Exercises Chapt. 3. Review of Stochastic Processes

1

$X = \{X_t\}_{t \in \mathbb{R}_+}$ is a Gaussian process iff for any $\{t_i\}_{i=1,\dots,n} \subset \mathbb{R}_+$, $n \in \mathbb{N}$ we have that $(X_{t_1}, \dots, X_{t_n})$ is multivariate Gaussian.

<u>Def (Lectures)</u> A standard B.m. is a process satisfying

1) W has continuous paths P-a.s.
2) $W_0 = 0$ P-a.s.
3) W has indep. increm.
4) For all $0 \le s < t$, the law of $W_t - W_s$ is a $N(0, t-s)$.

<u>Def (Gaussian)</u> A stand. B.m. is a process satisfying

a) W has contin. paths P-a.s.
b) W is a Gaussian process
c) W is centered ($E[W_t]=0$) and the covariance function

$$K(s,t) = E[(W_t - E[W_t])(W_s - E[W_s])] = E[W_t W_s] = \min(s,t)$$

<u>Def. Lect $\Rightarrow$ Def. Gaussian</u>

1) $\Rightarrow$ a) ✓. 3) and 4) yield that for any $0 \le t_1 < t_2 < \dots < t_n$, $n \in \mathbb{N}$ the vector

$(W_{t_n} - W_{t_{n-1}}, \dots, W_{t_2} - W_{t_1}, W_{t_1})$ has multiv. norm. distr.

and by a linear transform, we get that

$(W_{t_1}, W_{t_2}, \dots, W_{t_n})$ has a multiv. norm. distrib.

Alternatively, for all $\lambda_i \in \mathbb{R}$, $i = 1, \dots, n$, we have that

$$\sum_{i=1}^{n} \lambda_i W_i = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{i} (W_{t_j} - W_{t_{j-1}}) = \sum_{j=1}^{n} \left(\sum_{i=j}^{n} \lambda_i \right) (W_{t_j} - W_{t_{j-1}}).$$

which is a univariate normal by 3) and 4).

Therefore W is a Gaussian process b) ✓

4) $\Rightarrow$ $E[W_t] = 0$ $\forall t \geq 0$. Moreover if $s < t$ we have that

$$K(s,t) = E[W_s W_t] = E[W_s(W_t - W_s) + W_s^2]$$

$$= E[W_s(W_t - W_s)] + E[W_s^2]$$

$$\overset{3)+4)}{=} E[W_s]E[W_t - W_s] + s = s$$

A similar reasoning can be made if $t < s$, and we get

$K(s,t) = \min(s,t)$ $\Rightarrow$ c) ✓

<u>Def. Gaussi. $\Rightarrow$ Def. Lectures</u>

2) ✓

a) $\Rightarrow$ 1) ✓ c) $\Rightarrow$ $E[W_0] = 0$ and $E[W_0^2] = K(0,0) = 0$, which yield $W_0 = 0$ P a.s.

Note that if (Z_1, Z_2) is bivariate Gaussian then Z_1 is indep. of Z_2 iff $E[Z_1 Z_2] = E[Z_1]E[Z_2]$. In order to prove 3) we need to prove that $W_t - W_s$ is indep. of $W_v - W_u$, $0 \leq u \leq v \leq s \leq t$. b) implies that $(W_t - W_s, W_v - W_u)$ is bivariate Gaussian and it suffices to prove that

$$E[(W_t - W_s)(W_v - W_u)] = 0$$

But c) implies

$$E[(W_t - W_s)(W_v - W_u)] = K(t,v) - K(t,u) - K(s,v) + K(s,u) = v - u - v + u = 0.$$

Hence 3) ✓.

Finally, by b) we have that for all $0 \leq s \leq t$ the law of $W_t - W_s$ is Gaussian and using c) we get

$$E[W_t - W_s] = E[W_t] - E[W_s] = 0 - 0 = 0$$

$$\mathrm{Var}[W_t - W_s] = E[(W_t - W_s)^2] = K(t,t) - K(t,s) - K(s,t) + K(s,s)$$

$$= t - s - s + s = t - s.$$

Hence, 4) ✓.

