Exercises Chap. 2 : Review of Probability Theory

$\boxed{1}$ X discrete r.v. $X : \Omega \to \{x_i\}_{i \geq 1} \subseteq \mathbb{R}$

How is $\sigma(X)$? How are $\sigma(X)$-meas. functions?

Let $A_i := \{\omega : X(\omega) = x_i\} = X^{-1}(\{x_i\})$ $\quad i = 1, 2, \dots$

$\mathcal{P} := \{A_i\}_{i \in \mathbb{N}}$ is a countable partition of Ω.

This means: $A_i \cap A_j = \emptyset$ if $i \neq j$

and $\Omega = \biguplus_{i \geq 1} A_i$

Recall that $\sigma(X) = \sigma\{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$

By Exercise 2, in Chp 1, we know that $\{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$ is already a σ-algebra and, hence,

$$\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}.$$

But note that

$$X^{-1}(B) = \bigcup_{i : x_i \in B} A_i$$

Therefore, the elements of $\sigma(X)$ are countable (or finite) unions of elements of $\mathcal{P}$. That is,

$$B \in \sigma(X) \iff \exists\, J \subseteq \mathbb{N} \text{ s.t. } B = \bigcup_{i \in J} A_i.$$

Z is $\sigma(X)$-meas. $\overset{?}{\iff}$ Z is constant on A_i, $i \geq 1$.

$\Updownarrow$

$\forall B \in \mathcal{B}(\mathbb{R})$, $Z^{-1}(B) = \bigcup_{i \in J \subseteq \mathbb{N}} A_i$

• If Z is constant on $A_i \Rightarrow Z = \sum_{i=1}^{\infty} z_i \mathbb{1}_{A_i}$ and $\forall B \in \mathcal{B}(\mathbb{R})$, $Z^{-1}(B) = \bigcup_{j \in \{i : z_i \in B\}} A_i$ ✓

• Let $\omega_1, \omega_2 \in A_i$ s.t. $Z(\omega_1) = z_1 \neq z_2 = Z(\omega_2)$. This implies that for $\{z_1\} \in \mathcal{B}(\mathbb{R})$, $Z^{-1}(\{z_1\}) \cap A_i \neq A_i$ because $\omega_2 \notin Z^{-1}(\{z_1\})$. Then, $Z^{-1}(\{z_1\}) = \left(\biguplus_{j \neq i} Z^{-1}(\{z_1\}) \cap A_j\right) \uplus \left(Z^{-1}(\{z_1\}) \cap A_i\right) \Rightarrow Z^{-1}(\{z_1\})$ is not $\sigma(X)$-meas.

$\boxed{2}$ Let X be a r.v. and $Y = X^2$. Show that Y is also a r.v.. Is X measurable with respect to $\sigma(Y)$?

$X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is meas. (i.e. r.v.)

$g : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is meas. because it is continuous.
$\quad x \longmapsto x^2$

$Y = g \circ X$. Therefore Y is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-meas. because it is composition of meas. functions. $\Rightarrow Y$ is a r.v.

To check if X is $\sigma(Y)$-measurable we can use the Doob-Dynkin lemma.

X is $\sigma(Y)$-measurable $\iff \exists\, \xi : \mathbb{R} \to \mathbb{R}$ measurable s.t. $X = \xi(Y)$

As $Y = g(X)$, the desired ξ must be equal to g^{-1} and g^{-1} only exists if g is injective $(g(b) = g(a) \Rightarrow b = a)$ on $X(\Omega) \subset \mathbb{R}$, the image of X.

In this case, as $g(x) = x^2$, we have that

- If $X(\Omega) \subset \mathbb{R}_+$ or $X(\Omega) \subset \mathbb{R}_-$, then X is $\sigma(Y)$-measurable.
- In any other case, for instance $X(\Omega) = \mathbb{R}$, X is not $\sigma(Y)$-measurable.

3) Q distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$Q((a,b]) = \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz, \quad a < b \in \mathbb{R}.$$

Consider $(\Omega, \mathcal{F}, P)$ prob. space and Z r.v. s.t. $\mathcal{L}(Z) = Q$. Find $E[Z]$ and $Var(Z)$.

Define $X = \sigma Z + \mu$, $\sigma > 0$, $\mu \in \mathbb{R}$. Check that X is a r.v., find its distr. funct and compute $E[X]$, $Var[X]$.

First we need to check that $Q(\mathbb{R}) = 1$, to ensure that Q is a probability distribution function.

