathbb{1}_{A_+} d\mu = 0 \end{array}\right\} \xrightarrow{\text{Exercise 8}} \underset{(*)}{f \mathbb{1}_{A_+} = 0 \ \mu\text{-a.e.}}$$

As $f(x) > 0$ for $x \in A_+$ $\underset{(*)}{\Rightarrow}$ $\mathbb{1}_{A_+} \equiv 0$ μ-a.e. $\Rightarrow \mu(A_+) = 0$

We can repeat the same argument with

$A_- = \{x \in E : f(x) < 0\} \in \mathcal{E}$

and $-f \mathbb{1}_{A_-} \geq 0$ μ-a.e. to obtain that

$\mu(A_-) = 0$.

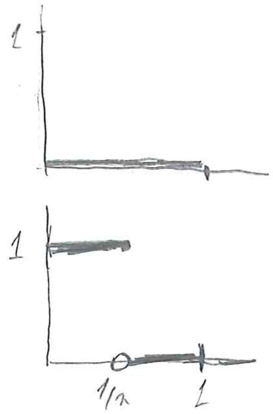
Then,

$\{f = 0\}^c = \{f \neq 0\} = A_+ \uplus A_-$ and

$\mu(\{f = 0\}^c) = \mu(A_+) + \mu(A_-) = 0 \Rightarrow$

$f = 0$ μ-a.e.

10 $([0,1], \mathcal{B}([0,1]), \lambda)$



$$f(x) = 0, \quad 0 \le x \le 1$$

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/n \\ 0 & \text{if } 1/n < x \le 1 \end{cases}$$

a) No.

$f_n(x) \xrightarrow[n\to\infty]{} f(x) \iff \forall \varepsilon > 0, \exists N = N(x,\varepsilon)$ s.t. $|f_n(x) - f(x)| < \varepsilon, \forall n > N$.

For $x \in (0,1]$ choose $N = \frac{1}{x}$, then for $n > N = \frac{1}{x}$

$$|f_n(x) - f(x)| = |\mathbb{1}_{\{0 \le x \le 1/n\}}| \underset{\substack{\uparrow \\ x > 1/n}}{=} 0$$

But for $x = 0$, $f_n(x) = 1, \forall n$ and $f(x) = 0 \Rightarrow f_n(0) \not\xrightarrow[n\to\infty]{} f(0)$

b) Yes. The set where f_n does not converge to f is $\{0\}$ which has Lebesgue measure 0.

c) $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ in measure $\iff \lim_{n\to\infty} \lambda(\{x \in [0,1] : |f_n(x) - f(x)| > \varepsilon\}) = 0$ $\forall \varepsilon > 0$.

$|f_n(x) - f(x)| = \mathbb{1}_{\{0 \le x \le 1/n\}}$ then

- $\varepsilon \ge 1$ $\quad \lambda(\{x \in [0,1] : \mathbb{1}_{\{0 \le x \le 1/n\}} > \varepsilon\}) = 0 \xrightarrow[n\to\infty]{} 0$ ✓
- $\varepsilon < 1$ $\quad \lambda(\{x \in [0,1] : \mathbb{1}_{\{0 \le x \le 1/n\}} > \varepsilon\}) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

d) $f_n \xrightarrow[n\to\infty]{} f$ in $L^p \iff \int_0^1 |f_n(x) - f(x)|^p \, d\lambda \xrightarrow[n\to\infty]{} 0$

$$\int_0^1 |f_n(x) - f(x)|^p \, d\lambda = \int_0^1 (\mathbb{1}_{\{0 \le x \le 1/n\}})^p \, d\lambda = \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \quad ✓$$

$\boxed{11}$ P, Q prob. meas defined on $(\Omega, \mathcal{F})$.

Show that
$$P \sim Q \iff Q \ll P \text{ and } P\left(\frac{dQ}{dP} = 0\right) = 0$$

Relationship between $\frac{dQ}{dP}$ and $\frac{dP}{dQ}$?