2 Let $W = \{W_t\}_{t \in \mathbb{R}_+}$ be a B.m and $a > 0$. Let $\mathbb{F}$ be the minimal augmented filtration generated by W.
Which of the following processes are B.m.?
" " " " " $\mathbb{F}$-B.m.?

a) $X_t = -W_t \ , t \in \mathbb{R}_+$

b) $X_t = W_{a+t} - W_a \ , t \in \mathbb{R}_+$

c) $X_t = W_{at^2} \ , t \in \mathbb{R}_+$.

We will use the def. of Exercise 1.

a) $X_t = -W_t, t \in \mathbb{R}_+$: X has cont. paths because W has cont paths.
For any $0 \le t_1 < t_2 < \dots < t_n$, $n \in \mathbb{N}$ the vector

$$(X_{t_1}, \dots X_{t_n}) = (-W_{t_1}, \dots, -W_{t_n}) \text{ is Gaussian because}$$

$(W_{t_1}, \dots W_{t_n})$ is Gaussian, as W is a Gaussian process. Moreover

$$E[X_t] = E[-W_t] = -E[W_t] = 0$$

$$K(s,t) = E[X_s X_t] = E[(-W_s)(-W_t)] = E[W_s W_t] = \min(s,t).$$

Therefore X is a B.m. Note that $\mathbb{F} = \mathbb{F}^X$ because $X_t = \varphi(W_t)$, where $\varphi(x) = -x$ is a bijection. This yields that $\sigma(X_t) = \sigma(W_t)$ and therefore $\mathcal{F}_t^X = \sigma(X_s : 0 \le s \le t) = \sigma(W_s : 0 \le s \le t) = \mathcal{F}_t \ \forall t \Rightarrow X$ is a $\mathbb{F}$-B.m.

b) $X_t = W_{a+t} - W_a, t \in \mathbb{R}_+$: X has cont. paths because W has cont. paths.
For any $0 \le t_1 < t_2 < \dots < t_n$, $n \in \mathbb{N}$, the vector $(W_a, W_{a+t_1}, \dots W_{a+t_n})$ is multiv. Gaussian because W is a Gaussian process. By a linear transf. of the previous vector we get that

$$(X_{t_1}, \dots, X_{t_n}) = (W_{a+t_1} - W_a, \dots, W_{a+t_n} - W_a) \text{ is MV Gaussian}$$

and, hence X is a Gaussian process. We also have that

$$E[X_t] = E[W_{a+t} - W_a] = 0$$

$$K(s,t) = E[X_s X_t] = E[(W_{a+s} - W_a)(W_{a+t} - W_a)]$$
$$= K(a+s, a+t) - K(a+s, a) - K(a, a+t) + K(a,a)$$

$$= \min(a+s, a+t) - \underbrace{\min(a+s, a) - \min(a, a+t) + \min(a,a)}_{= -a}$$

$$= \min(a+s, a+t) - a = \min(s,t)$$

Therefore we can conclude that X is a B.m.

However, X is not a $\mathbb{F}$-Brownmotion because X is not adapted to $\mathbb{F}$. Note that $X_t = W_{a+t} - W_a$ depends on W_{a+t} which is not $\mathcal{F}_t$-measurable

c) $X_t = W_{at^2}$, $t \in \mathbb{R}_+$: X is not a B.m. because although it has contin. paths, is Gaussian and centered, one has that

$$K(s,t) = E[X_s X_t] = E[W_{as^2} W_{at^2}] = \min(as^2, at^2) \neq \min(s,t)$$

As X is not a B.m., it cannot be a $\mathbb{F}$-B.m.

$\boxed{3}$ Let $W = \{W_t\}_{t\in\mathbb{R}_+}$ be a B.m. Prove that the following processes are $\mathbb{F}$-martingales

a) $X_t = \exp(\theta W_t - \frac{\theta^2}{2} t)$, $t \in [0,T]$.

b) $Z_t = W_t^2 - t$, $t \in [0,T]$.

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a) $X_t = \exp(\theta W_t - \frac{\theta^2}{2} t)$, $t\in[0,T]$: X is $\mathbb{F}$-adapted because $X_t = \xi_t(W_t)$ where ξ_t is Borel meas. (continu.) for all $t\in[0,T]$.
This means that X_t is $\sigma(W_t)$-meas. and, hence, $\mathcal{F}_t$-meas.