The usual way to check this is to use Fubini's theorem and a polar change of variable. Let $f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$

We have that

$$\left(\int_{\mathbb{R}} f(t)\,dt\right)^2 = \int_{\mathbb{R}} f(x)\,dx \int_{\mathbb{R}} f(y)\,dy \underset{\text{Fubini}}{=} \int_{\mathbb{R}^2} \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right) dx\,dy = (*)$$

Polar coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $J = \det \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$$\Rightarrow dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta \qquad \begin{aligned} 0 &\le r < +\infty \\ 0 &\le \theta \le 2\pi. \end{aligned}$$

$$(*) = \int_0^{+\infty} \int_0^{2\pi} \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right) r\,dr\,d\theta$$

$$\underset{\text{Fubini}}{=} \left(\int_0^{+\infty} r \exp\left(-\frac{r^2}{2}\right) dr\right) \underbrace{\left(\int_0^{2\pi} \frac{1}{2\pi}\,d\theta\right)}_{=1} = \left[-e^{-\frac{r^2}{2}}\right]_0^{+\infty} = 1$$

Next, set $(\Omega, \mathcal{F}, P) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), Q)$

and $Z : \Omega \to \mathbb{R}$, $\omega \mapsto \omega$. The law of Z is given

by $P_Z(A) = Q(Z^{-1}(A)) \underset{Z = Id}{=} Q(A)$ ✓

$$E[Z] = \int_\Omega Z\,dP \underset{\substack{\uparrow \\ \text{Image} \\ \text{measure}}}{=} \int_{\mathbb{R}} z\, P_Z(dz) = \int_{\mathbb{R}} z\; Q(dz) = \int_{-\infty}^{+\infty} z\, \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz = \left[\frac{e^{-z^2/2}}{\sqrt{2\pi}}\right]_{+\infty}^{+\infty} = 0$$

$$Var[Z] = E[(Z - E[Z])^2] = E[Z^2] \overset{\text{same reasonings}}{=} \int_{-\infty}^{+\infty} z^2 \frac{e^{-(z^2/2)}}{\sqrt{2\pi}} dz = \left\{ \begin{array}{l} u = z \\ dv = \frac{z}{\sqrt{2\pi}} e^{-z^2/2} \end{array} \right\}$$

Int. by parts

$$= \left[-z \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -\frac{e^{-z^2/2}}{\sqrt{2\pi}} dz = 0 - 0 + \int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = 1$$

Let $X = \varphi(Z)$, where $\varphi(z) = \sigma z + \mu$ a contin. (measurable) function

$\Rightarrow X$ is comp. of meas. functions $\Rightarrow X$ is a r.v.

$$\underbrace{\Omega \xrightarrow{Z} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}}_{X = \varphi \circ Z}$$

$$F_X(x) = P(X \le x) = Q(X \le x) = Q(\sigma Z + \mu \le x) = Q\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = \left\{ \begin{array}{l} z = \frac{y-\mu}{\sigma} \\ dz = \frac{dy}{\sigma} \end{array} \right\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

$$E[X] = E[\sigma Z + \mu] \underset{\substack{\uparrow \\ \text{lin. } E[\cdot]}}{=} \sigma E[Z] + \mu \underset{\substack{\uparrow \\ E[Z] = 0}}{=} \mu$$

$$Var[X] = E[(X - E[X])^2] = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] =$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2 = E[X^2] - \mu^2 \overset{?}{=} \sigma^2$$

$$\cdot\; E[X^2] = E[(\sigma Z + \mu)^2] = \sigma^2 E[Z^2] + 2\mu\sigma \overbrace{E[Z]}^{=0} + \mu^2 = \sigma^2 + \mu^2$$

4. $X \sim N(\mu, \sigma^2)$ defined on $(\Omega, \mathcal{F}, P)$. Compute $\psi(\theta) = E[e^{\theta X}]$ $\theta \in \mathbb{R}$.

Define $L(X;\theta) := \dfrac{e^{\theta X}}{\psi(\theta)}$ and state that

$Q_\theta(A) = E[L(X;\theta) 1_A]$, $A \in \mathcal{F}$ defines a prob. meas.

Show that $Q_\theta \ll P$.

Find the law of X under Q_θ.

- $$\psi(\theta) = E[e^{\theta X}] = \int_\Omega e^{\theta X} dP = \int_{\mathbb{R}} e^{\theta x} dP_X = \int_{\mathbb{R}} e^{\theta x} \underbrace{\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}}_{= \frac{dP_X}{d\lambda}} dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \theta x - \frac{(x-\mu)^2}{2\sigma^2} \right\} dx = (*)$$

$$\theta x - \frac{(x-\mu)^2}{2\sigma^2} = \frac{-1}{2\sigma^2}\left\{ -2\sigma^2\theta x + x^2 - 2x\mu + \mu^2 \right\}$$
$$= -\frac{1}{2\sigma^2}\left\{ x^2 - 2(\theta\sigma^2+\mu)x + \mu^2 \right\}$$
$$= -\frac{1}{2\sigma^2}\left\{ (x-(\theta\sigma^2+\mu))^2 + \mu^2 - (\theta\sigma^2+\mu)^2 \right\}$$
$$= -\frac{(x-(\theta\sigma^2+\mu))^2}{2\sigma^2} + \frac{2\mu\theta\sigma^2 + \theta^2\sigma^4}{2\sigma^2}$$