By the Radon-Nikodym
$$P \sim Q \iff \begin{cases} Q \ll P \iff Q(A) = \int_A \frac{dQ}{dP}\, dP, \forall A \in \mathcal{F}, \text{ with } \frac{dQ}{dP} \geq 0 \text{ P.a.s.} \\ P \ll Q \iff P(A) = \int_A \frac{dP}{dQ}\, dQ, \forall A \in \mathcal{F}, \text{ with } \frac{dP}{dQ} \geq 0, \text{Q.a.s.} \end{cases}$$

Note that if $P \sim Q$ then P and Q share the same null sets, that is, $P(A) = 0 \iff Q(A) = 0$.

$\Rightarrow$) Obviously $Q \ll P$. As we also have that $P \ll Q$ it suffices to prove $Q\left(\left\{\frac{dQ}{dP} = 0\right\}\right) = 0$. But

$$Q\left(\left\{\frac{dQ}{dP} = 0\right\}\right) \underset{Q \ll P}{=} \int_{\left\{\frac{dQ}{dP} = 0\right\}} \frac{dQ}{dP}\, dP = \int_\Omega \underbrace{\frac{dQ}{dP} \mathbb{1}_{\left\{\frac{dQ}{dP} = 0\right\}}}_{\equiv 0} dP =$$
$$= \int_\Omega 0\, dP = 0$$

$\Leftarrow$) We only need to prove that $P \ll Q$. This is equivalent to find f such that $P(A) = \int_A f\, dQ, \forall A \in \mathcal{F}, f \geq 0$.
As we know that $P\left(\frac{dQ}{dP} = 0\right) = 0$ and $\frac{dQ}{dP} \geq 0$ P-a.s. we have that $\frac{dQ}{dP} > 0$ P-a.s. Then it makes sense to define $f = \left(\frac{dQ}{dP}\right)^{-1}$.
Then, $\forall A \in \mathcal{F}$
$$\int_A f\, dQ = \int_A \left(\frac{dQ}{dP}\right)^{-1} dQ \underset{Q \ll P}{=} \int_A \left(\frac{dQ}{dP}\right)^{-1} \frac{dQ}{dP}\, dP = \int_A dP = P(A).$$
P-a.s. $\Rightarrow$ Q-a.s.: ≥ 0

$\boxed{12}$ $(E, \mathcal{E}, \mu)$ measure space. $I \subset \mathbb{R}$ open interval and $t_0 \in I$. $\{f_t\}_{t \in I}$ family of meas. functions from E to $\mathbb{R}$ satisfying

a) $f_\cdot(x) : I \longrightarrow \mathbb{R}$ different. on I, μ-a.e.

b) $\exists\ g \in L^1$ such that $|f_t| + |\frac{d}{dt} f_t| \leq g$, μ-a.e. $\forall t \in I$.

Show that

$$F : I \longrightarrow \mathbb{R}$$
$$t \longmapsto \int_E f_t \, d\mu$$

is well defined, differentiable at t_0 with

$$F'(t_0) = \int_E \frac{d}{dt} f_t \Big|_{t=t_0} d\mu .$$

- As $|f_t| \leq g \in L^1 \Rightarrow f_t \in L^1, \forall t \in I \Rightarrow F(t) = \int_E f_t \, d\mu < +\infty$ $\forall t \in I \Rightarrow F$ is well defined.

- $f_\cdot(x)$ is differentiable on I, μ-a.e. then we apply the mean value theorem for each $t \in I \setminus \{t_0\}$.

$$\left| \frac{f_t - f_{t_0}}{t - t_0} \right| = \left| \frac{d}{dt} f_t \Big|_{t=\xi \in I} \right| \leq \sup_{t \in I} \left| \frac{d f_t}{dt} \right| \leq g .$$

Let $\{t_n\}_{n \geq 1} \subset I$ such that $\lim_{n \to \infty} t_n = t_0$. Then,

$$\lim_{n \to \infty} \frac{f_{t_n} - f_{t_0}}{t_n - t_0} = \frac{d}{dt} f_t \Big|_{t=t_0} \quad \mu\text{-a.e.}$$

Moreover,

$$F'(t_0) = \lim_{n \to \infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim_{n \to \infty} \int_E \frac{f_{t_n} - f_{t_0}}{t_n - t_0} d\mu$$

$$\underset{\uparrow}{=} \int_E \lim_{n \to \infty} \frac{f_{t_n} - f_{t_0}}{t_n - t_0} d\mu = \int_E \frac{d f_t}{dt} \Big|_{t=t_0} d\mu$$

Dom. convergence.