- Integrability:

$$E[|X_t|] = E[\exp(\theta W_t - \frac{\theta^2}{2} t)] = e^{-\frac{\theta^2}{2}t} E[\exp(\theta W_t)]$$

$W_t \sim N(0,t)$ $\quad \underset{(*)}{=} e^{-\frac{\theta^2}{2}t} \cdot e^{\frac{\theta^2}{2}t} = 1 < +\infty$

$Z \sim N(\mu,\sigma^2)$
$E[e^{\theta Z}] = \exp(\mu\theta + \frac{\theta^2}{2}\sigma^2)$ (*)

- Martingale property.

To check that $E[X_t|\mathcal{F}_s] = X_s$ it suffices to check that $E[\frac{X_t}{X_s}|\mathcal{F}_s] = 1$, because X_s is $\mathcal{F}_s$-meas. and can go inside the cond. expect.
We have that

$$E[\frac{X_t}{X_s}|\mathcal{F}_s] = E[\exp(\theta(W_t - W_s) - \frac{\theta^2}{2}(t-s))|\mathcal{F}_s]$$

$$= E[\exp(\theta(W_t-W_s))|\mathcal{F}_s]\exp(-\frac{\theta^2}{2}(t-s))$$

($\exp(\theta(W_t-W_s))$ is indep. of $\mathcal{F}_s$.) $\quad = E[\exp(\theta(W_t-W_s))]\exp(-\frac{\theta^2}{2}(t-s))$

$(W_t - W_s \sim N(0,(t-s)))$ $\quad \underset{(*)}{=} \exp(\frac{\theta^2}{2}(t-s))\exp(-\frac{\theta^2}{2}(t-s)) = 1.$

b) $Z_t = W_t^2 - t$, $t\in[0,T]$. Z is $\mathbb{F}$-adapted by the same reasoning as in a).

- Integrability: $E[|Z_t|] = E[|W_t^2 - t|] \le E[W_t^2] + t = 2t < +\infty$.
- Martingale property:

$$E[Z_t|\mathcal{F}_s] = E[W_t^2 - t|\mathcal{F}_s] = E[(W_t - W_s + W_s)^2|\mathcal{F}_s] - t$$

linear. cond. exp. $\quad = E[(W_t-W_s)^2|\mathcal{F}_s] - E[2(W_t-W_s)W_s|\mathcal{F}_s] + E[W_s^2|\mathcal{F}_s] - t$

$W_t - W_s$ indep. of $\mathcal{F}_s$, W_s is $\mathcal{F}_s$-measurable $\quad = E[(W_t-W_s)^2] - 2W_s E[W_t - W_s|\mathcal{F}_s] + W_s^2 - t$

$$= t - s - 2W_s \underbrace{E[W_t - W_s]}_{0} + W_s^2 - t = W_s^2 - s = Z_s$$

✓

4) $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a filtered prob. space.
$M = \{M_t\}_{t\in\mathbb{R}_+}$ be a $\mathbb{F}$-martingale such that $E[|M_t|^2] < +\infty, t \in \mathbb{R}_+$
Prove that

$$E[(M_t - M_s)^2 | \mathcal{F}_s] = E[M_t^2 | \mathcal{F}_s] - E[M_s^2 | \mathcal{F}_s] \quad s \leq t$$

The square integrability of M ensures that the cond. expectation exist.
The result follows from the basic properties of the cond. expectation and expanding $(M_t - M_s)^2$.
We have that

$$E[(M_t - M_s)^2 | \mathcal{F}_s] = E[M_t^2 - 2M_tM_s + M_s^2 | \mathcal{F}_s]$$

(linearity of cond. exp.) $= E[M_t^2 | \mathcal{F}_s] - 2E[M_tM_s | \mathcal{F}_s] + E[M_s^2 | \mathcal{F}_s]$

(what is $\mathcal{F}_s$-meas goes out) $= E[M_t^2 | \mathcal{F}_s] - 2M_s E[M_t | \mathcal{F}_s] + M_s^2$

$$= E[M_t^2 | \mathcal{F}_s] - 2M_s^2 + M_s^2$$

$$= E[M_t^2 | \mathcal{F}_s] - E[M_s^2 | \mathcal{F}_s].$$

where we have used that M_s and M_s^2 are $\mathcal{F}_s$-meas.
and $E[M_t | \mathcal{F}_s] = M_s$.