$$(*) = \exp\left\{ \mu\theta + \frac{\theta^2\sigma^2}{2} \right\} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-(\theta\sigma^2+\mu))^2}{2\sigma^2} \right\} dx}_{= 1 \text{ Integral of the density of a } N(\theta\sigma^2+\mu, \sigma^2)}$$

$$= \exp\left\{ \mu\theta + \frac{\theta^2\sigma^2}{2} \right\}$$

- By construction $L(X;\theta) \geq 0$ and $E[L(X;\theta)] = 1$.

Let us check that $Q_\theta(\cdot)$ is a prob. measure.

- $Q_\theta(\Omega) = E[L(X;\theta)\mathbb{1}_\Omega] = E[L(X;\theta)] = 1$

- $Q_\theta(A) \geq 0$ Because $L(X;\theta)\mathbb{1}_A \geq 0 \Rightarrow E[L(X;\theta)\mathbb{1}_A] \geq 0$.

- σ-additivity: Let $\{A_n\}_{n\geq 1} \subset \mathcal{F}$ be pairwise disjoint.

  Note that $\mathbb{1}_{\{\bigcup_{n=1}^\infty A_n\}}(\omega) = \sum_{n=1}^\infty \mathbb{1}_{\{A_n\}}(\omega)$,

  and this series is well defined because A_n are disjoint

  if $\omega \in A_n \Rightarrow \mathbb{1}_{A_n}(\omega) = 1$ and $\mathbb{1}_{A_m}(\omega) = 0 \ \forall m \neq n$

  if $\omega \notin A_n, n \geq 1 \Rightarrow$ The series is equal to 0.

  Let $S_n := \sum_{k=1}^n L(X;\theta)\mathbb{1}_{\{A_k\}}$. S_n is an increasing sequence of positive random variables that converge pointwise to $\sum_{k=1}^\infty L(X;\theta)\mathbb{1}_{\{A_k\}}$.

  Then, $Q_\theta\left(\bigcup_{n=1}^\infty A_n\right) = E[L(X;\theta)\mathbb{1}_{\{\bigcup_{n=1}^\infty A_n\}}]$

  $$= E\left[\sum_{n=1}^\infty L(X;\theta)\mathbb{1}_{\{A_n\}}\right]$$

  $$(\text{monotone converg.}) = \sum_{n=1}^\infty E[L(X;\theta)\mathbb{1}_{\{A_n\}}]$$

  $$= \sum_{n=1}^\infty Q_\theta(A_n) \quad \checkmark$$

- $Q_\theta \ll P$?

Let $A \in \mathcal{F}$ s.t. $P(A) = 0$. Then,

$$Q_\theta(A) = E[L(X;\theta)\mathbb{1}_A] = \int_\Omega L(X;\theta)\mathbb{1}_A \, dP \underset{(*)}{=} 0$$

(*) Because $\mathbb{1}_A \equiv 0$ P-a.s. $\Rightarrow L(X;\theta)\mathbb{1}_A \equiv 0$, P-a.s.

Note that $\frac{dQ_\theta}{dP} = L(X;\theta)$.

The law of X under Q_θ is the image measure of Q_θ by X. We will denote it by $Q_{\theta,X}$.

$$\forall A \in \mathcal{B}(\mathbb{R}) \quad Q_{\theta,X}(A) \underset{\substack{\uparrow \\ \text{Def. of imag. meas.}}}{=} Q_\theta\left(X^{-1}(A)\right) \underset{\substack{\uparrow \\ \text{Def. of } Q_\theta}}{=} \int_{X^{-1}(A)} L(X;\theta)\, dP$$

$$\underset{\substack{\uparrow \\ \text{Image measure Theorem}}}{=} \int_A L(x;\theta)\, dP_X \underset{\substack{\uparrow \\ P_X \ll \lambda \\ \text{and } P_X \sim N(\mu,\sigma^2)}}{=} \int_A L(x;\theta) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \int_A \frac{e^{\theta x}}{\exp\left\{\mu\theta + \frac{\theta^2\sigma^2}{2}\right\}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \int_A \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\theta x - \mu\theta - \frac{\theta^2\sigma^2}{2} - \frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \int_A \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu-\theta\sigma^2)^2}{2\sigma^2}\right\} dx$$

$\Rightarrow$ We can conclude that, under Q_θ, the law of X is $N(\mu+\theta\sigma^2, \sigma^2)$.

The change of measure does not change the type of law of X, it remains Gaussian, but it changes the mean and keeps the same variance.

$\Rightarrow$ The simplest version of Girsanov's theorem.