$\boxed{5}$ $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a filt. prob. space.

An $\mathbb{F}$-adapted, integrable process X is a submartingale if

$$E[X_t \mid \mathcal{F}_s] \geq X_s \qquad 0 \leq s \leq t.$$

Let $M = \{M_t\}_{t \in \mathbb{R}_+}$ be a martingale and φ a convex fuct.

Show that if $Y_t = \varphi(M_t)$ is integrable the Y is a submarting.

- $Y_t \in L^1$, $t \in \mathbb{R}_+$ by assumption. ✓
- Y_t is $\mathbb{F}$-adapted ? ✓

  φ is convex on $\mathbb{R}$ (or open interval $\Rightarrow \varphi$ is continuous $\Rightarrow \varphi$ is Borel meas.

  $$\left.\begin{array}{l} \Rightarrow Y_t \text{ is } \sigma(M_t)\text{-measurable.} \\ \quad + \\ M \text{ is } \mathbb{F}\text{-adapted} \end{array}\right\} \Rightarrow Y_t \text{ is } \mathcal{F}_t\text{-meas}$$

- Submartingale property: ✓

$$E[Y_t \mid \mathcal{F}_s] = E[\varphi(M_t) \mid \mathcal{F}_s]$$

Cond. Jensen $$\geq \varphi(E[M_t \mid \mathcal{F}_s])$$

M is a martingale $$= \varphi(M_s) = Y_s$$

$\boxed{6}$ $X = \{X_t\}_{t\in\mathbb{R}_+}$ a stoch. process with indep. increm., $E[|X_t|] < +\infty, t \in \mathbb{R}_+$ and with $E[X_t] = m$ $t \in \mathbb{R}_+$. is a martingale.

- $\mathbb{F}^X = \{\mathcal{F}^X_t = \sigma(X_s : 0 \le s \le t)\}_{t\in\mathbb{R}_+}$
- $\mathbb{G}^X = \{\mathcal{G}^X_t = \sigma(X_v - X_u : 0 \le u \le v \le t)\}_{t\in\mathbb{R}_+}$

$\mathbb{F}^X = \mathbb{G}^X$ because for any $0 \le t_1 \le t_2 \le \dots < t_n \in \mathbb{R}_+, n \in \mathbb{N}$ we have $\sigma(X_{t_1}, \dots, X_{t_n}) = \sigma(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$

- Indep. increm. property can be written as follows:
For all $s \le t$, $X_t - X_s$ is indep. of all $X_v - X_u$, $0 \le u \le v \le s$.

Therefore, $X_t - X_s$ is indep. of $\mathcal{G}^X_s$ and we have

$$E[X_t - X_s | \mathcal{F}^X_s] = E[X_t - X_s | \mathcal{G}^X_s] = E[X_t - X_s] = E[X_t] - E[X_s] = m - m = 0$$

and we can conclude that X is a martingale.

$\boxed{7}$ $X = \{X_t\}_{t\in\mathbb{R}_+}$ a stoch. process with indep. increm., integrable.
Then X is a Markov process.

Using that $X_t - X_s$ is indep of $\mathcal{F}^X_s$ and X_s is $\mathcal{F}^X_s$-meas. we get that

$$P(X_t \in B | \mathcal{F}^X_s) = E[\mathbb{1}_{\{X_t \in B\}} | \mathcal{F}^X_s] = E[\mathbb{1}_{\{X_t - X_s + X_s \in B\}} | \mathcal{F}^X_s]$$

$$= E[\mathbb{1}_{\{X_t - X_s + x \in B\}}]\big|_{x = X_s} = \varphi(x)\big|_{x = X_s}$$

$X_t - X_s$ is indep. of $\mathcal{F}^X_s$
X_s is $\mathcal{F}^X_s$-meas.