$\boxed{5}$ $X \sim N(\mu, \sigma^2)$. Define $Y = \exp(X)$. Show that Y is a r.v.

$f(y)$ = Lognormal with mean μ and variance σ^2.

Show that $P_Y \ll \lambda$ and find $\frac{dP_Y}{d\lambda}$.

Compute $E[Y^n]$, $n \geq 1$ and $Var[Y]$.

- $(\Omega, \mathcal{F}) \xrightarrow{X} (\mathbb{R}, \mathcal{B}(\mathbb{R})) \xrightarrow{h(x) = \exp(x)} (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$Y = h \circ X$$

Y is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-meas because h is continuous (→Borel meas.) and X is a r.v.

$y > 0$

- $F_Y(y) = P_Y((-\infty, y]) = P(Y \leq y) = P(\exp(X) \leq y)$

$= P(X \leq \log(y)) = P_X((-\infty, \log(y)])$

log is increas.

$= \int_{-\infty}^{\log(y)} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \left\{ \begin{array}{l} x = \log(z) \\ dx = \frac{dz}{z} \end{array} \right\} =$

$= \int_0^y \frac{1}{z\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log(z)-\mu)^2}{2\sigma^2}\right) dz \implies P_Y \ll \lambda$ and

$$\frac{dP_Y}{d\lambda} = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log(y)-\mu)^2}{2\sigma^2}\right)$$

- You can use $\frac{dP_Y}{d\lambda}$ or use that $Y = \exp(X)$.

$E[Y^n] = E[\exp(nX)] = \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right)$ (By exercise 4)

$Var[Y] = E[(Y - E[Y])^2] = E[Y^2] - 2E[Y] \cdot E[Y] + E[Y]^2$

$= E[Y^2] - (E[Y])^2 = \exp(2\mu + 2\sigma^2) - \left(\exp\left(\mu + \frac{\sigma^2}{2}\right)\right)^2$

$= \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) =$

$= \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1) > 0$ because $\sigma^2 > 0$.

$\boxed{6}$ X and Y be 2 indep. id. d. r.v. with law $N(\mu,\sigma^2)$

Find the density of $(U,V) = (X+Y, X-Y)$.

Condi. on μ and σ^2 for U and V to be independent.

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{(x-\mu)^2+(y-\mu)^2}{2\sigma^2}\right\}$$

$(U,V) = g(X,Y)$ where $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto (u,v) = g(x,y) = (x+y, x-y)$

The map g is injective (actually is a bijection between $\mathbb{R}^2$ and $\mathbb{R}^2$)

$g^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(u,v) \longmapsto (x,y) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$

$$J_{g^{-1}} = \begin{pmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \Rightarrow \det J_{g^{-1}} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$f_{U,V}(u,v) = f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v)) \, |\det J_{g^{-1}}(u,v)| \, \mathbb{1}_{g(\mathbb{R}^2)}(u,v)$$

$$\Rightarrow f_{U,V}(u,v) = \frac{1}{4\pi\sigma^2} \exp\left(-\frac{\left(\frac{u+v}{2}-\mu\right)^2 + \left(\frac{u-v}{2}-\mu\right)^2}{2\sigma^2}\right) \mathbb{1}_{\mathbb{R}^2}(u,v)$$

$$= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{u^2+v^2+4\mu^2-4\mu u}{4\sigma^2}\right) \mathbb{1}_{\mathbb{R}^2}(u,v)$$

Then,

$$f_{U,V}(u,v) = f_U(u) f_V(v) \iff \mu = 0$$

If $\mu = 0$ then

$$f_U(u) = f_V(u) \quad \text{and} \quad f_U(u) = \frac{1}{\sqrt{2\pi \, 2\sigma^2}} \exp\left(-\frac{u^2}{4\sigma^2}\right)$$

$\Rightarrow U \sim V \sim N(0, 2\sigma^2)$ independent

$\boxed{7}$ X, Y i.i.d. with law $N(0, \sigma^2)$

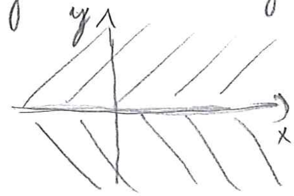
Find the density of $(U, V) = \left(\sqrt{X^2+Y^2},\ X/Y\right)$, where V is defined as 0 if $Y = 0$. Are U and V independent?

We have that $f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$

$(U, V) = g(X, Y)$ where $g: \mathbb{R}^2 \longrightarrow \mathbb{R}_+ \times \mathbb{R}$

$(x,y) \longmapsto (u,v) = \left(\sqrt{x^2+y^2},\ x/y\right)$

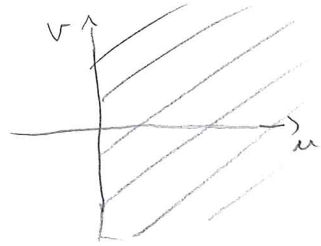
g is not injective.



$S_1 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$

$S_0 = \{(x,y) \in \mathbb{R}^2 : y = 0\}$

$S_2 = \{(x,y) \in \mathbb{R}^2 : y < 0\}$



$$g(x,y) = g_0(x,y) \mathbb{1}_{S_0}(x,y) + g_1(x,y) \mathbb{1}_{S_1}(x,y) + g_2(x,y) \mathbb{1}_{S_2}(x,y)$$

$$\lambda_2(S_0) = \int_{\mathbb{R}^2} \mathbb{1}_{S_0}\, dx\,dy \underset{\text{Fubini}}{=} \int_{\mathbb{R}} \left(\int_{\{0\}} dy \right) dx = 0$$

$$\left.\begin{array}{l} u = \sqrt{x^2+y^2} \\ v = x/y \end{array}\right\} \Rightarrow \left.\begin{array}{l} u = \sqrt{(vy)^2+y^2} \\ x = vy \end{array}\right\} \Rightarrow \left.\begin{array}{l} u^2 = y^2(1+v^2) \\ x = vy \end{array}\right\} \Rightarrow y = \frac{\pm u}{\sqrt{1+v^2}}$$

$$\Rightarrow \boxed{y = \frac{u}{\sqrt{1+v^2}},\ x = \frac{uv}{\sqrt{1+v^2}}} \text{ and } \boxed{y = \frac{-u}{\sqrt{1+v^2}},\ x = \frac{-uv}{\sqrt{1+v^2}}}$$

$$g_1^{-1}(u,v) = \left(\frac{uv}{\sqrt{1+v^2}}, \frac{u}{\sqrt{1+v^2}}\right) \Rightarrow J_{g_1^{-1}} = \begin{pmatrix} \frac{v}{\sqrt{1+v^2}} & \frac{u}{(1+v^2)^{3/2}} \\ \frac{1}{\sqrt{1+v^2}} & \frac{-uv}{(1+v^2)^{3/2}} \end{pmatrix}$$

$$\Rightarrow \det J_{g_1^{-1}} = \frac{-uv^2 - u}{(1+v^2)^2} = -\frac{u(1+v^2)}{(1+v^2)^2} = -\frac{u}{1+v^2}$$

$$g_2^{-1}(u,v) = \left(\frac{-uv}{\sqrt{1+v^2}}, \frac{-u}{\sqrt{1+v^2}}\right) \Rightarrow J_{g_2^{-1}} = \begin{pmatrix} \frac{-v}{\sqrt{1+v^2}} & \frac{-u}{(1+v^2)^{3/2}} \\ \frac{-1}{\sqrt{1+v^2}} & \frac{uv}{(1+v^2)^{3/2}} \end{pmatrix}$$

$$\Rightarrow \det J_{g_2^{-1}} = -\frac{u}{1+v^2}$$

Therefore,

$$f_{U,V}(u,v) = f_{X,Y}\left(g_1^{-1}(u,v)\right) \left|\det J_{g_1^{-1}}(u,v)\right| \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}(u,v)$$
$$+ f_{X,Y}\left(g_2^{-1}(u,v)\right) \left|\det J_{g_2^{-1}}(u,v)\right| \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}(u,v)$$

Note that $g_2^{-1}(u,v) = -g_1^{-1}(u,v)$, $\left|\det J_{g_1^{-1}}\right| = \left|\det J_{g_2^{-1}}\right|$

and $f_{X,Y}(x,y) = f_{X,Y}(-x,-y)$

Hence,

$$f_{U,V}(u,v) = 2\, f_{X,Y}\left(g_1^{-1}(u,v)\right) \left|\det J_{g_1^{-1}}(u,v)\right| \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}(u,v)$$
$$= \frac{1}{\pi\sigma^2} \exp\left(-\frac{\left(\frac{uv}{\sqrt{1+v^2}}\right)^2 + \left(\frac{u}{\sqrt{1+v^2}}\right)^2}{2\sigma^2}\right) \frac{u}{(1+v^2)} \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}(u,v)$$
$$= \frac{u}{\pi\sigma^2(1+v^2)} \exp\left(-\frac{u^2v^2 + u^2}{(1+v^2)2\sigma^2}\right) \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}(u,v)$$
$$= \frac{u}{\pi\sigma^2(1+v^2)} \exp\left(-\frac{u^2}{2\sigma^2}\right) \mathbb{1}_{\mathbb{R}_+ \times \mathbb{R}}(u,v)$$
$$= \frac{1}{\pi(1+v^2)} \mathbb{1}_{\mathbb{R}}(v) \quad \frac{u}{\sigma^2} \exp\left(-\frac{u^2}{2\sigma^2}\right) \mathbb{1}_{\mathbb{R}_+}(u)$$
$$= f_V(v) \cdot f_U(u)$$

The joint density factorizes $\Rightarrow$ U and V are independent.