On the other hand

$$P(X_t \in B | \sigma(X_s)) = E[\mathbb{1}_{\{X_t \in B\}} | \sigma(X_s)] = E[\mathbb{1}_{\{X_t - X_s + X_s \in B\}} | \sigma(X_s)]$$

$$= E[\mathbb{1}_{\{X_t - X_s + x \in B\}}]\big|_{x = X_s} = \varphi(x)\big|_{x = X_s}$$

$X_t - X_s$ is indep. of $\sigma(X_s) \subseteq \mathcal{F}^X_s$.
X_s is $\sigma(X_s)$-meas.

We can conclude that

$$P(X_t \in B | \mathcal{F}^X_s) = P(X_t \in B | \sigma(X_s))$$

$\boxed{8}$ $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a filtered probability space.

$X \in L^p$, $p \geq 1$

Show that $M_t = E[X \mid \mathcal{F}_t]$ is a $\mathbb{F}$-martingale and

$M_t \in L^p$, $t \in \mathbb{R}_+$

- M is $\mathbb{F}$-adapted ✓

Clearly $M_t = E[X \mid \mathcal{F}_t]$ is $\mathcal{F}_t$-meas. by the basic properties of the conditional expectation.

- $M_t \in L^p$, $t \in \mathbb{R}_+$ ✓

We have

$$E[|M_t|^p] = E[|E[X \mid \mathcal{F}_t]|^p] \leq E[E[|X|^p \mid \mathcal{F}_t]]$$

Jensen's ineq. for cond. expect. $\varphi(x) = |x|^p$

$$= E[|X|^p] < +\infty$$

↑ conservat. of expectation

In particular, $M_t \in L^1$, $t \in \mathbb{R}_+$.

- Martingale property. ✓ Let $s < t$ then

$$E[M_t \mid \mathcal{F}_s] = E[E[X \mid \mathcal{F}_t] \mid \mathcal{F}_s] = E[X \mid \mathcal{F}_s] = M_s$$

↑ Tower property.

9 $M = \{M_n\}_{n \in \mathbb{N}}$ $\mathcal{F}_n$-martingale.

$H = \{H_n\}_{n \geq 0}$ is predictable, i.e. $\mathcal{F}_{n-1}$-meas., and bounded, i.e.

$H_n \leq C_n < +\infty$, $\forall n \geq 1$, $H_0 = 0$

Prove that the process $G = \{G_n\}_{n \geq 0}$ defined by

$G_0 = 0$, $G_n = (H \cdot M)_n = \sum_{i=1}^{n} H_i (M_i - M_{i-1})$

is a $\mathbb{F}$-martingale

- G is $\mathbb{F}$-adapted ✓

$1 \leq i \leq n$, $H_i(M_i - M_{i-1})$ is $\mathcal{F}_i$-measurable $\Rightarrow$ G_n is $\mathcal{F}_n$-meas

- $E[|G_n|] < +\infty$, $n \geq 1$ ✓

$$E[|G_n|] = E\left[\left|\sum_{i=1}^{n} H_i(M_i - M_{i-1})\right|\right] \underset{\text{triang. ineq.}}{\leq} E\left[\sum_{i=1}^{n} |H_i| \, |M_i - M_{i-1}|\right]$$

$$\underset{\text{H-bounded}}{\leq} \sum_{i=1}^{n} C_i \, E[|M_i - M_{i-1}|] \leq \sum_{i=1}^{n} C_i \left(E[|M_i|] + E[|M_{i-1}|]\right)$$

$$\leq 2 E[|X_n|] \sum_{i=1}^{n} C_i$$

$|M_i|$ is a submartingale Exercise 5

$E[|M_n| \mid \mathcal{F}_{n-1}] \geq |M_{n-1}| \Rightarrow E[|M_n|] = E[E[|M_n| \mid \mathcal{F}_{n-1}]] \geq E[M_{n-1}]$

- Martingale property

$$E[G_n \mid \mathcal{F}_{n-1}] = E\left[\sum_{i=1}^{n} H_i(M_i - M_{i-1}) \mid \mathcal{F}_{n-1}\right]$$

$$= \sum_{i=1}^{n} E[H_i(M_i - M_{i-1}) \mid \mathcal{F}_{n-1}]$$

$1 \leq i \leq n-1$, $H_i(M_i - M_{i-1})$ is $\mathcal{F}_{n-1}$ meas.