$\boxed{8}$ $Y \sim N(0,1)$ and for $a > 0$ define
$$Z = Y \mathbb{1}_{\{|Y| \le a\}} - Y \mathbb{1}_{\{|Y| > a\}}$$
Show that Z is a Gaussian r.v. but (Y, Z) is not multivariate Gaussian.

First note that the density of Y is symmetric, i.e.,
$$f_Y(y) = \phi(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) = \phi(-y) = f_Y(-y).$$

We compute the law of Z.

$$F_Z(z) = P(Z \le z) = P\left(Y \mathbb{1}_{\{|y| \le a\}} - Y \mathbb{1}_{\{|Y| > a\}} \le z\right)$$

$$= E\left[\mathbb{1}_{\{Y \mathbb{1}_{\{|Y| \le a\}} - Y \mathbb{1}_{\{|Y| > a\}} \le z\}}\right] = \int_{-\infty}^{+\infty} \mathbb{1}_{\{y \mathbb{1}_{\{|y| \le a\}} - y \mathbb{1}_{\{|y| > a\}} \le z\}} \phi(y)\, dy$$

$$= \int_{|y| \le a} \mathbb{1}_{\{y \le z\}} \phi(y)\, dy + \int_{|y| > a} \mathbb{1}_{\{-y \le z\}} \phi(y)\, dy = (*)$$

$$\cdot \int_{|y| > a} \mathbb{1}_{\{-y \le z\}} \phi(y)\, dy = A_1 + A_2 = \int_{|y| > a} \mathbb{1}_{\{y \le z\}} \phi(y)\, dy$$

$$A_1 = \int_{a}^{+\infty} \mathbb{1}_{\{-y \le z\}} \phi(y)\, dy = \left\{\begin{array}{l} u = -y \\ du = -dy \end{array}\right\} = -\int_{-a}^{-\infty} \mathbb{1}_{\{u \le z\}} \phi(u)\, du = \int_{-\infty}^{-a} \mathbb{1}_{\{u \le z\}} \phi(u)\, du$$

$$A_2 = \int_{-\infty}^{-a} \mathbb{1}_{\{-y \le z\}} \phi(y)\, dy = \left\{\begin{array}{l} u = -y \\ du = -dy \end{array}\right\} = -\int_{+\infty}^{a} \mathbb{1}_{\{u \le z\}} \phi(u)\, du = \int_{a}^{+\infty} \mathbb{1}_{\{u \le z\}} \phi(u)\, du$$

$$(*) = \int_{\mathbb{R}} \mathbb{1}_{\{y \le z\}} \phi(y)\, dy = \int_{-\infty}^{z} \phi(y)\, dy \Rightarrow Z \sim N(0,1).$$

(Y, Z) is multivariate Gaussian $\Leftrightarrow$ $\forall a, b \in \mathbb{R}$ $aY + bZ$ is a Gauss. rand. vari.

Take $a = b = 1$, $Y + Z = Y + Y\mathbb{1}_{\{|Y| \le a\}} - Y \mathbb{1}_{\{|Y| > a\}} = 2Y \mathbb{1}_{\{|Y| \le a\}}$

Obviously, $2Y \mathbb{1}_{\{|Y| \le a\}}$ is not constant a.s. $\Rightarrow$ It is not a degenerate Gaussian.

But $P(2Y\mathbb{1}_{\{|Y| \le a\}} > 3a) = 0 \Rightarrow$ It cannot be Gaussian.

$\boxed{9}$ X be a r.v. and $f: \mathbb{R} \to \mathbb{R}_+$ an increasing, Borel measurable function. Let $a \in \mathbb{R}$ such that $f(a) > 0$. Prove that

$$P(X \geq a) \leq \frac{E[f(X)]}{f(a)}$$

~

First note that

$$f \circ X \geq f \circ X \, \mathbb{1}_{\{X \geq a\}} \qquad P\text{-a.s.}$$

Then,

$$E[f(X)] = \int_\Omega f \circ X \, dP \underset{\substack{\uparrow \\ \text{Monotonicity of the integral}}}{\geq} \int_\Omega f \circ X \, \mathbb{1}_{\{X \geq a\}} \, dP$$

$$\underset{\substack{\uparrow \\ f \text{ increasing.}}}{\geq} f(a) \int_\Omega \mathbb{1}_{\{X \geq a\}} \, dP = f(a) \, P(X \geq a)$$

$$\Rightarrow \quad P(X \geq a) \leq \frac{E[f(X)]}{f(a)} \qquad (f(a) > 0).$$

~

Chebyshev inequality

$$P(|X - E[X]| \geq a) \leq \frac{Var[X]}{a^2}$$

Apply the previous inequality (Markov's inequality) to the r.v. $|X - E[X]|$ and $f(a) = a^2$.