H_n is $\mathcal{F}_{n-1}$-meas. (gaerant)

$$= \sum_{i=1}^{n-1} H_i(M_i - M_{i-1}) + H_n E[M_n - M_{n-1} \mid \mathcal{F}_{n-1}]$$

M is a marting.

$$= G_{n-1} + H_n \cdot 0 = G_{n-1}$$

$\boxed{10}$ Let $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ be a filtered prob. space.

Assume $Q \ll P$ with $Z_T := \frac{dQ}{dP}$

Prove that $Q|_{\mathcal{F}_t} \ll P|_{\mathcal{F}_t}$, $t \in [0,T]$ and show that

$Z_t := \frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}}$ is a $\mathbb{F}$-martingale.

Prove also that an $\mathbb{F}$-adap. process Y is a martingale under Q

iff ZY is a martingale under P.

$P|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ are the restrictions on $\mathcal{F}_t$ of P and Q,

that is, $P|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ coin-cide with P and Q, respectively,

on all sets of $\mathcal{F}_t$. Hence,

$$\forall A \in \mathcal{F}_t \text{ s.t. } 0 = P|_{\mathcal{F}_t}(A) = P(A) \underset{Q \ll P}{\Rightarrow} 0 = Q(A) = Q|_{\mathcal{F}_t}(A)$$

We will check that, indeed, $Z_t = E_P[\frac{dQ}{dP} | \mathcal{F}_t]$ and,

hence, a martingale by exercise 8.

Let $A \in \mathcal{F}_t$.

On the one hand

$$Q(A) = Q|_{\mathcal{F}_t}(A) \underset{\substack{\uparrow \\ \text{R.N. theorem} \Rightarrow \frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} \text{ is } \mathcal{F}_t\text{-meas.}}}{=} E_{P|_{\mathcal{F}_t}}\left[\frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} 1_A\right] = E_{P|_{\mathcal{F}_t}}[Z_t 1_A]$$

$$= E_P[Z_t 1_A] \quad (\text{because if } N \text{ is } \mathcal{F}_t\text{-meas. } E_{P|_{\mathcal{F}_t}}[N] = E_P[N])$$

On the other hand

$$Q(A) = E_Q[1_A] = E_P[\frac{dQ}{dP} 1_A] = E[E_P[\frac{dQ}{dP} 1_A | \mathcal{F}_t]]$$

$$= E[E_P[\frac{dQ}{dP} | \mathcal{F}_t] 1_A]$$

As $A \in \mathcal{F}_t$ is arbitrary $\underset{\text{Ex 9, Chap 1}}{\Rightarrow}$ $Z_t = E_P[\frac{dQ}{dP} | \mathcal{F}_t]$ P-a.s.

To prove the last statement we need a preliminary lemma.

Lemma

Let $W \in L^1(Q, \mathcal{F}_s)$, then for $0 \le t \le s \le T$

$$E_Q[W \mid \mathcal{F}_t] Z_t = E_P[W Z_s \mid \mathcal{F}_t] \quad P\text{-a.s. } (Q\text{-a.s.})$$

where $Z_t = E_P[\frac{dQ}{dP} \mid \mathcal{F}_t] \geq 0$.

Note that $P(Z_t = 0)$ may be strictly positive, so we cannot define Z_t^{-1} like in Exercise 11 in chap. 1.

Proof

We have that $\forall A \in \mathcal{F}_t$

$$E_Q[1\!\!1_A E_Q[W \mid \mathcal{F}_t]] \overset{\text{Def. of } E_Q[\cdot \mid \mathcal{F}_t]}{=} E_Q[1\!\!1_A W] \overset{\text{def of } E_Q[\cdot]}{=} E_P[1\!\!1_A W \frac{dQ}{dP}]$$

$$\underset{\text{Law of total expectation}}{=} E_P[1\!\!1_A E_P[W \frac{dQ}{dP} \mid \mathcal{F}_s]] \underset{\substack{W \text{ is } \mathcal{F}_s\text{-meas} \\ \text{and def of } Z}}{=} E_P[1\!\!1_A W Z_s] \underset{\text{law of total expect.}}{=} E_P[1\!\!1 E_P[W Z_s \mid \mathcal{F}_t]]$$

On the other hand

$$E_Q[1\!\!1_A E_Q[W \mid \mathcal{F}_t]] \overset{\text{Def. of } E_Q[\cdot]}{=} E_P[1\!\!1_A E_Q[W \mid \mathcal{F}_t] \frac{dQ}{dP}] = E_P[1\!\!1_A E_P[E_Q[W \mid \mathcal{F}_t] \frac{dQ}{dP} \mid \mathcal{F}_t]]$$

$$= E_P[1\!\!1_A E_Q[W \mid \mathcal{F}_t] Z_t]$$

$E_Q[W \mid \mathcal{F}_t]$ is $\mathcal{F}_t$-meas. and def of Z.

Therefore,

$$E_P[1\!\!1_A (E_Q[W \mid \mathcal{F}_t] Z_t - E_P[W Z_s \mid \mathcal{F}_t])] = 0 \quad \forall A \in \mathcal{F}_t$$

which yields the result by applying Exercise 9 in chap 1 with $\mathcal{E} = \mathcal{F}_t$. $\square$

Finally, we have to prove

Y is a $\mathbb{F}$-marting. under $Q \iff ZY$ is a $\mathbb{F}$-mart. under P.

$\Rightarrow$) • $Z_t Y_t$ is $\mathcal{F}_t$-meas. because Z_t is $\mathcal{F}_t$-meas. and Y_t is $\mathcal{F}_t$-meas. by assumption ✓

• $E_P[|Z_t Y_t|] \underset{Z_t \geq 0}{=} E_P[Z_t |Y_t|] \underset{\text{def of } Z}{=} E_P[E_P[\frac{dQ}{dP} | \mathcal{F}_t] |Y_t|]$

$\underset{|Y_t| \text{ is } \mathcal{F}_t\text{-meas.}}{=} E_P[E_P[\frac{dQ}{dP} |Y_t| | \mathcal{F}_t]] \underset{\text{Law of total exped.}}{=} E_P[\frac{dQ}{dP} |Y_t|] \underset{\text{Def. of } E_Q[\cdot]}{=} E_Q[|Y_t|] \underset{\substack{Y \in L^1(Q) \\ \text{by assumption}}}{<} +\infty$ ✓

• Martingale property:

$E_P[Z_t Y_t | \mathcal{F}_s] \underset{\text{Lemma}}{=} E_Q[Y_t | \mathcal{F}_s] Z_s \underset{Y \text{ is a } Q\text{-marting.}}{=} Y_s Z_s$ ✓

$\Leftarrow$) • Note that

$$\tilde{Y}_t = \begin{cases} \frac{Z_t Y_t}{Z_t} & \text{if } \{Z_t > 0\} \\ 0 & \text{if } \{Z_t = 0\} \end{cases}$$

$\{Z_t > 0\}$ and $\{Z_t = 0\}$ are $\mathcal{F}_t$-meas. $\frac{Z_t Y_t}{Z_t}$ is $\mathcal{F}_t$-meas. on $\{Z_t > 0\}$ $\Big\} \Rightarrow \tilde{Y}_t$ is $\mathcal{F}_t$-meas

and $Q(Z_t = 0) = 0 \Rightarrow \tilde{Y}_t = Y_t$ Q-a.s. Assuming $(\Omega, \mathcal{F}_T, Q)$ is complete.

• By assumption $E_P[Z_t |Y_t|] < +\infty$, $0 \leq t \leq T$, by reasoning as before we get that this is equivalent to $E_Q[|Y_t|] < +\infty$, $0 \leq t \leq T$. ✓

• $E_Q[Y_t | \mathcal{F}_s] \overset{?}{=} Y_s$ Q-a.s. ✓

By the previous Lemma que know that

(Q-a.s.) $E_Q[Y_t | \mathcal{F}_s] Z_s = E_P[Y_t Z_t | \mathcal{F}_s] \underset{ZY \text{ is a } P\text{-marting.}}{=} Y_s Z_s$

$$\Downarrow (*)$$

$$E_Q[Y_t | \mathcal{F}_s] = Y_s \quad Q\text{-a.s.}$$

(*) Note that this may fail for $\omega \in \{Z_s = 0\}$, but $Q(Z_s = 0) = 0$.

• Also note that $P(Z_s = 0) > 0$, if it was 0, then by Exercise 1L, Chap 1 we would have $P \sim Q$