$\boxed{10}$ Let $Y \in L^2(\Omega, \mathcal{F}, P)$. Show that if $E[Y|X] = X$ and $E[Y^2|X] = X^2$ then $X = Y$, P-a.s.

First we will prove that $E[Y-X] = Var[Y-X] = 0$.

- $E[Y-X] = E[E[Y-X|X]] \underset{\text{lin. of cond. exp.}}{=} E[E[Y|X] - E[X|X]]$

$= \underset{\substack{\text{By hypothesis} \\ \text{and } E[X|X]=X}}{E[X-X]} = 0$

- $Var[Y-X] \underset{E[Y-X]=0}{=} E[(Y-X)^2] \underset{\text{linear. of exped.}}{=} E[Y^2] - 2E[XY] + E[X^2]$

Conservat. of exped. $= E[E[Y^2|X]] - 2E[E[XY|X]] + E[X^2]$

Hypoth. + what is meas. goes out $= E[X^2] - 2E[X E[Y|X]] + E[X^2]$

Hypoth. $= 2E[X^2] - 2E[X^2] = 0$

If a rand. var. Z has $Var[Z] = 0 \Rightarrow Z \equiv E[Z]$ P-a.s.

This follows from Chap 1, Ex. 8. with

$f = (Z - E[Z])^2 \geq 0$, $E = \Omega$.

As $0 = \int_E f \, d\mu = \int_\Omega (Z - E[Z])^2 dP \Rightarrow (Z - E[Z])^2 = 0$ P-a.s.

$\Leftrightarrow Z = E[Z]$, P-a.s.

As $Z = Y - X$ and $E[Z] = E[Y-X] = 0$ and $Var[Z] = Var[Y-X] = 0$

$\Rightarrow Y - X = 0$ P-a.s. $\Leftrightarrow Y = X$, P-a.s.

$(\Omega, \mathcal{F}, P)$

$\boxed{11}$ $X \sim N(\mu, \sigma^2)$, $L(X;\theta) = \exp\left(\theta(X-\mu) - \theta^2 \frac{\sigma^2}{2}\right)$

$\mathcal{G} \subseteq \mathcal{F}$. $\dfrac{dQ_\theta}{dP} = L(X;\theta)$

Show that for any $Y \in L^1(\Omega, \mathcal{F}, Q_\theta)$ we have

$$E_{Q_\theta}[Y \mid \mathcal{G}] = \frac{E[Y L(X,\theta) \mid \mathcal{G}]}{E[L(X;\theta) \mid \mathcal{G}]}$$

- Recall that $L(X,\theta) > 0$, P-a.s. and $E[L(X;\theta)] = 1 \Rightarrow L(X;\theta) \in L^1(P)$.
  $\Rightarrow E[L(X;\theta) \mid \mathcal{G}]$ exists.
- To show that $E[L(X;\theta) \mid \mathcal{G}] > 0$ Q_θ-a.s. we can show first that the r.v. $\mathbb{1}_{\{E[L(X;\theta)\mid\mathcal{G}]=0\}} = 0$ P-a.s.

  Note that $\{E[L(X;\theta)\mid\mathcal{G}] = 0\} \in \mathcal{G}$.

  $$E[L(X;\theta)\, \mathbb{1}_{\{E[L(X;\theta)\mid\mathcal{G}]=0\}}] \underset{\text{def. of cond. exped.}}{=} E[\,E[L(X;\theta)\mid\mathcal{G}]\, \mathbb{1}_{\{E[L(X;\theta)\mid\mathcal{G}]=0\}}]$$

  $$= E[0 \cdot \mathbb{1}_{\{E[L(X;\theta)\mid\mathcal{G}]=0\}}] = 0.$$

  By exercise 8, Chapt. 1. we have that $L(X;\theta)\mathbb{1}_{\{E[L(X;\theta)\mid\mathcal{G}]=0\}} = 0$, P-a.s.

  But as $L(X;\theta) > 0$ P-a.s. $\Rightarrow \mathbb{1}_{\{E[L(X;\theta)\mid\mathcal{G}]=0\}} = 0$, P-a.s

  $\Rightarrow P(E[L(X;\theta)\mid\mathcal{G}] = 0) = 0$

  As $Q_\theta \ll P$, $\Rightarrow Q_\theta(\{E[L(X;\theta)\mid\mathcal{G}] = 0\}) = 0$

  $\Rightarrow E[L(X;\theta)\mid\mathcal{G}] > 0$, Q_θ-a.s.
- Next note that $Y L(X,\theta) \in L^1(P) \Leftrightarrow Y \in L^1(Q_\theta)$. So by assumption $E[Y L(X,\theta)\mid\mathcal{G}]$ is well defined.

The previous two points show that $\dfrac{E[Y L(X;\theta)\mid\mathcal{G}]}{E[L(X,\theta)\mid\mathcal{G}]}$ exists and it is finite Q_θ-a.s.

Obviously, $\dfrac{E[Y L(X;\theta) \mid \mathcal{G}]}{E[L(X;\theta) \mid \mathcal{G}]}$ is $\mathcal{G}$- measurable.

Let's check that satisfies the conditional expectation property

$$\forall B \in \mathcal{G} \quad E_{Q_\theta}[Y \mathbb{1}_B] = E_{Q_\theta}\left[\frac{E[Y L(X;\theta) \mid \mathcal{G}]}{E[L(X;\theta) \mid \mathcal{G}]} \mathbb{1}_B \right]$$

- $$E_{Q_\theta}\left[\frac{E[Y L(X;\theta) \mid \mathcal{G}]}{E[L(X;\theta) \mid \mathcal{G}]} \mathbb{1}_B \right] \underset{\substack{\uparrow \\ \text{R-N. Th.}}}{=} E\left[L(X;\theta) \frac{E[Y L(X;\theta) \mid \mathcal{G}]}{E[L(X;\theta) \mid \mathcal{G}]} \mathbb{1}_B \right]$$

conserv. of expect.

$$= E\left[E\left[L(X;\theta) \frac{E[Y L(X;\theta) \mid \mathcal{G}]}{E[L(X;\theta) \mid \mathcal{G}]} \mathbb{1}_B \mid \mathcal{G} \right] \right]$$

what is meas. goes out

$$= E\left[\cancel{E[L(X;\theta) \mid \mathcal{G}]} \frac{E[Y L(X;\theta) \mid \mathcal{G}]}{\cancel{E[L(X;\theta) \mid \mathcal{G}]}} \mathbb{1}_B \right]$$

what is meas. goes in

$$= E\left[E\left[Y L(X;\theta) \mathbb{1}_B \mid \mathcal{G} \right] \right]$$

$$= E\left[Y L(X;\theta) \mathbb{1}_B \right] \quad \text{(conserv. of expectation)}$$

$$\underset{\substack{\uparrow \\ \frac{dQ_\theta}{dP} = L(X,\theta)}}{=} E_{Q_\theta}\left[Y \mathbb{1}_B \right]$$

12 Let $X \sim N(\mu, \sigma^2)$ and $K > 0$. Compute $E[\max(0, e^X - K)]$ in terms of
$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

Recall that $X \sim N(\mu, \sigma^2) \Rightarrow X = \mu + \sigma Z$ where $Z \sim N(0,1)$

and let $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) = \frac{dP_Z}{d\lambda}$

$$E[\max(0, e^X - K)] = E[\max(0, e^{\mu+\sigma Z} - K)]$$

$$= \int_\Omega \max(0, e^{\mu+\sigma z} - K)\, dP_Z = \int_{\mathbb{R}} \max(0, e^{\mu+\sigma z} - K)\phi(z)\, dz \underset{(*)}{=}$$

Image measure

Note that
$$e^{\mu+\sigma z} - K \leq 0 \iff \mu + \sigma z \leq \log(K) \iff z \leq \frac{\log(K) - \mu}{\sigma}$$

$$(*) = \int_{\frac{\log(K)-\mu}{\sigma}}^{+\infty} (e^{\mu+\sigma z} - K)\phi(z)\, dz = A_1 - A_2$$

$$A_2 = K \int_{\frac{\log(K)-\mu}{\sigma}}^{+\infty} \phi(z)\, dz = K\left(1 - \Phi\left(\frac{\log(K)-\mu}{\sigma}\right)\right)$$

$$A_1 = \int_{\frac{\log(K)-\mu}{\sigma}}^{+\infty} e^{\mu+\sigma z}\phi(z)\, dz = \int_{\frac{\log(K)-\mu}{\sigma}}^{+\infty} e^{\mu+\frac{\sigma^2}{2}} \phi(z-\sigma)\, dz$$

$$= \left\{ \begin{matrix} u = z - \sigma \\ du = dz \end{matrix} \right\} = e^{\mu+\frac{\sigma^2}{2}} \int_{\frac{\log(K)-\mu-\sigma^2}{\sigma}}^{+\infty} \phi(u)\, du = e^{\mu+\frac{\sigma^2}{2}}\left(1 - \Phi\left(\frac{\log(K)-\mu-\sigma^2}{\sigma}\right)\right)$$

But note that $\forall u \in \mathbb{R}$ $\Phi(u) = 1 - \Phi(-u)$

Therefore,

$$E\left[\max(0, e^X - K)\right] = e^{\mu + \frac{\sigma^2}{2}} \Phi\left(\frac{\mu + \sigma^2 - \log(K)}{\sigma}\right) - K\,\Phi\left(\frac{\mu - \log(K)}{\sigma}\right)$$